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A NEW INERTIAL-PROJECTION METHOD FOR SOLVING SPLIT GENERALIZED MIXED EQUILIBRIUM AND HIERARCHICAL FIXED POINT PROBLEMS

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Abstract. In this paper, we introduce a new iterative algorithm of inertial form for approximating the common solution of Split Generalized Mixed Equilibrium Problem (SGMEP) and Hierarchical Fixed Point Problem (HFPP) in real Hilbert spaces. Motivated by the subgradient extragradient method, we incorporate the inertial technique to accelerate the convergence of the proposed method. Under standard and mild assumption of monotonicity and lower semicontinuity of the SGMEP and HFPP associated mappings, we establish the strong convergence of the iterative algorithm. Some numerical experiments are presented to illustrate the performance and behaviour of our method as well as comparing it with some related methods in the literature.

1. Introduction

Let C be a nonempty, closed and convex subset of a real Hilbert space H and $T : C \to C$ be a nonlinear mapping. T is said to be:

(i) firmly nonexpansive, if for each $x, y \in C$
$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle;$$
(ii) a contraction, if for every $x, y \in C$ and $c \in (0, 1)$
$$\|Tx - Ty\| \leq c\|x - y\|.$$
If $c = 1$, then T is called nonexpansive.

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We denote by $Fix(T)$, the set of fixed points of the mapping T, that is $Fix(T) = \{x \in C : x = Tx\}$. The mapping T is called quasi nonexpansive if $Fix(T) \neq \emptyset$ and

$$\|Tx - p\| \leq \|x - p\|, \quad \text{for all } p \in Fix(T),\ x \in C.$$

It is known that if T is quasinonexpansive, then $Fix(T)$ is closed and convex (see [45]).

Let $F : C \times C \to \mathbb{R}$ be a bifunction. The Equilibrium Problem (EP) in the sense of Blum and Oetlli [9], is to find a point $x^* \in C$ such that

$$F(x^*, y) \geq 0, \quad \text{for all } y \in C. \tag{1.1}$$

We denote by $EP(F, C)$, the set of solutions of EP (1.1). The EP unifies many important mathematical problems, such as optimization problems, complementary problems, fixed point problems, variational inequality problems, see [4, 6, 9, 25, 36, 37]. Let $B : C \to H$ be a nonlinear mapping. The Variational Inequality Problem (VIP) is to obtain a point $x^* \in C$ such that

$$\langle Bx^*, y - x^* \rangle \geq 0, \quad \text{for all } y \in C. \tag{1.2}$$

The set of solutions of the VIP is denoted $VIP(B, C)$. Solution to these class of problems, fixed point problems and related optimization problems have been investigated and iterative algorithm for approximating them have been proposed and studied by several authors, see [2, 5, 10, 14, 15, 17, 19, 20, 27, 28, 32, 35]. Let $\phi : C \to \mathbb{R}$ be a real valued function, then the Minimization Problem (MP), consists of finding a point $x^* \in C$ such that

$$\phi(x^*) \leq \phi(y), \quad \text{for all } y \in C. \tag{1.3}$$

The set of solutions of MP (1.3) will be denoted by $MP(\phi, C)$. For more on MP (see [1, 8, 23, 42]) and the references therein.

Let $F : C \times C \to \mathbb{R}$ be a bifunction, $B : C \to H$ a nonlinear mapping and $\phi : C \to \mathbb{R}$ a proper, convex and lower semicontinuous function. The Generalized Mixed Equilibrium Problem (GMEP) [10, 24, 26, 33, 38, 48] is the problem of finding a point $x^* \in C$ such that

$$F(x^*, y) + \langle Bx^*, y - x^* \rangle + \phi(y) - \phi(x^*) \geq 0, \quad \text{for all } y \in C. \tag{1.4}$$

We use $GMEP(F, B, \phi)$ to denote the set of solutions of GMEP (1.4). The GMEP includes several optimization problems as special cases. The relationship with the VIP and MP are easily observed by setting some maps to the zero map in inequality (1.4). Numerous problems in economics, science and engineering can be reduced to the problem of finding a solution to the GMEP (see [26, 34, 37]).

Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively and $L : H_1 \to H_2$ a bounded linear operator. In 1994, Censor and Elfving [12] introduced the notion of Split Feasibility Problem (SFP), which is defined as follows: find a point

$$x^* \in C \text{ such that } Lx^* \in Q. \tag{1.5}$$

The SFP is a special case of the Split Inverse Problem (SIP) first studied by Censor et al. [13]. In SIP, there are two given vector spaces X and Y and a linear operator $L : X \to Y$. The first Inverse Problem, IP_X say, is formulated in space X and the second one IP_Y formulated in space Y. Given this information, the SIP is formulated as follows: find $x^* \in X$ that solves IP_X, such that $y^* = Lx^* \in Y$ solves IP_Y. The SIP is used as a model for sensor networks, radiation therapy treatment planning, color imaging and other image restoration problems, see [11].

Furthermore, SFP over EP have been studied by some authors in the literature. For example, Moudafi [30] considered a SFP over EP and called this the Split Equilibrium Problem (SEP), see [22]. Let $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ be two bifunctions and $L : H_1 \to H_2$ be a bounded linear operator. The SEP is given as follows: find $x^* \in C$ such that

$$F(x^*, x) \geq 0, \quad \text{for all } x \in C,$$

and such that

$$y^* = Lx^* \in Q \text{ solves } G(y^*, y) \geq 0, \quad \text{for all } y \in Q.$$

For more, see [37, 46] and the references therein.

Since then, there have been several research in this direction where both bifunctions have same mononotonicity property and others with different monotonicity properties. Dinh et al. [16], studied the SEP involving pseudomonotone and monotone bifunctions. Also, in 2017 Rattanaseeha et al. [40], studied a split generalized equilibrium problem which involves both pseudomonotone bifunction and a monotone bifunction. For more literature on this class of problems (see [16, 40, 43]) and the references therein.

Moudafi and Mainge [31] introduced and studied the following Hierarchical Fixed Point Problem (HFPP) for a nonexpansive mapping S with respect to another non-expansive mapping T on C. The HFPP consists of finding a point $x^* \in Fix(S)$ such that

$$\langle (I - T)x^*, y - x^* \rangle \geq 0, \quad \text{for all } y \in Fix(S). \tag{1.6}$$

It is easy to see that the HFPP is equivalent to the problem of finding the fixed point of a map $A = P_{Fix(S)} \circ T$. Let Ω denote the solution set of the HFPP (1.6). Note that if $\Omega \neq \emptyset$, then Ω is closed and convex. The HFPP is general in the sense that it includes as special case the monotone VIP on fixed point sets, MP over equilibrium constraints, hierarchical MP... Very recently, Alansari et al. [7], studied an hybrid iterative scheme for approximating a common solution of a split EP involving both monotone and pseudomonotone bifiunction and a HFPP for a nonexpansive and quasi nonexpansive mappings. They proved a weak convergence theorem for their proposed algorithm.

Inspired by the works above and current research interest in this direction, in particular, in order to provide a partial answer to the future research posed by Alansari et al. [7] in conclusion of their work. We propose an iterative algorithm which combines the inertial technique, projection method, diagonal subgradient method and viscosity approach [8, 24], see Section 3. We prove a strong convergence theorem using the proposed algorithm to a solution of a SGMEP involving a pseudomonotone bifunction and a monotone bifunction which is also a solution of a HFPP. In our proposed method, the inertial extrapolation step was included to accelerate the rate of convergence of the algorithm, (see [1,3,25,39]) for more literature on inertial algorithms. We present some numerical examples to illustrate the behaviour and performance of our method as well as comparing it with some related methods in the literature.

2. preliminaries

We denote by $x_n \rightharpoonup v$ and $x_n \to v$ the weak and strong convergence respectively of a sequence $\{x_n\}$ in H to a point $v \in H$.

For each $x \in H$, there exists a unique nearest point $y = P_C x \in C$ such that

$$\|x - y\| \leq \|x - z\|, \quad \text{for all } z \in C.$$

The mapping $P_C : H \to C$ is called the metric projection from H onto C. It is well known that P_C satisfies the following conditions.

(i) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle$ for all $x, y \in H$.

(ii) For $x \in H$ and $y \in C$, $y = P_C x$ if and only if

$$\langle x - y, y - z\rangle \geq 0, \quad \text{for all } z \in C. \tag{2.1}$$

Definition 2.1. A mapping $T : C \to C$ is said to be demiclosed at 0, if for any sequence $\{x_n\} \subset C$ which converges weakly to $x \in C$ with $\|x_n - Tx_n\| = 0$, then $Tx = x$.

It is well known (see [21]) that the nonexpansive mapping is demiclosed.

Definition 2.2. A bifunction $f : C \times C \to \mathbb{R}$ is said to be

(a) strongly monotone on C, if there exists a constant $\gamma > 0$ such that

$$f(x, y) + f(y, x) \leq -\gamma\|x - y\|^2, \quad \text{for all } x, y \in C;$$

(b) monotone on C, if $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

(c) pseudomonotone on C, if $f(x, y) \geq 0$ implies $f(y, x) \leq 0$ for all $x, y \in C$.

It is obvious from above that a strongly pseudomonotone bifunction is contained in the class of monotone bifunctions and a monotone bifunction is pseudomonotone.

Definition 2.3 ([18])**.** Let $f : C \times C \to \mathbb{R}$ be a bifunction, where $f(x, \cdot)$ is convex for each $x \in C$. Then for $\epsilon \geq 0$ the ϵ-subdifferential (ϵ-diagonal subdifferential) of f at x, denoted by $\partial_\epsilon f(x, \cdot)(x)$ is given by

$$\partial_\epsilon f(x, \cdot)(x) = \{z \in H_1 : f(x, y) + \epsilon \geq f(x, x) + \langle z, y - x\rangle \text{ for all } y \in C\}.$$

For solving the GMEP, we assume $\phi : Q \to \mathbb{R}$ is proper, convex and lower semicontinuous, the nonlinear mapping, $B : Q \to H_2$ is continuous and monotone and the bifunction $F : Q \times Q \to \mathbb{R}$ satisfies the following restrictions:

(R1) $F(x,x)=0$ for all $x\in Q$;
(R2) F is monotone, i.e., $F(x,y)+F(y,x)\le 0$ for all $x,y\in Q$;
(R3) $\lim_{t\downarrow 0} F(x+t(z-x),y)\le F(x,y)$ for all $x,y,z\in Q$;
(R4) for each $x\in Q$, the function $y\mapsto F(x,y)$ is convex and lower semicontinuous.

The following lemmas are used in the sequel.

Lemma 2.1 ([44]). *In a real Hilbert space H, the following hold:*

(i) $\|x-y\|^2=\|x\|^2-2\langle x,y\rangle+\|y\|^2$ *for all* $x,y\in H$;
(ii) $\|x+y\|^2\le\|x\|^2+2\langle y,x+y\rangle$ *for all* $x,y\in H$;
(iii) $\|tx+(1-t)y\|^2=t\|x\|^2+(1-t)\|y\|^2-t(1-t)\|x-y\|^2$ *for all* $x,y\in H$ *and* $t\in(0,1)$.

Lemma 2.2 ([48]). *Let $B:Q\to H_2$ be a continuous and monotone mapping, $\phi:Q\to\mathbb{R}$ be a proper, lower semicontinuous and convex function, and $F:Q\times Q\to\mathbb{R}$ be a bifunction satisfying the conditions* (R1)-(R4). *Let $r>0$ be any given number and $x\in H_2$ be any given point. Then, the following hold.*

(i) *There exists $w\in Q$ such that*

$$F(w,y)+\langle B(w),y-w\rangle+\phi(y)-\phi(w)+\frac{1}{r}\langle y-w,w-x\rangle\ge 0,\quad \text{for all } y\in Q.$$

(ii) *Define a mapping $K_r^{F,B,\phi}:Q\to Q$ by $K_r^{F,B,\phi}(x)=\Big\{w\in Q: F(w,y)+\langle B(w),y-w\rangle+\phi(y)-\phi(w)+\frac{1}{r}\langle y-w,w-x\rangle\ge 0, y\in Q\Big\}$, $x\in Q$.*

The mapping $K_r^{F,B,\phi}$ satisfies the following characteristics:

(a) *$K_r^{F,B,\phi}$ is single valued;*
(b) *$K_r^{F,B,\phi}$ is fimrly nonexpansive, i.e., for all $z,y\in H$*

$$\|K_r^{F,B,\phi}z-K_r^{F,B,\phi}y\|^2\le\langle K_r^{F,B,\phi}z-K_r^{F,B,\phi}y,z-y\rangle;$$

(c) *$Fix(K_r^{F,B,\phi})=GMEP(F,B,\phi)$;*
(d) *$GMEP(F,B,\phi)$ is a closed and convex subset of Q.*

The following restrictions are assumed to be satisfied by the pseudomonotone bifunction $f:C\times C\to\mathbb{R}$:

(F1) $f(x,x)=0$ for all $x\in C$;
(F2) f is pseudomonotone on C with respect $x\in EP(f,C)$, that is, for $x\in EP(f,C)$, $f(x,y)\ge 0$ implies $f(y,x)\le 0$ for all $y\in C$;
(F3) f is strict paramonotone, that is the following holds

$$x\in EP(f,C),\quad y\in C,\quad f(y,x)\le 0 \text{ implies } y\in EP(f,C);$$

(F4) f is jointly weakly upper semicontinuous on $C\times C$ in the sense that, if $x,y\in C$ and $\{x_n\},\{y_n\}\subseteq C$ converges weakly to x and y, respectively, then $f(x_n,y_n)\to f(x,y)$ as $n\to+\infty$.

The following lemmas are very useful in obtaining the strong convergence of the sequence considered in this work.

Lemma 2.3 ([47]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following inequality*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\beta_n + \gamma_n, \quad n \in \mathbb{N},$$

where $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy the restrictions:

(i) $\sum_{n=1}^{+\infty} \alpha_n = +\infty$, $\lim_{n\to+\infty} \alpha_n = 0$;
(ii) $\limsup_{n\to+\infty} \beta_n \leq 0$;
(iii) $\gamma_n \geq 0$, $\sum_{n=1}^{+\infty} \gamma_n < +\infty$.

Then $\lim_{n\to+\infty} a_n = 0$.

Lemma 2.4 ([29,41]). *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_j\}$ of $\{n\}$ with $a_{n_j} \leq a_{n_j+1}$ for all $j \in \mathbb{N}$. Consider the integer $\{\tau(n)\}_{n\geq n_0}$ defined by*

$$\tau(n) := \max\{j \leq n : a_j \leq a_{j+1}\}.$$

Then $\{\tau(n)\}_{n\geq n_0}$ is a non-decreasing sequence satisfying $\lim_{n\to+\infty} \tau(n) = +\infty$ *and for all $n \geq n_0$, the following estimates hold:*

$$a_{\tau(n)} \leq a_{\tau(n)+1} \quad \textit{and} \quad a_n \leq a_{\tau(n)+1}.$$

3. Main Result

In this section, we state and prove our main result. First, we give an explicit statement of the proposed problem in this study. Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively and $L : H_1 \to H_2$ be a bounded linear operator. Let $f : C \times C \to \mathbb{R}$ and $F : Q \times Q \to \mathbb{R}$ be pseudomonotone and monotone bifunctions respectively satisfying restrictions (F1)-(F4) and (R1)-(R4). Let $B : C \to H_2$ be a nonlinear mapping and $\phi : Q \to \mathbb{R}$ a proper, convex and lower semicontinuous function. Let S be a nonexpansive mapping and T a quasinonexpansive mapping such that $I - T$ is monotone. We consider the problem of finding a point $x^* \in C$ such that

$$x^* \in EP(f, C) \cap Fix(P_{Fix(S)} \circ T) \tag{3.1}$$

and such that

$$y^* = Lx^* \in Q \text{ solves } GMEP(F, B, \phi). \tag{3.2}$$

We assume that the solution set of Problem (3.1)–(3.2) denoted by Γ is nonempty.

Remark 3.1 ([18]). If f is pseudomonotone on C with respect to $EP(f, C)$, then by restrictions (F1) and (F4), $EP(f, C)$ is closed and convex. From Lemma 2.2 (d), we have that $GMEP(F, B, \phi)$ is closed and convex. Also, if $Fix(P_{Fix(S)} \circ T) \neq \emptyset$, then the solution set of the HFPP is closed and convex see [31]. We assume $\Gamma \neq \emptyset$, hence Γ is well defined.

Algorithm 3.1. **Initialization.** Choose $x_0, x_1 \in C$. Take the sequence of real numbers $\{\mu_n\}$, $\{\beta_n\}$, $\{r_n\}$, $\{\theta_n\}$, $\{\gamma_n\}$, $\{\sigma_n\}$, $\{\epsilon_n\}$, $\{\alpha_n\}$ and $\{\lambda_n\}$ satisfying

(i) $0 < r < r_n$, $0 < a < \alpha_n < b < 1$, $0 < \acute{a} < \lambda_n < \acute{b} < 1$, $0 < \bar{a} < \sigma_n < \bar{b} < 1$, $\beta_n \geq 0$, $\gamma_n \in (0, 2/\|L\|^2)$ and $\epsilon_n \to 0$ as $n \to +\infty$;
(ii) $\sum_{n=1}^{+\infty} \mu_n^2 < +\infty$;
(iii) $\sum_{n=1}^{+\infty} \beta_n = +\infty$, $\lim_{n\to+\infty} \beta_n = 0$;
(iv) $\{\theta_n\} \subset [0, \theta]$, where $\theta \in [0, 1)$ and $\sum_{n=1}^{+\infty} \theta_n \|x_n - x_{n-1}\| < +\infty$;
(v) $\lim_{n\to+\infty} \frac{\theta_n}{\beta_n} = 0$.

[**Step 1.** Given x_{n-1} and x_n, $n \geq 1$, compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}). \tag{3.3}$$

Step 2. Take $g(w_n) \in \partial_{\epsilon_n}(f(w_n, \cdot))(w_n)$, $n \geq 1$. Calculate $\eta_n = \max\{1, \|g(w_n)\|\}$, $\lambda_n = \frac{\mu_n}{\eta_n}$ and

$$z_n = P_C(w_n - \lambda_n g(w_n)). \tag{3.4}$$

Step 3. If $w_n = z_n$ ($w_n \in EP(f, C)$), then stop. Otherwise, evaluate

$$\begin{cases} t_n = (1 - \sigma_n)Tz_n + \sigma_n z_n \;\; y_n = (1 - \alpha_n)w_n + \alpha_n St_n, \\ u_n = K_{r_n}^{F,B,\phi} L y_n, \\ v_n = y_n + \gamma_n L^*(u_n - Ly_n). \end{cases} \tag{3.5}$$

Step 4. Compute

$$x_{n+1} = \beta_n h(x_n) + (1 - \beta_n)v_n, \tag{3.6}$$

where h is a contraction.
Step 5. Set $n := n + 1$ and go to step 1.

Lemma 3.1. *Let $\{x_n\}$ be the sequence given by Algorithm* 3.1*, then $\{x_n\}$ is bounded. Consequently, the sequences $\{y_n\}$, $\{z_n\}$, $\{v_n\}$ and $\{u_n\}$ are bounded.*

Proof. Let $u \in \Gamma$, then from Lemma 2.1 (i) and (3.3), we have

$$\begin{aligned} \|w_n - u\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - u\|^2 \\ &= \|x_n - u\|^2 + 2\theta_n\langle x_n - u, x_n - x_{n-1}\rangle + \theta_n^2\|x_n - x_{n-1}\|^2 \\ &\leq \|x_n - u\|^2 + 2\theta_n\|x_n - u\|\|x_n - x_{n-1}\| + \theta_n^2\|x_n - x_{n-1}\|^2 \\ &= \|x_n - u\|^2 + \theta_n\|x_n - x_{n-1}\|(2\|x_n - u\| + \theta_n\|x_n - x_{n-1}\|). \end{aligned} \tag{3.7}$$

It also follows from Lemma 2.1 (iii), that

$$\begin{aligned} \|t_n - u\|^2 &= \|(1 - \sigma_n)(Tz_n - u) + \sigma_n(z_n - u)\|^2 \\ &= (1 - \sigma_n)\|Tz_n - u\|^2 + \sigma_n\|z_n - u\|^2 - \sigma_n(1 - \sigma_n)\|Tz_n - z_n\|^2 \\ &\leq \|z_n - u\|^2 - \sigma_n(1 - \sigma_n)\|Tz_n - z_n\|^2 \\ &\leq \|z_n - u\|^2. \end{aligned} \tag{3.8}$$

Next,

$$\|y_n - u\|^2 = \|(1 - \alpha_n)(w_n - u) + \alpha_n(St_n - u)\|^2$$

$$
\begin{aligned}
&\leq (1-\alpha_n)\|w_n-u\|^2+\alpha_n\|St_n-u\|^2-\alpha_n(1-\alpha_n)\|St_n-w_n\|^2\\
&\leq (1-\alpha_n)\|w_n-u\|^2+\alpha_n\|t_n-u\|^2-\alpha_n(1-\alpha_n)\|St_n-w_n\|^2\\
&\leq (1-\alpha_n)\|w_n-u\|^2+\alpha_n\|z_n-u\|^2-\alpha_n(1-\alpha_n)\|St_n-w_n\|^2\\
&\leq \|w_n-u\|^2+2\alpha_n\langle w_n-z_n,u-z_n\rangle-\alpha_n(1-\alpha_n)\|St_n-w_n\|^2, \qquad (3.9)
\end{aligned}
$$

but from the definition of z_n, we get

$$\langle w_n-z_n,u-z_n\rangle \leq \lambda_n\langle g(w_n),u-z_n\rangle.$$

Using this in (3.9), we obtain

$$
\begin{aligned}
\|y_n-u\|^2 \leq& \|w_n-u\|^2+2\lambda_n\alpha_n\langle g(w_n),u-z_n\rangle-\alpha_n(1-\alpha_n)\|St_n-w_n\|^2\\
=& \|w_n-u\|^2+2\lambda_n\alpha_n[\langle g(w_n),u-w_n\rangle+\langle g(w_n),w_n-z_n\rangle]\\
&-\alpha_n(1-\alpha_n)\|St_n-w_n\|^2\\
\leq& \|w_n-u\|^2+2\lambda_n\alpha_n\langle g(w_n),u-w_n\rangle+2\lambda_n\alpha_n\|g(w_n)\|\|w_n-z_n\|\\
&-\alpha_n(1-\alpha_n)\|St_n-w_n\|^2. \qquad (3.10)
\end{aligned}
$$

Note that by the definition of z_n and $w_n\in C$, we have

$$\|w_n-z_n\|^2\leq\lambda_n\langle g(w_n),w_n-z_n\rangle\leq\lambda_n\|g(w_n)\|\|w_n-z_n\|,$$

thus $\|w_n-z_n\|\leq\lambda_n\|g(w_n)\|$ and

$$
\begin{aligned}
\lambda_n\|g(w_n)\|\|w_n-z_n\| \leq& \lambda_n^2\|g(w_n)\|^2\\
=& \left(\frac{\mu_n}{\eta_n}\right)^2\|g(w_n)\|^2=\mu_n^2\left(\frac{\|g(w_n)\|}{\max(1,\|g(w_n)\|)}\right)^2\\
\leq& \mu_n^2, \qquad (3.11)
\end{aligned}
$$

which implies

$$\|w_n-z_n\|^2\leq\mu_n^2. \qquad (3.12)$$

Since $\sum_{n=1}^{+\infty}\mu_n^2<+\infty$, we obtain from above inequality, that

$$\|w_n-z_n\|\to 0 \quad \text{as} \quad n\to+\infty. \qquad (3.13)$$

Using (3.11) in (3.10), we have

(3.14)

$$\|y_n-u\|^2\leq\|w_n-u\|^2+2\lambda_n\alpha_n\langle g(w_n),u-w_n\rangle+2\alpha_n\mu_n^2-\alpha_n(1-\alpha_n)\|St_n-w_n\|^2.$$

By using Lemma 2.2, we have

$$
\begin{aligned}
\|u_n-Lu\|^2 =& \|K_{r_n}^{F,B,\phi}Ly_n-Lu\|^2=\|K_{r_n}^{F,B,\phi}Ly_n-K_{r_n}^{F,B,\phi}Lu\|^2\\
\leq& \langle K_{r_n}^{F,B,\phi}Ly_n-K_{r_n}^{F,B,\phi}Lu,Ly_n-Lu\rangle\\
=& \langle K_{r_n}^{F,B,\phi}Ly_n-Lu,Ly_n-Lu\rangle\\
=& \frac{1}{2}\left(\|K_{r_n}^{F,B,\phi}Ly_n-Lu\|^2+\|Ly_n-Lu\|^2-\|K_{r_n}^{F,B,\phi}Ly_n-Ly_n\|^2\right).
\end{aligned}
$$

Hence, $\|u_n - Lu\|^2 \leq \|Ly_n - Lu\|^2 - \|u_n - Ly_n\|^2$, which implies

$$2\langle Ly_n - Lu, u_n - Ly_n\rangle \leq -2\|u_n - Ly_n\|^2. \tag{3.15}$$

Now, from (3.5) and (3.15), we have

$$\begin{aligned}
\|v_n - u\|^2 =& \|y_n + \gamma_n L^*(u_n - Ly_n) - u\|^2 \\
=& \|y_n - u\|^2 + 2\gamma_n\langle y_n - u, L^*(u_n - Ly_n)\rangle + \gamma_n^2\|L^*(u_n - Ly_n)\|^2 \\
=& \|y_n - u\|^2 + 2\gamma_n\langle Ly_n - Lu, u_n - Ly_n\rangle + \gamma_n^2\|L^*(u_n - Ly_n)\|^2 \\
\leq& \|y_n - u\|^2 - \gamma_n(2 - \gamma_n\|L\|^2)\|u_n - Ly_n\|^2, \qquad (3.16)
\end{aligned}$$

which implies

$$\begin{aligned}
\|v_n - u\|^2 \leq& \|w_n - u\|^2 + 2\lambda_n\alpha_n\langle g(w_n), u - w_n\rangle + 2\alpha_n\mu_n^2 \\
& - \alpha_n(1-\alpha_n)\|St_n - w_n\|^2 - \gamma_n(2 - \gamma_n\|L\|^2)\|u_n - Ly_n\|^2. \qquad (3.17)
\end{aligned}$$

Since $w_n \in C$ and $g(w_n) \in \partial_{\epsilon_n} f(w_n, \cdot)(w_n)$, we obtain

$$f(w_n, u) + \epsilon_n = f(w_n, u) - f(w_n, w_n) + \epsilon_n \geq \langle g(w_n), u - w_n\rangle.$$

Using this in (3.17), we get

$$\begin{aligned}
\|v_n - u\|^2 \leq& \|w_n - u\|^2 + 2\lambda_n\alpha_n(f(w_n, u) + \epsilon_n) + 2\alpha_n\mu_n^2 \\
& - \alpha_n(1-\alpha_n)\|St_n - w_n\|^2 - \gamma_n(2 - \gamma_n\|L\|^2)\|u_n - Ly_n\|^2.
\end{aligned}$$

From the definition of λ_n and η_n, we obtain

$$\lambda_n = \frac{\mu_n}{\eta_n} \leq \mu_n.$$

Therefore, we get from above, that

$$\begin{aligned}
\|v_n - u\|^2 \leq& \|w_n - u\|^2 + 2\lambda_n\alpha_n f(w_n, u) + 2\alpha_n(\mu_n\epsilon_n + \mu_n^2) \\
& - \alpha_n(1-\alpha_n)\|St_n - w_n\|^2 - \gamma_n(2 - \gamma_n\|L\|^2)\|u_n - Ly_n\|^2. \qquad (3.18)
\end{aligned}$$

Since $u \in \Gamma$ and $w_n \in C$, we have $f(u, w_n) \geq 0$, then it follows from the monotonicity of f that $f(w_n, u) \leq 0$ and

$$\begin{aligned}
\|v_n - u\|^2 \leq& \|w_n - u\|^2 + 2\alpha_n(\mu_n\epsilon_n + \mu_n^2) - \alpha_n(1-\alpha_n)\|St_n - w_n\|^2 \\
& - \gamma_n(2 - \gamma_n\|L\|^2)\|u_n - Ly_n\|^2 \\
\leq& \|w_n - u\|^2 + 2\alpha_n(\mu_n\epsilon_n + \mu_n^2), \qquad (3.19)
\end{aligned}$$

which implies that

$$\|v_n - u\| \leq \|w_n - u\| + \sqrt{(2\alpha_n(\mu_n\epsilon_n + \mu_n^2)}. \tag{3.20}$$

Furthermore, we have from (3.6) and some $M_1, M_2 > 0$, that

$$\begin{aligned}
\|x_{n+1} - u\| =& \|\beta_n h(x_n) + (1-\beta_n)v_n - u\| \\
\leq& \beta_n\|h(x_n) - u\| + (1-\beta_n)\|v_n - u\| \\
\leq& \beta_n\|h(x_n) - h(u)\| + \beta_n\|h(u) - u\| + (1-\beta_n)\|v_n - u\|
\end{aligned}$$

$$
\begin{aligned}
&\le c\beta_n\|x_n-u\|+\beta_n\|h(u)-u\|+(1-\beta_n)\Big(\|w_n-u\|\\
&\quad+\sqrt{2\alpha_n(\mu_n\epsilon_n+\mu_n^2)}\Big)\\
&\le c\beta_n\|x_n-u\|+(1-\beta_n)\Big(\|x_n-u\|+\theta_n\|x_n-x_{n-1}\|\\
&\quad+\sqrt{2\alpha_n(\mu_n\epsilon_n+\mu_n^2)}\Big)+\beta_n\|h(u)-u\|\\
&\le[1-\beta_n(1-c)]\|x_n-u\|+\theta_n(1-\beta_n)\|x_n-x_{n-1}\|+\beta_n\|h(u)-u\|\\
&\quad+\sqrt{2\alpha_n(\mu_n\epsilon_n+\mu_n^2)}\\
&=[1-\beta_n(1-c)]\|x_n-u\|+\theta_n(1-\beta_n)\|x_n-x_{n-1}\|\\
&\quad+\frac{\beta_n(1-c)}{1-c}\|h(u)-u\|+\sqrt{2\alpha_n(\mu_n\epsilon_n+\mu_n^2)}\\
&\le\max\left\{\|x_n-u\|,\frac{\|h(u)-u\|}{(1-c)}\right\}+\theta_n(1-\beta_n)\|x_n-x_{n-1}\|\\
&\quad+\sqrt{2\alpha_n(\mu_n\epsilon_n+\mu_n^2)}\\
&\le\max\left\{\max\left\{\|x_{n-1}-u\|,\frac{\|h(u)-u\|}{(1-c)}\right\}\right\}\\
&\quad+\theta_{n-1}(1-\beta_{n-1})\|x_{n-1}-x_{n-2}\|\\
&\quad+\sqrt{2\alpha_{n-1}(\mu_{n-1}\epsilon_{n-1}+\mu_{n-1}^2)}+\theta_n(1-\beta_n)\|x_n-x_{n-1}\|\\
&\quad+\sqrt{2\alpha_n(\mu_n\epsilon_n+\mu_n^2)}\\
&\vdots\\
&\le\max\left\{\|x_1-u\|,\frac{\|h(u)-u\|}{1-c}\right\}+M_1+M_2\\
&<+\infty,
\end{aligned}
$$

where

$$M_1=\sum_{i=1}^{n}\theta_i(1-\beta_i)\|x_i-x_{i-1}\|<+\infty,$$

by condition (iv) and

$$M_2=\sum_{i=1}^{n}\sqrt{2\alpha_i(\mu_i\epsilon_i+\mu_i^2)}.$$

Hence, $\{x_n\}$ is bounded. Consequently, all other sequences in Algorithm 3.1 are bounded. ☐

Lemma 3.2. *The following inequality is satisfied from* (3.6) *and all* $u\in\Gamma$

$$\|x_{n+1}-u\|^2\le\left(1-\frac{2\beta_n(1-c)}{1-c\beta_n}\right)\|x_n-u\|^2$$

$$+\frac{2\beta_n(1-c)}{1-c\beta_n}\left(\frac{\langle h(u)-u,x_{n+1}-u\rangle}{1-c}+\frac{\beta_n M_3}{1-c}\right)$$
$$+\frac{\theta_n(1-\beta_n)}{1-c\beta_n}(\|x_n-x_{n-1}\|)(M_4+\|x_n-x_{n-1}\|)+2\alpha_n(\mu_n\epsilon_n+\mu_n^2),$$

for some $M_3, M_4 > 0$.

Proof. Let $u\in\Gamma$, then from Lemma 2.1 (ii), (3.4) and some $M_3, M_4 > 0$, we have

$$\begin{aligned}\|x_{n+1}-u\|^2 =&\|\beta_n(h(x_n)-u)+(1-\beta_n)(v_n-u)\|^2\\ \le&(1-\beta_n)^2\|v_n-u\|^2+2\beta_n\langle h(x_n)-u,x_{n+1}-u\rangle\\ \le&(1-\beta_n)^2\|y_n-u\|^2+2\beta_n\langle h(x_n)-h(u),x_{n+1}-u\rangle\\ &+2\beta_n\langle h(u)-u,x_{n+1}-u\rangle\\ \le&(1-\beta_n)^2\|w_n-u\|^2+2\beta_n\|h(x_n)-h(u)\|\|x_{n+1}-u\|\\ &+2\beta_n\langle h(u)-u,x_{n+1}-u\rangle+(1-\beta_n)^2(2\alpha_n(\mu_n\epsilon_n+\mu_n^2))\\ \le&(1-\beta_n)^2\left(\|x_n-u\|^2+\theta_n\|x_n-x_{n-1}\|(2\|x_n-u\|+\theta_n\|x_n-x_{n-1}\|)\right)\\ &+2\beta_n\langle h(u)-u,x_{n+1}-u\rangle+2\alpha_n(\mu_n\epsilon_n+\mu_n^2)\\ &+c\beta_n(\|x_n-u\|^2+\|x_{n+1}-u\|^2)\\ =&[1-2\beta_n+c\beta_n]\|x_n-u\|^2+c\beta_n\|x_{n+1}-u\|^2+\beta_n^2\|x_n-u\|^2\\ &+2\beta_n\langle h(u)-u,x_{n+1}-u\rangle+2\alpha_n(\mu_n\epsilon_n+\mu_n^2)\\ &+\theta_n(1-\beta_n)^2\|x_n-x_{n-1}\|\Big(2\|x_n-u\|+\theta_n\|x_n-x_{n-1}\|\Big),\end{aligned}$$

which implies

$$\begin{aligned}(1-c\beta_n)\|x_{n+1}-u\|^2\le&(1-2\beta_n+c\beta_n)\|x_n-u\|^2+\beta_n^2\|x_n-u\|^2\\ &+2\beta_n\langle h(u)-u,x_{n+1}-u\rangle+2\alpha_n(\mu_n\epsilon_n+\mu_n^2)\\ &+\theta_n(1-\beta_n)\|x_n-x_{n-1}\|\Big(2\|x_n-u\|+\theta_n\|x_n-x_{n-1}\|\Big)\end{aligned}$$

and

$$\begin{aligned}\|x_{n+1}-u\|^2\le&\left(\frac{1-2\beta_n+c\beta_n}{1-c\beta_n}\right)\|x_n-u\|^2\\ &+\frac{\theta_n(1-\beta_n)}{1-c\beta_n}\|x_n-x_{n-1}\|\Big(2\|x_n-u\|+\theta_n\|x_n-x_{n-1}\|\Big)\\ &+\frac{\beta_n^2}{1-c\beta_n}\|x_n-u\|^2+\frac{2\beta_n}{1-c\beta_n}\langle h(u)-u,x_{n+1}-u\rangle\\ &+\frac{2\alpha_n}{1-c\beta_n}(\mu_n\epsilon_n+\mu_n^2)\\ \le&\left(1-\frac{2\beta_n(1-c)}{1-c\beta_n}\right)\|x_n-u\|^2\end{aligned}\tag{3.21}$$

$$+\frac{2\beta_n(1-c)}{1-c\beta_n}\left(\frac{\langle h(u)-u,x_{n+1}-u\rangle}{1-c}+\frac{\beta_n M_3}{1-c}\right)$$
$$+\frac{\theta_n(1-\beta_n)}{1-c\beta_n}(\|x_n-x_{n-1}\|)\left(M_4+\theta_n\|x_n-x_{n-1}\|\right)$$
$$+2\alpha_n\left(\mu_n\epsilon_n+\mu_n^2\right). \qquad \square$$

Theorem 3.2. *Let $\{x_n\}$ be given by Algortihm 3.1, then $\{x_n\}$ converges strongly to $u=P_\Gamma h(u)$, where P_Γ is the metric projection of H_1 onto Γ.*

Proof. We consider the following two possible cases for the sequence $\{\|x_n-u\|\}$.

Case 1. Suppose there exists $n\in\mathbb{N}$ such that $\{\|x_n-u\|^2\}$ is nonincreasing. Then $\{\|x_n-u\|^2\}$ converges and

$$\|x_n-u\|^2-\|x_{n+1}-u\|^2\to 0 \quad \text{as} \quad n\to+\infty.$$

From (3.3) and condition (iv), we get

(3.22)
$$\|w_n-x_n\|=\|x_n+\theta_n(x_n-x_{n-1})-x_n\|\le\theta_n\|x_n-x_{n-1}\|\to 0 \quad \text{as} \quad n\to+\infty.$$

Observe from (3.7) and (3.18), that

$$\begin{aligned}\|v_n-u\|^2\le&\|w_n-u\|^2+2\alpha_n(\mu_n\epsilon_n+\mu_n^2)\\&-\alpha_n(1-\alpha_n)\|St_n-w_n\|^2-\gamma_n(2-\gamma_n\|L\|^2)\|u_n-Ly_n\|^2\\\le&\|x_n-u\|^2+\theta_n\|x_n-x_{n-1}\|\Big(2\|x_n-u\|+\theta_n\|x_n-x_{n-1}\|\Big)\\&-\alpha_n(1-\alpha_n)\|St_n-w_n\|^2\\&-\gamma_n(2-\gamma_n\|L\|^2)\|u_n-Ly_n\|^2+2\alpha_n(\mu_n\epsilon_n+\mu_n^2),\end{aligned}$$

using this in (3.21), we get

$$\begin{aligned}\|x_{n+1}-u\|^2\le&(1-\beta_n)\Big(\|x_n-u\|^2+\theta_n\|x_n-x_{n-1}\|(2\|x_n-u\|+\theta_n\|x_n-x_{n-1}\|)\\&+2\alpha_n(\mu_n\epsilon_n+\mu_n^2)\Big)+2\beta_n\langle h(x_n)-h(u),x_{n+1}-u\rangle\\&-\alpha_n(1-\alpha_n)\|St_n-w_n\|^2-\gamma_n(2-\gamma_n\|L\|^2)\|u_n-Ly_n\|^2.\end{aligned}$$

This implies

$$\begin{aligned}\alpha_n(1-\alpha_n)(1-\beta_n)\|St_n-w_n\|^2\le&(1-\beta_n)\Big(\theta_n\|x_n-x_{n-1}\|(2\|x_n-u\|\\&+\theta_n\|x_n-x_{n-1}\|)+2\alpha_n(\mu_n\epsilon_n+\mu_n^2)\Big)\\&+2\beta_n\langle h(x_n)-h(u),x_{n+1}-u\rangle\\&+\|x_n-u\|^2-\|x_{n+1}-u\|^2-\beta_n\|x_n-u\|^2.\end{aligned}$$

Using conditions (i)-(iv), we get

$$\lim_{n\to+\infty}\|St_n-w_n\|=0. \tag{3.23}$$

Similarly, one gets,

$$\begin{aligned}\gamma_n(2-\gamma_n\|L\|^2)\|u_n-Ly_n\|^2 \leq &(1-\beta_n)\Big[\theta_n\|x_n-x_{n-1}\|\Big(2\|x_n-u\| \\ &+\theta_n\|x_n-x_{n-1}\|\Big)+2\alpha_n(\mu_n\epsilon_n+\mu_n^2)\Big] \\ &+2\beta_n\langle h(x_n)-h(u),x_{n+1}-u\rangle+\|x_n-u\|^2 \\ &-\|x_{n+1}-u\|^2-\beta_n\|x_n-u\|^2.\end{aligned}$$

Since $\gamma_n \in \left(0, \frac{2}{\|L\|^2}\right)$, we have

$$\|u_n-Ly_n\|\to 0 \quad \text{as} \quad n\to+\infty. \tag{3.24}$$

Recall from (3.18) that

$$\begin{aligned}\|v_n-u\|^2 \leq &\|w_n-u\|^2+2\lambda_n\alpha_n f(w_n,u)+2\alpha_n(\mu_n\epsilon_n+\mu_n^2) \\ &-\alpha_n(1-\alpha_n)\|St_n-w_n\|^2-\gamma_n(2-\gamma_n\|L\|^2)\|u_n-Ly_n\|^2,\end{aligned}$$

using this in (3.21), we obtain

$$\begin{aligned}2(1-\beta_n)\lambda_n\alpha_n(f(-w_n,u)) \leq &\|x_n-u\|^2-\|x_{n+1}-u\|^2 \\ &+(1-\beta_n)\theta_n\|x_n-x_{n-1}\|\Big(2\|x_n-u\|+\theta_n\|x_n-x_{n-1}\|\Big) \\ &-\beta_n\|x_n-u\|^2+2\beta_n\langle h(x_n)-u,x_{n+1}-u\rangle \\ &+2\alpha_n(\mu_n\epsilon_n+\mu_n^2).\end{aligned}$$

Taking limit as $n\to+\infty$ and using (iv), we get

$$2\lim_{n\to+\infty}(1-\beta_n)\lambda_n\alpha_n(-f(w_n,u))=0.$$

Since $0<\acute{a}<\lambda_n<\acute{b}<1$, $0<a<\alpha_n<b<1$ and $-f(w_n,u)\geq 0$, we have that

$$\limsup_{n\to+\infty} f(w_n,u)=0. \tag{3.25}$$

Next we show $\|Tz_n-z_n\|\to 0$ as $n\to+\infty$. Observe that

$$\|z_n-u\|^2=\|z_n-w_n+w_n-u\|^2\leq\|w_n-u\|^2+2\langle z_n-u,z_n-w_n\rangle.$$

It follows from this, (3.8), (3.9) and (3.16), that

$$\begin{aligned}\|v_n-u\|^2 \leq &\|w_n-u\|^2+2\alpha_n\langle z_n-u,z_n-w_n\rangle \\ &-\alpha_n\sigma_n(1-\sigma_n)\|Tz_n-z_n\|^2-\alpha_n(1-\alpha_n)\|St_n-w_n\|^2 \\ &-\gamma_n(2-\gamma_n\|L\|^2)\|u_n-Ly_n\|^2.\end{aligned} \tag{3.26}$$

Substituting (3.26) into (3.21), we get

$$\begin{aligned}\sigma_n\alpha_n(1-\beta_n)(1-\alpha_n)\|Tz_n-z_n\|^2 \leq &\|x_n-u\|^2-\|x_{n+1}-u\|^2 \\ &+(1-\beta_n)\theta_n\|x_n-x_{n-1}\|\Big(2\|x_n-u\| \\ &+\theta_n\|x_n-x_{n-1}\|\Big)-\beta_n\|x_n-u\|^2 \\ &+2\beta_n\langle h(x_n)-u,x_{n+1}-u\rangle\end{aligned}$$

$$+ 2\alpha_n\|z_n - u\|\|z_n - w_n\|.$$

Again, since $0 < a < \alpha_n < b < 1$, $0 < \bar{a} < \sigma_n < \bar{b} < 1$, it follows that

$$\lim_{n\to+\infty} \|Tz_n - z_n\| = 0. \tag{3.27}$$

Observe that

$$\begin{aligned}\|St_n - z_n\|^2 \leq& \|St_n - w_n\|^2 + 2\langle w_n - z_n, St_n - z_n\rangle \\ \leq& \|St_n - w_n\|^2 + 2\|w_n - z_n\|\|St_n - z_n\|,\end{aligned}$$

which implies by condition, (3.13) and (3.23), that

$$\lim_{n\to+\infty} \|St_n - z_n\|^2 = 0. \tag{3.28}$$

The following holds by triangular inequality, (3.27) and (3.28)

$$\|Tz_n - St_n\| \leq \|Tz_n - z_n\| + \|z_n - St_n\| \to 0 \quad \text{as} \quad n \to +\infty. \tag{3.29}$$

Also,

$$\lim_{n\to+\infty} \|t_n - z_n\| = \lim_{n\to+\infty} (1 - \sigma_n)\|z_n - Tz_n\| = 0. \tag{3.30}$$

It follows again by triangular inequality, that

(3.31)

$$\begin{cases} \lim_{n\to+\infty} \|t_n - w_n\| \leq \lim_{n\to+\infty}(\|t_n - z_n\| + \|z_n - w_n\|) = 0, \\ \lim_{n\to+\infty} \|St_n - t_n\| \leq \lim_{n\to+\infty}(\|St_n - z_n\| + \|z_n - t_n\|) = 0, \\ \lim_{n\to+\infty} \|y_n - t_n\| \leq \lim_{n\to+\infty}(1 - \alpha_n)\|w_n - t_n\| + \lim_{n\to+\infty} \alpha_n\|St_n - t_n\| = 0, \\ \lim_{n\to+\infty} \|y_n - w_n\| \leq \lim_{n\to+\infty}(\|y_n - t_n\| + \|t_n - w_n\|) = 0, \\ \lim_{n\to+\infty} \|y_n - x_n\| \leq \lim_{n\to+\infty}(\|y_n - w_n\| + \|x_n - w_n\|) = 0. \end{cases}$$

Again,

$$\begin{aligned}\|Sz_n - z_n\|^2 =& \|Sz_n - St_n + St_n - z_n\|^2 \\ \leq& \|Sz_n - St_n\|^2 + 2\langle St_n - z_n, Sz_n - z_n\rangle \\ \leq& \|z_n - t_n\|^2 + 2(\|St_n - z_n\| \times \|Sz_n - z_n\|),\end{aligned}$$

we obtain by (3.28) and (3.30), that

$$\|Sz_n - z_n\|^2 \to 0 \quad \text{as} \quad n \to +\infty. \tag{3.32}$$

Finally, we show that $\|x_{n+1} - x_n\| \to 0$ as $n \to +\infty$. Indeed, we have from (3.5) and (3.24), that

$$\begin{aligned}\lim_{n\to+\infty} \|v_n - y_n\| =& \lim_{n\to+\infty} \|y_n + \gamma_n L^*(u_n - Ly_n) - y_n\| \\ \leq& \lim_{n\to+\infty} \gamma_n\|L\|\|u_n - Ly_n\| = 0 \end{aligned} \tag{3.33}$$

and

$$\|x_{n+1} - v_n\| = \|\beta_n h(x_n) + (1 - \beta_n)v_n - v_n\| \leq \beta_n\|h(x_n) - v_n\|,$$

which by condition (iii), implies that

$$\|x_{n+1} - v_n\| \to 0 \quad \text{as} \quad n \to +\infty. \tag{3.34}$$

Hence, by (3.22), (3.31), (3.33) and (3.34), we obtain

$$\lim_{n\to+\infty} \|x_{n+1} - x_n\| \le \lim_{n\to+\infty} \Big(\|x_{n+1} - v_n\| + |v_n - y_n\| + \|y_n - w_n\| + \|w_n - x_n\|\Big) = 0.$$

Since $\{x_n\}$ is bounded, then there exists a subsequence $\{x_{n_j}\}$ such that $x_{n_j} \rightharpoonup v$ and $\limsup_{n\to+\infty} f(x_n, u) = \lim_{j\to+\infty} f(x_{n_j}, u)$. It follows from (3.13), (3.23), (3.30) and (3.31), that the sequences $\{w_n\}$, $\{t_n\}$, $\{z_n\}$ and $\{y_n\}$ all converge weakly to v. Consequently, $Lz_{n_j} \rightharpoonup Lv$ and $Ly_{n_j} \rightharpoonup Lv$. It follows from the demiclosedness of $I - S$ and (3.32), that $v \in Fix(S)$. Next we show that $v = (P_{Fix(S)} \circ T)v$. It follows from (3.5), that

$$t_n - St_n = \sigma_n(I - T)z_n + (Tz_n - St_n),$$

which implies

$$\frac{1}{\sigma_n}(t_n - St_n) = (I - T)z_n + \frac{1}{\sigma_n}(Tz_n - St_n), \tag{3.35}$$

thus for all $w \in Fix(S)$, the monotonicity of $(I - T)$ and (3.35), we have

$$\begin{aligned}\Big\langle \frac{t_n - St_n}{\sigma_n}, z_n - w\Big\rangle =&\langle (I - T)z_n - (I - T)w, z_n - w\rangle \\ &+ \langle (I - T)w, z_n - w\rangle + \frac{1}{\sigma_n}\langle Tz_n - St_n, z_n - w\rangle \\ \ge&\langle (I - T)w, z_n - w\rangle + \frac{1}{\sigma_n}\langle Tz_n - St_n, z_n - w\rangle.\end{aligned} \tag{3.36}$$

Since $\{z_n\}$ and $\{z_n - w\}$ are bounded, it follows from (3.29), (3.31) and (3.36), that

$$\limsup_{n\to+\infty} \langle (I - T)w, z_n - w\rangle \le 0, \quad \text{for all } w \in Fix(S). \tag{3.37}$$

Replacing n with n_j and letting $j \to +\infty$ in (3.37), we obtain

$$\langle (I - T)w, v - w\rangle \le 0, \quad \text{for all } w \in Fix(S).$$

Note that $tw + (1 - t)v \in$ F(S) for $t \in (0, 1)$, since $Fix(S)$ is convex. Hence,

$$\langle (I - T)(tw + (1 - t)v), v - w\rangle \le 0, \quad \text{for all } w \in Fix(S).$$

Setting $t \to 0_+$ and using the continuity of $(I - T)$, we obtain

$$\langle (I - T)v, v - w\rangle \le 0, \quad \text{for all } w \in Fix(S).$$

Thus $v \in F(P_{Fix(S)} \circ T)$. Next, we show that $v \in EP(f, C)$. Since $x_{n_j} \rightharpoonup v$, $\|w_{n_j} - x_{n_j}\| \to 0$ and $\limsup_{n\to+\infty} f(w_n, u) = \lim_{j\to+\infty} f(w_{n_j}, u)$, by the upper weakly continuity of $f(\cdot, u)$ and (3.25), we have

$$f(v, u) \ge \limsup_{j\to+\infty} f(w_{n_j}, u) = \lim_{j\to+\infty} f(w_{n_j}, u) = \limsup_{n\to+\infty} f(w_n, u) = 0.$$

Since $u \in \Gamma$ and $v \in C$, we have $f(u,v) \geq 0$. By the pseudomonotone property of f, we have $f(v,u) \leq 0$. Consequently, we obtain $f(v,u) = 0$, and by restriction $F3$, we get $v \in EP(f,C)$. Furthermore, we show that $Lv \in Fix(K_{r_n}^{F,B,\phi}) = GMEP(F,B,\phi)$. Since $\lim_{n\to+\infty} \|y_n - x_n\| = 0$ and $x_{n_j} \rightharpoonup v$, it is easy to see that $y_{n_j} \rightharpoonup v$. It therefore follows from the continuity of L, that $Ly_{n_j} \rightharpoonup Lv$ and by (3.24), we get $u_{n_j} \rightharpoonup Lv$.

Now since $u_n = K_{r_n}^{F,B,\phi} Ly_n$, we have

$$F(u_n,w)+\langle B(u_n),w-u_n\rangle+\phi(w)-\phi(u_n)+\frac{1}{r_n}\langle w-u_n,u_n-Ly_n\rangle \geq 0, \quad \text{for all } w \in Q.$$

It follows from the monotonicity of F, that

$$\phi(w) - \phi(u_n) + \langle B(u_n), w - u_n\rangle + \frac{1}{r_n}\langle w - u_n, u_n - Ly_n\rangle \geq F(w,u_n), \quad \text{for all } w \in Q,$$

and

$$\phi(w) - \phi(u_{n_j}) + \langle B(u_{n_j}), w - u_{n_j}\rangle + \left\langle w - u_{n_j}, \frac{u_{n_j} - Ly_{n_j}}{r_{n_j}} \right\rangle \geq F(w,u_{n_j}), \tag{3.38}$$

for all $w \in Q$. This implies

$$\begin{aligned}\langle B(Ly_{n_j}), w - u_{n_j}\rangle \geq& \phi(u_{n_j}) - \phi(w) + \langle B(Ly_{n_j}), w - u_{n_j}\rangle - \langle B(u_{n_j}), w - u_{n_j}\rangle \\ &- \left\langle w - u_{n_j}, \frac{u_{n_j} - L_{n_j}}{r_{n_j}} \right\rangle + F(w,u_{n_j}) \\ =& \phi(u_{n_j}) - \phi(w) + \langle B(Ly_{n_j}) - B(u_{n_j}), w - u_{n_j}\rangle \\ &- \left\langle w - u_{n_j}, \frac{u_{n_j} - Ly_{n_j}}{r_{n_j}} \right\rangle + F(w,u_{n_j}).\end{aligned} \tag{3.39}$$

Since B is continuous and $\lim_{n\to+\infty} \|Ly_n - u_n\| = 0$, it follows that $\lim_{n\to+\infty} \|B(Ly_n) - B(u_n)\| = 0$. From the monotonicity of B, the weakly lower semicontinuity of ϕ and $u_{n_j} \rightharpoonup Lv$, it follows from (3.39), that

$$\langle B(Lv), w - Lv\rangle \geq \phi(Lv) - \phi(w) + F(w,Lv), \quad \text{for all } w \in Q. \tag{3.40}$$

For any $t \in (0,1]$ and $w \in Q$, set $z_t = tw + (1-t)Lv$ we have $z_t \in Q$ and thus satisfies (3.40). Using assumptions (R1) and (R4), we get

$$\begin{aligned}0 =& F(z_t,z_t) + \phi(z_t) - \phi(z_t) \\ \leq& tF(z_t,w) + (1-t)F(z_t,Lv) + t\phi(w) + (1-t)\phi(Lv) - \phi(z_t) \\ =& t[F(z_t,w) + \phi(w) - \phi(z_t)] + (1-t)[F(z_t,Lv) + \phi(Lv) - \phi(z_t)] \\ \leq& t[F(z_t,w) + \phi(w) - \phi(z_t)] + (1-t)t\langle B(Lv), w - Lv\rangle.\end{aligned}$$

This implies

$$F(z_t,w) + \phi(w) - \phi(z_t) + (1-t)\langle B(Lv), w - Lv\rangle \geq 0.$$

Letting $t \to 0_+$, we get

$$F(Lv,w) + \phi(w) - \phi(Lv) + \langle B(Lv), w - Lv\rangle \geq 0, \quad \text{for all } w \in C,$$

which implies $Lv \in GMEP(F, B, \phi)$.

To end Case 1, we show that $\{x_n\}$ converges strongly to $u = P_\Gamma h(u)$. To do this, it suffices to show that $\limsup_{n\to+\infty}\langle h(u) - u, x_{n+1} - u\rangle \le 0$ and apply Lemma 2.3. Indeed, choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup v \in H_1$ and

$$\limsup_{n\to+\infty}\langle h(u) - u, x_{n+1} - u\rangle = \lim_{j\to+\infty}\langle h(u) - u, x_{n_j+1} - u\rangle.$$

We have that $x_{n+1} \rightharpoonup v$ since $\|x_{n+1} - x_n\| \to 0$ as $n \to +\infty$. By applying (2.1), we have

$$\limsup_{n\to+\infty}\langle h(u) - u, x_{n+1} - u\rangle = \lim_{j\to+\infty}\langle h(u) - u, x_{n_j+1} - u\rangle = \langle h(u) - u, v - u\rangle \le 0.$$

Using (2.1), (3.21) and Lemma 2.3, we conclude that $\|x_n - u\| \to 0$ as $n \to +\infty$.

Case 2. Assume that $\{\|x_n - u\|\}$ is non monotone. For some n_0 large enough, let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined for all $n \ge n_0$ by

$$\tau(n) := \max\{j \in \mathbb{N} : j \le n, \|x_j - u\| \le \|x_{j+1} - u\|\}.$$

By Lemma 2.4, $\tau(n)$ is nondecreasing sequence such that $\tau(n) \to +\infty$ as $n \to +\infty$ and $0 \le \|x_{\tau(n)} - u\| \le \|x_{\tau(n)+1} - u\|$ for all $n \ge n_0$. Just by using similar argument as in Case 1, we have

$$\begin{aligned}\lim_{n\to+\infty}\|w_{\tau(n)} - x_{\tau(n)}\| &= \lim_{n\to+\infty}\|z_{\tau(n)} - w_{\tau(n)}\| = \lim_{n\to+\infty}\|y_{\tau(n)} - w_{\tau(n)}\| \\ &= \lim_{n\to+\infty}\|u_{\tau(n)} - Ly_{\tau(n)}\| = \lim_{n\to+\infty}\|Tz_{\tau(n)} - z_{\tau(n)}\| \\ &= \lim_{n\to+\infty}\|Sz_{\tau(n)} - z_{\tau(n)}\| = \lim_{n\to+\infty}\|x_{\tau(n)+1} - x_{\tau(n)}\| = 0\end{aligned}$$

and

$$\lim_{n\to+\infty}\langle h(u) - u, x_{\tau(n)+1} - u\rangle \le 0.$$

Since $\{x_{\tau(n)}\}$ is bounded, there exists a subsequence of $\{x_{\tau(n)}\}$ still denoted by $\{x_{\tau(n)}\}$ such that $x_{\tau(n)} \rightharpoonup v \in C$. Following similar argument as in Case 1, we obtain $v \in \Gamma$.

From (3.21), we get

$$\begin{aligned}\|x_{\tau(n)+1} - u\|^2 \le{}& \left(1 - \frac{2\beta_{\tau(n)}(1-c)}{1 - c\beta_{\tau(n)}}\right)\|x_{\tau(n)} - u\|^2 \\ &+ \frac{2\beta_{\tau(n)}(1-c)}{1 - c\beta_{\tau(n)}}\left(\frac{\langle h(u) - u, x_{\tau(n)+1} - u\rangle}{1-c} + \frac{\beta_{\tau(n)}M_3}{1-c}\right) \\ &+ \frac{\theta_{\tau(n)}(1 - \beta_{\tau(n)})}{1 - c\beta_{\tau(n)}}\left(\|x_{\tau(n)} - x_{\tau(n)-1}\|\right)\left(M_4 + \theta_{\tau(n)}\|x_{\tau(n)} - x_{\tau(n)-1}\|\right) \\ &+ 2\alpha_{\tau(n)}\left(\mu_{\tau(n)}\epsilon_{\tau(n)} + \mu_{\tau(n)}^2\right).\end{aligned}$$

Since $\|x_{\tau(n)} - u\| \le \|x_{\tau(n)+1} - u\|$ and $\beta_{\tau(n)} > 0$, we have

$$\frac{2\beta_{\tau(n)}(1-c)}{1 - c\beta_{\tau(n)}}\|x_{\tau(n)} - u\|^2 \le \frac{2\beta_{\tau(n)}(1-c)}{1 - c\beta_{\tau(n)}}\left(\frac{\langle h(u) - u, x_{\tau(n)+1} - u\rangle}{1-c} + \frac{\beta_{\tau(n)}M_3}{1-c}\right)$$

$$+ 2\alpha_{\tau(n)}\left(\mu_{\tau(n)}\epsilon_{\tau(n)} + \mu^2_{\tau(n)}\right) + \frac{\theta_{\tau(n)}(1-\beta_{\tau(n)})}{1-c\beta_{\tau(n)}} \times \left(\|x_{\tau(n)} - x_{\tau(n)-1}\|\right)\left(M_4 + \theta_{\tau(n)}\|x_{\tau(n)} - x_{\tau(n)-1}\|\right).$$

Hence,

$$\begin{aligned}\frac{2(1-c)}{1-c\beta_{\tau(n)}}\|x_{\tau(n)} - u\|^2 \leq & \frac{2(1-c)}{1-c\beta_{\tau(n)}}\left(\frac{\langle h(u)-u, x_{\tau(n)+1}-u\rangle}{1-c} + \frac{\beta_{\tau(n)}M_3}{1-c}\right) \\ & + \frac{2\alpha_{\tau(n)}}{\beta_{\tau(n)}}(\mu_{\tau(n)}\epsilon_{\tau(n)} + \mu^2_{\tau(n)}) \\ & + \frac{\theta_{\tau(n)}(1-\beta_{\tau(n)})}{\beta_{\tau(n)}(1-c\beta_{\tau(n)})}\left(\|x_{\tau(n)} - x_{\tau(n)-1}\|\right) \\ & \times \left(M_4 + \theta_{\tau(n)}\|x_{\tau(n)} - x_{\tau(n)-1}\|\right).\end{aligned}$$

This implies that $\limsup_{n\to+\infty}\|x_{\tau(n)} - u\|^2 \leq 0$ and

$$\lim_{n\to+\infty}\|x_{\tau(n)} - u\| = 0. \tag{3.41}$$

From (3.34) and (3.41), we obtain

$$\|x_{\tau(n)+1} - u\| \leq \|x_{\tau(n)} - u\| + \|x_{\tau(n)} - x_{\tau(n)+1}\| \to 0 \quad \text{as} \quad n \to +\infty.$$

Furthermore, for $n \geq n_0$, it is obvious that $\|x_n - u\| \leq \|x_{\tau(n)} - u\|$. Consequently, we get for all $n \geq n_0$, that

$$0 \leq \|x_n - u\| \leq \max\{\|x_{\tau(n)} - u\|, \|x_{\tau(n)+1} - u\|\} = \|x_{\tau(n)+1} - u\|.$$

Therefore, $\|x_n - u\| \to 0$ as $n \to +\infty$, that is $x_n \to u$. Thus completing the proof. □

If we set $B = \phi = 0$ in (3.1)–(3.2), we obtain the following method for obtaining a common solution of split EP and HFPP considered in [7].

Algorithm 3.3. **Initialization.** Choose $x_0, x_1 \in C$. Take the sequence of real numbers $\{\mu_n\}$, $\{\beta_n\}$, $\{r_n\}$, $\{\theta_n\}$, $\{\gamma_n\}$, $\{\sigma_n\}$, $\{\epsilon_n\}$, $\{\alpha_n\}$ and $\{\lambda_n\}$ satisfying

(i) $0 < r < r_n$, $0 < a < \alpha_n < b < 1$, $0 < \acute{a} < \lambda_n < \acute{b} < 1$, $0 < \bar{a} < \sigma_n < \bar{b} < 1$, $\beta_n \geq 0$, $\gamma_n \in (0, 2/\|L\|^2)$ and $\epsilon_n \to 0$ as $n \to +\infty$;
(ii) $\sum_{n=1}^{+\infty}\mu_n^2 < +\infty$;
(iii) $\sum_{n=1}^{+\infty}\beta_n = +\infty$, $\lim_{n\to+\infty}\beta_n = 0$;
(iv) $\{\theta_n\} \subset [0, \theta]$, where $\theta \in [0,1)$ and $\sum_{n=1}^{+\infty}\theta_n\|x_n - x_{n-1}\| < +\infty$;
(v) $\lim_{n\to+\infty}\frac{\theta_n}{\beta_n} = 0$.

Step 1. Given x_{n-1} and x_n, $n \geq 1$, compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}). \tag{3.42}$$

Step 2. Take $g(w_n) \in \partial_{\epsilon_n}(f(w_n, \cdot))(w_n)$, $n \geq 1$. Calculate $\eta_n = \max\{1, \|g(w_n)\|\}$, $\lambda_n = \frac{\mu_n}{\eta_n}$ and $z_n = P_C(w_n - \lambda_n g(w_n))$.

Step 3. If $w_n = z_n$ ($w_n \in EP(f, C)$), then go to step 3. Otherwise, evaluate

$$\begin{cases} t_n = (1-\sigma_n)Tz_n + \sigma_n z_n, \\ y_n = (1-\alpha_n)w_n + \alpha_n Stn, \\ u_n = K_{r_n}^F Ly_n, \\ v_n = y_n + \gamma_n L^*(u_n - Ly_n). \end{cases}$$

Step 4. Compute $x_{n+1} = \beta_n h(x_n) + (1-\beta_n)v_n$, where h is a contraction.
Step 5. Set $n := n+1$ and go to step 1.

We therefore give the following result as a consequence of our main theorem.

Corollary 3.4. Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively. Let $L : H_1 \to H_2$ be a bounded linear operator. Let $f : C \times C \to \mathbb{R}$ and $F : Q \times Q \to \mathbb{R}$ be bifunctions satisfying restrictions $F1$-$F4$ and $R1$-$R4$ respectively. Let $S : C \to C$ be a nonexpansive mapping and $T : C \to C$ be a quasinonexpansive mapping such that $I - T$ is monotone. Assume that $\Gamma = EP(f, C) \cap EP(F, Q) \cap \Omega \neq \emptyset$. Then the sequence $\{x_n\}$ given by Algorithm 3.3 converges strongly to $u = P_\Gamma h(u)$, where P_Γ is the metric projection of H_1 onto Γ.

4. Numerical Examples

We give some numerical examples to illustrate the behaviour and performance of our method as well as comparing it with some related methods in the literature.

Example 4.1. Let $H_1 = H_2 = C = Q = \mathbb{R}$ with inner product $\langle x, y\rangle = xy$ for all $x, y \in \mathbb{R}$ and the induced usual norm $|\cdot|$. Let $f : C \times C \to \mathbb{R}$ be defined by $f(x, y) = 2xy(y-x) + xy|y-x|$, for all $x, y \in H_1$. Define the bifunction $F : Q \times Q \to \mathbb{R}$ by $F(u, v) = -u^2 + v^2$, for all $u, v \in Q$, $B : Q \to H_2$ by $B(u) = \frac{u}{5}$ for all $u \in Q$ and $\phi : Q \to \mathbb{R}$ by $\phi(u) = 0$ for all $u \in Q$. For each $x \in H_1$, define the mapping $L : H_1 \to H_2$ by $Lx = x$ for all $x \in H_1$. Also define the mappings S and T respectively by $Sx = \frac{x}{2}$ and $Tx = x$. It is easy to see that f, F, S and T satisfy the conditions of Theorem 3.2 and that $\Gamma = \{0\}$. From Theorem 3.2, we can conclude that the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge to 0. Let $r_n = 1$, for all $n \geq 1$, it is easy to find that the resolvent $K_{r_n}^{F,B,\phi} Ly_n = \frac{5y_n}{16}$.

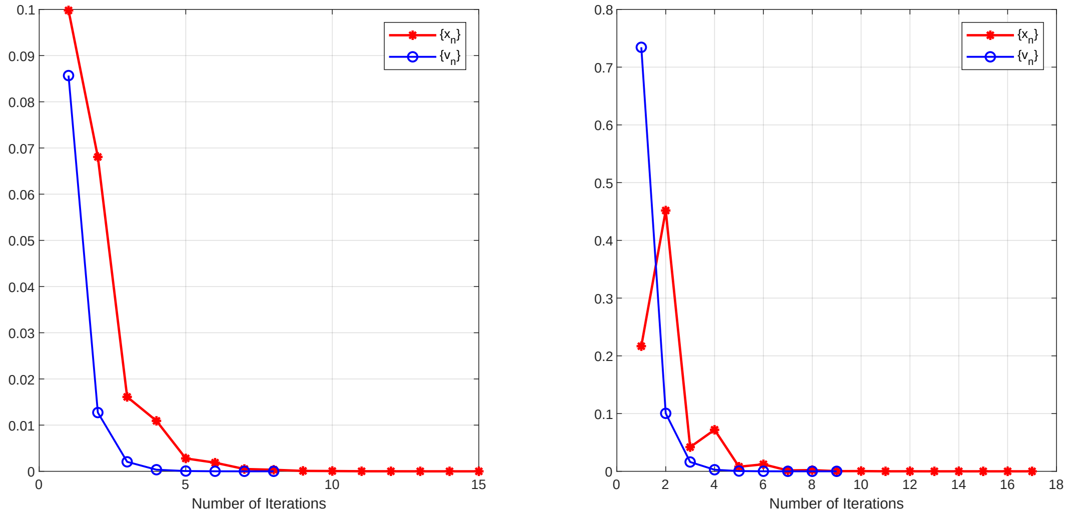

FIGURE 1. Numerical results for Example 4.1. Left: $x_0 = 0.8$, $x_1 = 0.5$; right: $x_0 = -1$, $x_1 = -3$.

Set $\sigma_n = \frac{1}{2n^2+3}$, $\alpha_n = \frac{1}{2n^2+5}$, $\epsilon_n = 0$, $\gamma_n = \frac{1}{5}$, $\lambda_n = \frac{1}{2}$. Also, let $\mu_n = \frac{1}{n}$, $\beta_n = \frac{1}{n+1}$ and $\theta_n = \frac{1}{4n^2+1}$. Then, after simplification Algorithm 3.1, becomes

$$\begin{cases} \text{Given } x_0 \text{ and } x_1 \in H_1, \\ w_n = x_n + \theta_n(x_n - x_{n-1}), \\ w_n \in H_1 \text{ such that } g(w_n) \in \partial_{\epsilon_n} f(w_n, \cdot)(w_n) = [w_n^2, 3w_n^2], \\ z_n = P_C(w_n - \lambda_n g(w_n)), \\ t_n = (1 - \sigma_n)Tz_n + \sigma_n z_n, \\ y_n = (1 - \alpha_n)w_n + \alpha_n St_n, \\ u_n = \frac{5y_n}{16}, \\ v_n = \frac{1}{15}(3u_n + 2y_n), \\ x_{n+1} = \beta_n h(x_n) + (1 - \beta_n)v_n. \end{cases}$$

We test our algorithm with varying values of initial terms x_0 and x_1, see Figure 1.

In this example, we set $B = \phi = 0$.

Example 4.2. Let $H_1 = H_2 = Q = \ell_2(\mathbb{R})$ be the linear spaces whose elements are all 2-summable sequences $\{x_i\}_{i=1}^{+\infty}$ of scalars in $\mathbb{R}$, that is

$$\ell_2(\mathbb{R}) := \left\{ x = (x_1, x_2 \dots, x_i \dots),\ x_i \in \mathbb{R} \text{ and } \sum_{i=1}^{+\infty} |x_i|^2 < +\infty \right\},$$

with an inner product $\langle \cdot, \cdot \rangle : \ell_2 \times \ell_2 \to \mathbb{R}$ defined by $\langle x, y \rangle := \sum_{i=1}^{+\infty} x_i y_i$, where $x = \{x_i\}_{i=1}^{+\infty}$, $y = \{y_i\}_{i=1}^{+\infty}$ and the norm $\| \cdot \| : \ell_2 \to \mathbb{R}$ by $\|x\|_2 := (\sum_{i=1}^{+\infty} |x_i|^2)^{\frac{1}{2}}$, where $x = \{x_i\}_{i=1}^{+\infty}$. Let $C = \{z \in \ell_2(\mathbb{R}) : \langle a, z\rangle \leq b\}$, where $0 \neq a \in \ell_2$ and $b \in \mathbb{R}$. Let $f : C \times C \to \mathbb{R}$ be defined by $f(x, y) = 2xy(y - x) + xy\|y - x\|$ for all $x = \{x_i\}_{i=1}^{+\infty}, y = \{y_i\}_{i=1}^{+\infty} \in \ell_2$. Define the bifunction $F : Q \times Q \to \mathbb{R}$ by $F(u, v) = u(v - u)$ for all $u = \{u_i\}_{i=1}^{+\infty}, v = \{v_i\}_{i=1}^{+\infty} \in \ell_2$. For each $x \in \ell_2$, define the mapping $L : \ell_2 \to \ell_2$ by $Lx = (x_1, x_2, \dots, x_i, \dots)$ for all $x = \{x_i\}_{i=1}^{+\infty} \in \ell_2$. Let $r_n = 0.5$ for all $n \geq 1$, then it is easy to see that $K_{r_n}^F Ly_n = \frac{2Ly_n}{3}$. Also, define the mappings S and T ,respectively by $Sx = (\frac{x_1}{2}, \frac{x_2}{2}, \dots, \frac{x_i}{2}, \dots)$ for all $x = \{x_i\}_{i=1}^{+\infty} \in \ell_2$ and $Tx = (x_1 \cos x_1, x_2 \cos x_2, \dots, x_i \cos x_i, \dots)$ for all $x = \{x_i\}_{i=1}^{+\infty} \in \ell_2$. It is easy to see that f, F, S and T satisfy the conditions of Corollary 3.4 and that $\Gamma = \{0\}$. We define the control parameters as in Example 4.1 above and obtain the figures for varying initial values. Using $\|x_{n+1} - x_n\|_{\ell_2} < 10^{-3}$ as the stopping criterion, we compare our Algorithm 3.3 with Algorithm Theorem 3.1 in [7], see Figure 2.

Case (i) $x_1 = (3.568, -5.8091, 0, \dots, 0, \dots)^T$, $x_0 = (1.521, -7.5647, 0, \dots, 0, \dots)^T$.

Case (ii) $x_1 = (1.7601, -2.1594, 0, \dots, 0, \dots)^T$, $x_0 = (0.3456, -4.1031, 0, \dots, 0, \dots)^T$.

Case (iii) $x_1 = (10.5613, 7.2610, 0, \dots, 0, \dots)^T$, $x_0 = (5.1063, 2.1687, 0, \dots, 0, \dots)^T$. We then plot the graphs of error $\|x_{n+1} - x_n\|_{\ell_2}$ against the number of iteration in each case.

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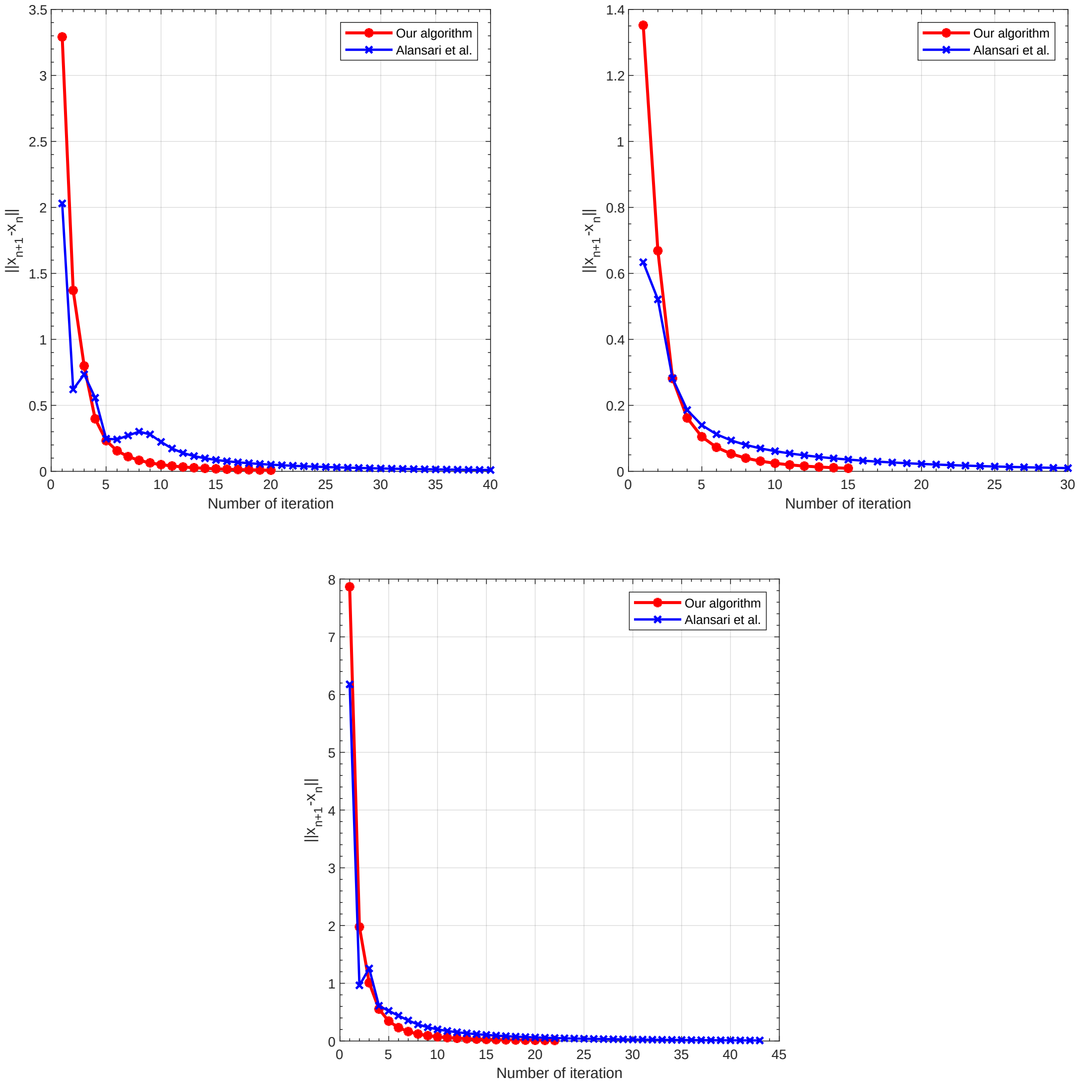

FIGURE 2. Numerical results for Example 4.2. Top left: Case (i); top right: Case (ii); bottom: Case (iii).

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