

LORENTZIAN PARA-SASAKIAN MANIFOLDS AND *-RICCI SOLITONS

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ABSTRACT. We study the properties of Lorentzian para-Sasakian manifolds endowed with *-Ricci solitons and gradient *-Ricci solitons. Finally, the existence of *-Ricci soliton on a 4-dimensional Lorentzian para-Sasakian manifold is proved by constructing a non-trivial example.

1. INTRODUCTION

A Ricci soliton (g, F, λ) [12] on a semi-Riemannian manifold (M, g) is a generalization of Einstein metric such that

$$\frac{1}{2}\mathcal{L}_F g + S + \lambda g = 0,$$

where S is the Ricci tensor, \mathcal{L}_F is the Lie derivative operator along the vector field F on M , g represents the semi-Riemannian metric of M and λ is a real number. The Ricci soliton is said to be shrinking, steady and expanding according to λ being less than 0, 0 and greater than 0, respectively.

In 1959, the notion of *-Ricci tensor on almost Hermitian manifolds was introduced by Tachibana [23] and further studied by Hamada [11] on real hypersurfaces of non-flat complex space forms. A semi-Riemannian metric g on a smooth manifold M is called a *-Ricci soliton [16] if there exists a smooth vector field F (called soliton vector field) and a real number λ , such that

$$(1.1) \quad \mathcal{L}_F g + 2S^* = -2\lambda g,$$

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where

$$S^*(U, V) = g(Q^*U, V) = \text{Trace} \{ \phi \circ R(U, \phi V) \},$$

for all vector fields U, V on M [6]. Here, ϕ is the $(1, 1)$ tensor field and Q^* is the $(1, 1)$ $*$ -Ricci operator. If we choose λ as a smooth function in (1.1), then the soliton (g, F, λ) satisfying equation (1.1) is known as an almost $*$ -Ricci soliton on M . In this connection, we recommend the papers [4, 10, 13, 15, 17, 21, 22, 24, 25] for more details about the study of Ricci solitons, η -Ricci solitons and $*$ -Ricci solitons in the context of contact Riemannian geometry. As far as our knowledge goes, the study of $*$ -Ricci solitons in the context of Lorentzian para-Sasakian manifolds is left. The main motive of this article is to fill this gap.

In 1989, K. Matsumoto [18] introduced the notion of LP -Sasakian manifolds, while in 1992, the same notion was independently studied by I. Mihai and R. Rosca [19] and they obtained several results on this manifold. The Lorentzian para-Sasakian manifolds have also been studied by various authors such as [1, 2, 7–9, 14, 26] and many others.

We present our work as follows. In Section 2, we collect the basic results and some basic definitions of Lorentzian para-Sasakian manifolds. The $*$ -Ricci solitons and gradient $*$ -Ricci solitons on Lorentzian para-Sasakian manifolds are discussed in Section 3 and Section 4, respectively. We present a 4-dimensional non-trivial example of Lorentzian para-Sasakian manifold admitting a $*$ -Ricci soliton in Section 5.

2. PRELIMINARIES

Let M be an n -dimensional smooth manifold equipped with a quartet (ϕ, ξ, η, g) , where ϕ is a tensor field of type $(1, 1)$, ξ is the unit timelike vector field, η is a 1-form and a Lorentzian metric g on M such that [5, 20]

$$(2.1) \quad \phi^2 = I + \eta \otimes \xi, \quad \eta(\xi) = -1,$$

which implies

$$(2.2) \quad \phi\xi = 0, \quad \eta(\phi U) = 0, \quad \text{rank}(\phi) = n - 1,$$

for all $U \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the collection of all smooth vector fields of M . The manifold M is said to have an almost para-contact metric structure (ϕ, ξ, η, g) when it admits a Lorentzian metric g , such that

$$(2.3) \quad g(\phi U, \phi V) = g(U, V) + \eta(U)\eta(V), \quad g(U, \xi) = \eta(U),$$

for all $U, V \in \mathfrak{X}(M)$.

If moreover,

$$(2.4) \quad (\nabla_U \phi)V = \eta(V)\phi^2 U + g(\phi U, \phi V)\xi,$$

$$(2.5) \quad \nabla_U \xi = \phi X \Leftrightarrow (\nabla_U \eta)V = g(\phi U, V) = g(U, \phi V),$$

where ∇ denotes the Levi-Civita connection of the manifold.

An n -dimensional Lorentzian para-Sasakian manifold satisfies the following relations (see [9]):

$$(2.6) \quad g(R(U, V)W, \xi) = g(V, W)\eta(U) - g(U, W)\eta(V),$$

$$(2.7) \quad R(U, V)\xi = \eta(V)U - \eta(U)V,$$

$$(2.8) \quad S(U, \xi) = (n - 1)\eta(U) \Leftrightarrow Q\xi = (n - 1)\xi,$$

for all $U, V, W \in \mathfrak{X}(M)$, where R denotes the curvature tensor and S denotes the Ricci tensor of M such that $S(U, V) = g(QU, V)$ for all $U, V \in \mathfrak{X}(M)$.

A Lorentzian para-Sasakian manifold M is said to be a generalized η -Einstein [3] if its non-vanishing Ricci tensor S is of the form

$$(2.9) \quad S(U, V) = \rho_1 g(U, V) + \rho_2 \eta(U)\eta(V) + \rho_3 g(\phi U, V),$$

where ρ_1, ρ_2 and ρ_3 are smooth functions on M . If $\rho_3 = 0$ (resp. $\rho_2 = \rho_3 = 0$), then M is called an η -Einstein (resp. Einstein) manifold.

Lemma 2.1. *An n -dimensional Lorentzian para-Sasakian manifold satisfies the following relations*

$$(2.10) \quad (\nabla_U Q)\xi = (n - 1)\phi U - Q\phi U,$$

$$(2.11) \quad (\nabla_\xi Q)U = -2Q\phi U + 2aU + 2a\eta(U)\xi,$$

where Q is the Ricci operator.

Proof. Differentiating $Q\xi = (n - 1)\xi$ along U and using (2.5), we get (2.10). Next differentiating (2.7) then using (2.5), we find

$$(2.12) \quad (\nabla_E R)(V, W)\xi = -R(V, W)\phi E + g(\phi E, W)V - g(\phi E, V)W.$$

Let $\{e_i\}_{i=1}^n$ be a local orthonormal basis on M . Putting $V = E = e_i$ in (2.12) and summing over i leads to

$$(2.13) \quad \sum_{i=1}^n \epsilon_i g((\nabla_{e_i} R)(e_i, W)\xi, U) = S(W, \phi U) + (n - 1)g(\phi W, U) - 2ag(W, U) - 2a\eta(V)\eta(W),$$

where $\epsilon_i = g(e_i, e_i)$ and $a = \text{tr } \phi$. Here tr stands for trace. From Bianchi's second identity, we can easily obtain that

$$(2.14) \quad \sum_{i=1}^n \epsilon_i g((\nabla_{e_i} R)(U, \xi)W, e_i) = (\nabla_U S)(\xi, W) - (\nabla_\xi S)(U, W).$$

By considering (2.13) in (2.14), equation (2.11) follows. □

On a Lorentzian para-Sasakian manifold (M, ϕ, ξ, η, g) , we have the following lemmas.

Lemma 2.2. *On a Lorentzian para-Sasakian manifold (M, ϕ, ξ, η, g) , we have*

$$(2.15) \quad \begin{aligned} \bar{R}(U, V, \phi W, \phi E) = & \bar{R}(U, V, W, E) - g(U, W)g(V, E) + g(V, W)g(U, E) \\ & + 2[g(V, W)\eta(U)\eta(E) - g(U, W)\eta(V)\eta(E) \\ & + g(U, E)\eta(V)\eta(W) - g(V, E)\eta(U)\eta(W)] \\ & + g(U, \phi W)g(V, \phi E) - g(V, \phi W)g(U, \phi E), \end{aligned}$$

for any U, V, W, E on M , where $\bar{R}(U, V, W, E) = g(R(U, V)W, E)$.

Proof. By virtue of the well-known definition of curvature tensor, we can write

$$(2.16) \quad \bar{R}(U, V, \phi W, \phi E) = g(\nabla_U \nabla_V \phi W, \phi E) - g(\nabla_V \nabla_U \phi W, \phi E) - g(\nabla_{[U, V]} \phi W, \phi E).$$

By making use of (2.2), (2.4) and (2.5), (2.16) takes the form

$$\begin{aligned} \bar{R}(U, V, \phi W, \phi E) = & g(R(U, V)W, E) + \eta(R(U, V)W)\eta(E) \\ & + g(V, W)g(\phi U, \phi E) - g(U, W)g(\phi V, \phi E) \\ & + 2g(U, E)\eta(V)\eta(W) - 2g(V, E)\eta(U)\eta(W) \\ & + g(U, \phi W)g(V, \phi E) - g(V, \phi W)g(U, \phi E), \end{aligned}$$

which in view of (2.3) and (2.6) leads to (2.15). This completes the proof. \square

Lemma 2.3. *The *-Ricci tensor of an n -dimensional Lorentzian para-Sasakian manifold (M, ϕ, ξ, η, g) is given by*

$$(2.17) \quad S^*(V, W) = S(V, W) + (n - 2)g(V, W) - g(V, \phi W)a + (2n - 3)\eta(V)\eta(W),$$

for any $V, W \in \mathfrak{X}(M)$.

Proof. Let $\{e_i\}_{i=1}^n$ be an orthonormal basis of the tangent space at each point of the manifold. By the definition of *-Ricci tensor, from (2.15), we have

$$\begin{aligned} S^*(V, W) &= \sum_{i=1}^n \epsilon_i \bar{R}(e_i, V, \phi W, \phi e_i) \\ &= \sum_{i=1}^n \epsilon_i \bar{R}(e_i, V, W, e_i) + \sum_{i=1}^n \epsilon_i [g(V, W)g(e_i, e_i) - g(e_i, W)g(V, e_i)] \\ &\quad + 2 \sum_{i=1}^n \epsilon_i [g(V, W)\eta(e_i)\eta(e_i) - g(e_i, W)\eta(V)\eta(e_i) \\ &\quad + g(e_i, e_i)\eta(V)\eta(W) - g(V, e_i)\eta(e_i)\eta(W)] \\ &\quad + \sum_{i=1}^n \epsilon_i [g(e_i, \phi W)g(V, \phi e_i) - g(V, \phi W)g(e_i, \phi e_i)], \end{aligned}$$

which leads to (2.17), where $\epsilon_i = g(e_i, e_i)$, i.e., $\epsilon_1 = \epsilon_2 = \dots = \epsilon_{n-1} = 1, \epsilon_n = -1$. \square

3. LORENTZIAN PARA-SASAKIAN MANIFOLDS ADMITTING *-RICCI SOLITONS

In this section, we characterize the properties of Lorentzian para-Sasakian manifold endowed with *-Ricci solitons. Now, we prove the following.

Theorem 3.1. *If an n-dimensional Lorentzian para-Sasakian manifold admits a *-Ricci soliton (g, F, λ) , then the *-Ricci soliton is steady.*

Proof. By using (2.17) in (1.1), we have

$$(3.1) \quad (\mathcal{L}_F g)(U, V) = -2S(U, V) - 2[\lambda + (n - 2)]g(U, V) - 2(2n - 3)\eta(U)\eta(V) + 2g(U, \phi V)a.$$

Taking covariant differentiation of (3.1) with respect to W , we get

$$(3.2) \quad (\nabla_W \mathcal{L}_F g)(U, V) = -2(\nabla_W S)(U, V) - 2(2n - 3)[g(\phi W, U)\eta(V) + g(\phi W, V)\eta(U)] + 2[g(V, W)\eta(U) + g(U, W)\eta(V) + 2\eta(U)\eta(V)\eta(W)]a.$$

Following Yano [27], the following formula

$$(\mathcal{L}_F \nabla_U g - \nabla_U \mathcal{L}_F g - \nabla_{[F, U]}g)(V, W) = -g((\mathcal{L}_F \nabla)(U, V), W) - g((\mathcal{L}_F \nabla)(U, W), V)$$

is well-known for any U, V, W on M . As g is parallel with respect to ∇ , the above relation becomes

$$(3.3) \quad (\nabla_U \mathcal{L}_F g)(V, W) - g((\mathcal{L}_F \nabla)(U, V), W) - g((\mathcal{L}_F \nabla)(U, W), V) = 0,$$

for any U, V, W . Since $\mathcal{L}_F \nabla$ is a symmetric tensor of type $(1, 2)$, then from (3.3) it follows that

$$(3.4) \quad g((\mathcal{L}_F \nabla)(U, V), W) = \frac{1}{2}(\nabla_V \mathcal{L}_F g)(U, W) + \frac{1}{2}(\nabla_U \mathcal{L}_F g)(V, W) - \frac{1}{2}(\nabla_W \mathcal{L}_F g)(U, V).$$

Using (3.2) in (3.4), we have

$$g((\mathcal{L}_F \nabla)(U, V), W) = (\nabla_W S)(U, V) - (\nabla_V S)(W, U) - (\nabla_U S)(V, W) - 2(2n - 3)g(\phi U, V)\eta(W) + 2g(\phi U, \phi V)\eta(W)a,$$

which by putting $V = \xi$ reduces to

$$(3.5) \quad g((\mathcal{L}_F \nabla)(U, \xi), W) = (\nabla_W S)(U, \xi) - (\nabla_U S)(\xi, W) - (\nabla_\xi S)(W, U).$$

By considering (2.10) and (2.11) in (3.5), we obtain

$$(3.6) \quad (\mathcal{L}_F \nabla)(U, \xi) = 2Q\phi U - 2aU - 2a\eta(U)\xi.$$

Taking the covariant derivative of (3.6) with respect to V , we have

$$(\nabla_V \mathcal{L}_F \nabla)(U, \xi) = 2(\nabla_V Q)\phi U - (\mathcal{L}_F \nabla)(U, \phi V) + 2Q(\nabla_V \phi)U - 2ag(U, \phi V)\xi - 2a\eta(U)\phi V.$$

Again from [27], we have

$$(\mathcal{L}_F R)(U, V)W + (\nabla_V \mathcal{L}_F \nabla)(U, W) - (\nabla_U \mathcal{L}_F \nabla)(V, W) = 0.$$

Thus the last two equations give

$$(3.7) \quad (\mathcal{L}_F R)(U, V)\xi = 2(\nabla_U Q)\phi V - 2(\nabla_V Q)\phi U \\ + 2Q(\eta(V)U - \eta(U)V) + 2a(\eta(U)\phi V - \eta(V)\phi U) \\ + (\mathcal{L}_F \nabla)(U, \phi V) - (\mathcal{L}_F \nabla)(V, \phi U).$$

Setting $V = \xi$ in (3.7) and making use of (2.11), it follows that

$$(3.8) \quad (\mathcal{L}_F R)(U, \xi)\xi = 2QU + 2Q\eta(U)\xi - 2a\phi U - (\mathcal{L}_F \nabla)(\xi, \phi U).$$

Taking the Lie derivative of $R(U, \xi)\xi = -U - \eta(U)\xi$ along F , we have

$$(3.9) \quad (\mathcal{L}_F R)(U, \xi)\xi - g(U, \mathcal{L}_F \xi)\xi + 2\eta(\mathcal{L}_F \xi)U = -(\mathcal{L}_F \eta)(U)\xi.$$

By using (3.9), (3.8) takes the form

$$(3.10) \quad (\mathcal{L}_F \eta)(U)\xi = -2QU - 2Q\eta(U)\xi + 2a\phi U + (\mathcal{L}_F \nabla)(\xi, \phi U) + g(U, \mathcal{L}_F \xi)\xi \\ - 2\eta(\mathcal{L}_F \xi)U.$$

Now taking the Lie derivative of $g(U, \xi) = \eta(U)$, we find

$$(3.11) \quad (\mathcal{L}_F \eta)U = g(U, \mathcal{L}_F \xi) + (\mathcal{L}_F g)(U, \xi).$$

By putting $V = \xi$ in (3.1) and using (2.1)–(2.3), we find

$$(3.12) \quad (\mathcal{L}_F g)(U, \xi) = -2\lambda\eta(U).$$

Again putting $U = \xi$ in (3.12), we arrive

$$(3.13) \quad \eta(\mathcal{L}_F \xi) = -\lambda.$$

By making use of (3.11)–(3.13), we get from (3.10) that

$$(\lambda I - Q)\phi^2 U = -a\phi U - \frac{1}{2}(\mathcal{L}_F \nabla)(\xi, \phi U),$$

which by virtue of (3.6) leads to $\lambda = 0$, where $\phi^2 U \neq 0$. This shows that $*$ -Ricci soliton on M is steady. This completes the proof. \square

Theorem 3.2. *An n -dimensional Lorentzian para-Sasakian manifold endowed with an almost $*$ -Ricci soliton (g, ξ, λ) is a generalized η -Einstein. Also, the soliton is steady.*

Proof. Let the Lorentzian metric of an n -dimensional Lorentzian para-Sasakian manifold be an almost $*$ -Ricci soliton (g, ξ, λ) , then (1.1) turns into

$$(3.14) \quad g(\nabla_U \xi, V) + g(U, \nabla_V \xi) + 2S^*(U, V) + 2\lambda g(U, V) = 0,$$

for all vector fields U and V on M . By making use of equations (2.5) and (2.17), equation (3.14) transforms to

$$S = \rho_1 g + \rho_2 \eta \otimes \eta + \rho_3 g(\cdot, \phi \cdot),$$

where $\rho_1 = -(\lambda + n - 2)$, $\rho_2 = -(2n - 3)$ and $\rho_3 = a - 1$. Also, in view of (2.1)–(2.3), (2.8) and the above equation, we can easily find that $\lambda = 0$. This gives the statement of Theorem 3.2. \square

Particularly, if we suppose that $a = \text{tr } \phi = 1$, then from Theorem 3.2, we infer that

$$(3.15) \quad S = \rho_1 g + \rho_2 \eta \otimes \eta.$$

Let us consider an orthonormal frame field on a Lorentzian para-Sasakian manifold and contracting (3.15), we lead

$$r = n\rho_1 - \rho_2 = -n^2 + 4n - 3.$$

Now, we state the following.

Corollary 3.1. *If an n -dimensional Lorentzian para-Sasakian manifold admits an almost *-Ricci soliton (g, ξ, λ) , with $\text{tr } \phi = 1$, then it has constant scalar curvature.*

A non-flat semi-Riemannian manifold is called pseudo Ricci symmetric and denoted by $(PRS)_n$ if the non-zero Ricci tensor S of type $(0, 2)$ of the manifold satisfies the condition [28]

$$(3.16) \quad (\nabla_U S)(V, W) = 2A(U)S(V, W) + A(V)S(U, W) + A(W)S(U, V),$$

where A is a non-zero 1-form such that $g(U, \sigma) = A(U)$, for all vector fields $U; \sigma$ being the vector field corresponding to the associated 1-form A . In particular, if $A = 0$, then the manifold is called Ricci symmetric.

Taking the covariant derivative of (3.15) leads to

$$(3.17) \quad (\nabla_U S)(V, W) = \rho_2[g(\phi U, V)\eta(W) + g(\phi U, W)\eta(V)].$$

Now using (3.15) and (3.17), (3.16) becomes

$$(3.18) \quad \begin{aligned} \rho_2[g(\phi U, V)\eta(W) + g(\phi U, W)\eta(V)] = & 2A(U)[\rho_1 g(V, W) + \rho_2 \eta(V)\eta(W)] \\ & + A(V)[\rho_1 g(U, W) + \rho_2 \eta(U)\eta(W)] \\ & + A(W)[\rho_1 g(U, V) + \rho_2 \eta(U)\eta(V)]. \end{aligned}$$

Taking $U = W = \xi$ in (3.18), we get $A(V) = 3A(\xi)\eta(V)$, which by putting $V = \xi$ gives $A(\xi) = 0$. This implies that $A(V) = 0$. Thus we have the following.

Theorem 3.3. *A pseudo Ricci symmetric Lorentzian para-Sasakian manifold admitting an almost *-Ricci soliton (g, ξ, λ) , with $\text{tr } \phi = 1$ is Ricci symmetric.*

4. GRADIENT *-RICCI SOLITONS ON η -EINSTEIN LORENTZIAN PARA-SASAKIAN MANIFOLDS

This section is concerned with the study of gradient *-Ricci solitons within the context of η -Einstein Lorentzian para-Sasakian manifolds.

Let an n -dimensional Lorentzian para-Sasakian manifold be η -Einstein, then it is noticed that the equation (2.9) takes the form

$$(4.1) \quad S = \rho_1 g(U, V) + \rho_2 \eta(U) \otimes \eta(V).$$

Setting $V = U = e_i$ in (4.1), where $\{e_i\}_{i=1}^n$ represents a set of orthonormal frame field of M , and taking the summation over i , $1 \leq i \leq n$, we have

$$(4.2) \quad r = \rho_1 n - \rho_2.$$

On the other hand, putting $U = V = \xi$ in (4.1) and making use of (2.1) and (2.3), we also have

$$(4.3) \quad -(n-1) = -\rho_1 + \rho_2.$$

Hence, it follows from (4.2) and (4.3) that

$$\rho_1 = \frac{r}{n-1} - 1, \quad \rho_2 = \frac{r}{n-1} - n.$$

Thus, the Ricci tensor S of an η -Einstein Lorentzian para-Sasakian manifold is given by

$$(4.4) \quad S(U, V) = \left(\frac{r}{n-1} - 1\right) g(U, V) + \left(\frac{r}{n-1} - n\right) \eta(U)\eta(V).$$

Definition 4.1. A semi-Riemannian metric g of a semi-Riemannian manifold M is called a gradient $*$ -Ricci soliton if it satisfies

$$(4.5) \quad \text{Hess}f + S^* + \lambda g = 0,$$

for some smooth function f , where $\text{Hess}f$ (Hessian f) is defined by $\text{Hess}f = \nabla\nabla f$. It is noticed that if we choose $F = Df$ in equation (1.1), where D denotes the gradient operator of g , then we get (4.5).

Let the η -Einstein Lorentzian para-Sasakian manifold M admit a gradient $*$ -Ricci soliton. Then from (4.5) it follows that

$$(4.6) \quad \nabla_U Df + Q^*U + \lambda U = 0,$$

for all U on M . First we prove the following lemmas for later use.

Lemma 4.1. *An n -dimensional η -Einstein Lorentzian para-Sasakian manifold satisfies*

$$(4.7) \quad (\nabla_U Q^*)\xi - (\nabla_\xi Q^*)U = -\left(\frac{r}{n-1} + n - 3\right)\phi U + \left(a - \frac{\xi(r)}{n-1}\right)(U + \eta(U)\xi),$$

for all X on M .

Proof. By using (4.4) in (2.17), we find

$$S^*(V, W) = \left(\frac{r}{n-1} + n - 3\right)(g(V, W) + \eta(V)\eta(W)) - g(V, \phi W)a.$$

It yields

$$(4.8) \quad Q^*V = \left(\frac{r}{n-1} + n - 3\right)(V + \eta(V)\xi) - \phi Va.$$

Differentiating (4.8) along U , we get

$$(4.9) \quad (\nabla_U Q^*)V = \left(\frac{r}{n-1} + n - 3\right) [(\nabla_U \eta)(V)\xi + \eta(V)\nabla_U \xi] \\ - (g(U, V)\xi + \eta(V)U + 2\eta(U)\eta(V)\xi)a + \frac{U(r)}{n-1}(V + \eta(V)\xi),$$

which by replacing V by ξ and using (2.1), (2.3) and (2.5) reduces to

$$(4.10) \quad (\nabla_U Q^*)\xi = -\left(\frac{r}{n-1} + n - 3\right)\phi U + (U + \eta(U)\xi)a.$$

Again replacing U by ξ in (4.9) and using same equations, we find

$$(4.11) \quad (\nabla_\xi Q^*)U = \frac{\xi r}{n-1}(U - \eta(U)\xi).$$

By subtracting (4.11) from (4.10), (4.7) follows. □

Lemma 4.2. *If an η -Einstein Lorentzian para-Sasakian manifold admits a gradient *-Ricci soliton, then we have*

$$(4.12) \quad R(U, V)Df = (\nabla_V Q^*)U - (\nabla_U Q^*)V.$$

Proof. Differentiating (4.6) covariantly along Y , we have

$$(4.13) \quad \nabla_V \nabla_U Df + \nabla_V Q^*U + \lambda \nabla_V U = 0,$$

which by interchanging U and V becomes

$$(4.14) \quad \nabla_U \nabla_V Df + \nabla_U Q^*V + \lambda \nabla_U V = 0.$$

Also from (4.6), we find

$$(4.15) \quad \nabla_{[U, V]} Df = -Q^*[U, V] - \lambda[U, V].$$

By making use of (4.13)–(4.15), Lemma 4.2 follows. □

Theorem 4.1. *Let the metric of an η -Einstein Lorentzian para-Sasakian manifold M admit a gradient *-Ricci soliton. Then the gradient of the potential function is pointwise collinear with the potential vector field of M .*

Proof. Putting $U = \xi$ in (4.12), we have

$$R(\xi, V)Df = (\nabla_V Q^*)\xi - (\nabla_\xi Q^*)V,$$

which by virtue of the Lemma 4.1 leads to

$$(4.16) \quad g(R(\xi, V)Df, \xi) = 0.$$

By using (2.8), we have

$$(4.17) \quad g(R(\xi, V)Df, \xi) = -(Vf) - \eta(V)(\xi f).$$

From (4.16) and (4.17), we find $(Vf) = -\eta(V)(\xi f)$. This implies that

$$Df = -(\xi f)\xi.$$

This completes the proof. □

Taking the covariant derivative of $Df = -(\xi f)\xi$ along U , we have

$$(4.18) \quad \nabla_U Df = -(U(\xi f))\xi - (\xi f)\phi U,$$

which gives

$$g(\nabla_U Df, \xi) = U(\xi f),$$

where (2.1) and (2.2) are used. Using the last equation in (4.18), we obtain

$$(4.19) \quad \nabla_U Df = -g(\nabla_U Df, \xi)\xi - (\xi f)\phi U.$$

From equations (2.17) and (4.6), we conclude that

$$(4.20) \quad \nabla_U Df = -QU - (\lambda + n - 2)U - (2n - 3)\eta(U)\xi + \phi Ua,$$

which implies that

$$(4.21) \quad g(\nabla_U Df, \xi) = -\lambda\eta(U).$$

Thus from the equations (2.1), (2.2), (2.8), and (4.19)–(4.21), we obtain

$$QU = -(\lambda + n - 2)U - (\lambda + 2n - 3)\eta(U)\xi + (a + (\xi f))\phi U,$$

which informs that the manifold M under the consideration is generalized η -Einstein. Hence, we can state the following.

Corollary 4.1. *Every η -Einstein Lorentzian para-Sasakian manifold of dimension n endowed with a gradient $*$ -Ricci metric is generalized η -Einstein.*

5. EXAMPLE

In this section, we construct a non-trivial example of a Lorentzian para-Sasakian manifold.

We consider the 4-dimensional manifold $M = \{(u, v, w, t) \in \mathbb{R}^4\}$, where (u, v, w, t) are the standard coordinates in \mathbb{R}^4 . Let $\zeta_1, \zeta_2, \zeta_3$ and ζ_4 be the vector fields on M given by

$$\zeta_1 = e^t \frac{\partial}{\partial u}, \quad \zeta_2 = e^t \frac{\partial}{\partial v}, \quad \zeta_3 = e^t \left(\frac{\partial}{\partial v} + \frac{\partial}{\partial w} \right), \quad \zeta_4 = -\frac{\partial}{\partial t}.$$

Let g be the semi-Riemannian metric defined by

$$g(e_i, e_j) = \begin{cases} 1, & 1 \leq i = j \leq 3, \\ -1, & i = j = 4, \\ 0, & 1 \leq i \neq j \leq 4. \end{cases}$$

Let η be the 1-form on M defined by $\eta(U) = g(U, \zeta_4) = g(U, \xi)$ for all $U \in \mathfrak{X}(M)$. Let ϕ be the $(1, 1)$ tensor field on M defined by

$$\phi\zeta_1 = \zeta_1, \quad \phi\zeta_2 = \zeta_2, \quad \phi\zeta_3 = \zeta_3, \quad \phi\zeta_4 = 0.$$

By applying the linearity of ϕ and g , we have

$$\begin{aligned}\eta(\xi) &= -1, & \phi^2 U &= U + \eta(U)\xi, & \eta(\phi U) &= 0, \\ g(U, \xi) &= \eta(U), & g(\phi U, \phi V) &= g(U, V) + \eta(U)\eta(V),\end{aligned}$$

for all $U, V \in \mathfrak{X}(M)$. Then we have

$$\begin{aligned}[\zeta_1, \zeta_2] &= [\zeta_1, \zeta_3] = [\zeta_2, \zeta_3] = 0, \\ [\zeta_1, \zeta_4] &= \zeta_1, & [\zeta_2, \zeta_4] &= \zeta_2, & [\zeta_3, \zeta_4] &= \zeta_3.\end{aligned}$$

Using Koszul's formula, we can easily calculate

$$\begin{aligned}\nabla_{\zeta_1} \zeta_1 &= \zeta_4, & \nabla_{\zeta_1} \zeta_2 &= 0, & \nabla_{\zeta_1} \zeta_3 &= 0, & \nabla_{\zeta_1} \zeta_4 &= \zeta_1, \\ \nabla_{\zeta_2} \zeta_1 &= 0, & \nabla_{\zeta_2} \zeta_2 &= \zeta_4, & \nabla_{\zeta_2} \zeta_3 &= 0, & \nabla_{\zeta_2} \zeta_4 &= \zeta_2, \\ \nabla_{\zeta_3} \zeta_1 &= 0, & \nabla_{\zeta_3} \zeta_2 &= 0, & \nabla_{\zeta_3} \zeta_3 &= \zeta_4, & \nabla_{\zeta_3} \zeta_4 &= \zeta_3, \\ \nabla_{\zeta_4} \zeta_1 &= 0, & \nabla_{\zeta_4} \zeta_2 &= 0, & \nabla_{\zeta_4} \zeta_3 &= 0, & \nabla_{\zeta_4} \zeta_4 &= 0.\end{aligned}$$

From the above values it can be easily verified that for $\zeta_4 = \xi$, M is a Lorentzian para-Sasakian manifold. We found that the non-vanishing components of curvature tensor are given by

$$\begin{aligned}R(\zeta_1, \zeta_2)\zeta_1 &= -\zeta_2, & R(\zeta_1, \zeta_3)\zeta_1 &= -\zeta_3, & R(\zeta_1, \zeta_4)\zeta_1 &= -\zeta_4, \\ R(\zeta_1, \zeta_2)\zeta_2 &= \zeta_1, & R(\zeta_2, \zeta_3)\zeta_2 &= -\zeta_3, & R(\zeta_2, \zeta_4)\zeta_2 &= -\zeta_4, \\ R(\zeta_1, \zeta_3)\zeta_1 &= \zeta_1, & R(\zeta_2, \zeta_3)\zeta_3 &= \zeta_2, & R(\zeta_3, \zeta_4)\zeta_3 &= -\zeta_4, \\ R(\zeta_1, \zeta_4)\zeta_4 &= -\zeta_1, & R(\zeta_2, \zeta_4)\zeta_4 &= -\zeta_2, & R(\zeta_3, \zeta_4)\zeta_3 &= -\zeta_3.\end{aligned}$$

From the above expressions of curvature tensors, we obtain

$$S(\zeta_1, \zeta_1) = S(\zeta_2, \zeta_2) = S(\zeta_3, \zeta_3) = 3, \quad S(\zeta_4, \zeta_4) = -3.$$

In view of 2.17, L.H.S. of (1.1) can be expressed as

$$\begin{aligned}(\mathcal{L}_F g)(V, W) + 2S^*(V, W) + 2\lambda g(V, W) &= g(\nabla_V F, W) + g(V, \nabla_W F) \\ &\quad + 2S(V, W) + 4g(V, W) \\ &\quad - 6g(V, \phi W)a + 10\eta(V)\eta(W).\end{aligned}$$

Let $V = \sum_{i=1}^4 V^i e_i$, $W = \sum_{i=1}^4 W^i e_i$ and $F = \sum_{i=1}^4 F^i e_i$, where V^i, W^i and F^i are scalars for $i = 1, 2, 3, 4$ such that

$$F^4 = \frac{F^1(V^1 W^4 + W^1 V^4) + F^2(V^2 W^4 + W^2 V^4) + F^3(V^3 W^4 + W^3 V^4)}{2(V^1 W^1 + V^2 W^2 + V^3 W^3)} - 2,$$

provided $V^1 W^1 + V^2 W^2 + V^3 W^3 \neq 0$. Then by the straight forward calculations, we can notice that

$$\begin{aligned}2(V^1 W^1 F^4 + V^2 W^2 F^4 + V^3 W^3 F^4) &- (V^1 F^1 W^4 + V^2 F^2 W^4 + V^3 F^3 W^4 \\ &+ W^1 F^1 V^4 + W^2 F^2 V^4 + W^3 F^3 V^4) + 4(V^1 W^1 + V^2 W^2 + V^3 W^3) = 0,\end{aligned}$$

for $a = 3$ and hence we have $\mathcal{L}_F g + 2S^* + 2\lambda g = 0$, provided $\lambda = 0$. Thus, we can say that the Lorentzian para-Sasakian manifold of dimension 4 admits a steady type $*$ -Ricci soliton, which proves Theorem 3.1.

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