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APPLICATIONS POISSON DISTRIBUTION AND RUSCHEWEYH DERIVATIVE OPERATOR FOR BI-UNIVALENT FUNCTIONS

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ABSTRACT. In this paper we establish upper bounds for the second and third coefficients of holomorphic and bi-univalent functions in a new family which involve the Bazilevič functions and β-pseudo-starlike functions under a new operator joining Poisson distribution with Ruscheweyh derivative operator. Also, we discuss Fekete-Szegö problem of functions in this family.

1. INTRODUCTION

Let $\mathcal{A}$ be the collection of functions f that are holomorphic in the unit disk $\mathbb{D} = \{|z| < 1\}$ in the complex plane $\mathbb{C}$ and that have the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1.1}$$

Further, let $\mathcal{S}$ be the sub-collection of $\mathcal{A}$ containing of functions which are univalent in $\mathbb{D}$. According to the Koebe one-quarter theorem (see [3]), every function $f \in \mathcal{S}$ has an inverse f^{-1} such that $f^{-1}(f(z)) = z$, $z \in \mathbb{D}$, and $f(f^{-1}(w)) = w$, $|w| < r_0(f)$, $r_0(f) \geq \frac{1}{4}$. If f is of the form (1.1), then

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots, \quad |w| < r_0(f). \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{D}$ if both f and f^{-1} are univalent in $\mathbb{D}$. We denote by Σ the set of bi-univalent functions in $\mathbb{D}$. Srivastava et al. [19] have apparently resuscitated the study of holomorphic and bi-univalent functions in

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recent years. It was followed by such works as those by Frasin and Aouf [5], Goyal and Goswami [6], Srivastava and Bansal [15] and others (see, for example [2,16–18,20]).

For the polynomials $M(x)$ and $N(x)$ with real coefficients, the (M,N)-Lucas Polynomials $L_{M,N,k}(x)$ are defined by the following recurrence relation (see [8]):

$$L_{M,N,k}(x) = M(x)L_{M,N,k-1}(x) + N(x)L_{M,N,k-2}(x), \quad k \geq 2,$$

with

$$L_{M,N,0}(x) = 2, \quad L_{M,N,1}(x) = M(x) \quad \text{and} \quad L_{M,N,2}(x) = M^2(x) + 2N(x). \tag{1.3}$$

The Lucas Polynomials play an important role in a diversity of disciplines in the mathematical, statistical, physical and engineering sciences (see, for example [4,9,21]). The generating function of the (M,N)-Lucas Polynomial $L_{M,N,k}(x)$ (see [9]) is given by

$$T_{M(x),N(x)}(z) = \sum_{k=2}^{\infty} L_{M,N,k}(x)z^k = \frac{2 - M(x)z}{1 - M(x)z - N(x)z^2}. \tag{1.4}$$

Let the functions f and g be holomorphic in $\mathbb{D}$, we say that the function f is subordinate to g, if there exists a function w, holomorphic in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$, $z \in \mathbb{D}$, such that $f(z) = g(w(z))$. This subordination is indicated by $f \prec g$ or $f(z) \prec g(z)$, $z \in \mathbb{D}$. Furthermore, if the function g is univalent in $\mathbb{D}$, then we have the following equivalence (see [10])

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{D}) \subset g(\mathbb{D}).$$

A function $f \in \mathcal{A}$ is called Bazilevič function of order α, $\alpha \geq 0$, if (see [14])

$$\operatorname{Re}\left\{\frac{z^{1-\alpha}f'(z)}{(f(z))^{1-\alpha}}\right\} > 0, \quad z \in \mathbb{D}.$$

A function $f \in \mathcal{A}$ is called β-pseudo-starlike function of order β, $\beta \geq 1$, if (see [1])

$$\operatorname{Re}\left\{\frac{z\,(f'(z))^{\beta}}{f(z)}\right\} > 0, \quad z \in \mathbb{D}.$$

Recall that a random variable X has the Poisson distribution with parameter θ, if

$$P(X = r) = \frac{\theta^r e^{-\theta}}{r!}, \quad r = 0, 1, 2, 3, \ldots$$

Recently, Porwal [11] introduced a power series whose coefficients are probabilities of "Poisson distribution"

$$K(\theta, z) = z + \sum_{n=2}^{\infty} \frac{\theta^{n-1}}{(n-1)!}e^{-\theta}z^n, \quad z \in \mathbb{D},$$

where $\theta > 0$. By ratio test the radius of convergence of the above series is infinity.

In 2016, Porwal and Kumar [12] introduced and investigated a linear operator $I(\theta, z) : \mathcal{A} \to \mathcal{A}$, $\theta > 0$, by using the Hadamard product (or convolution) and defined as follows

$$I(\theta, z)f(z) = K(\theta, z) * f(z) = z + \sum_{n=2}^{\infty} \frac{\theta^{n-1}}{(n-1)!} e^{-\theta} a_n z^n, \quad z \in \mathbb{D},$$

where "$*$" indicate the Hadamard product (or convolution) of two power series.

In this paper, for $f \in \mathcal{A}$ we introduce a new linear operator $\mathfrak{J}_\theta^\delta : \mathcal{A} \to \mathcal{A}$ by

$$\mathfrak{J}_\theta^\delta f(z) = I(\theta, z) * \mathcal{R}^\delta, \tag{1.5}$$

where $\mathcal{R}^\delta$, $\delta \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, denote the Ruscheweyh derivative operator [13] given by

$$\mathcal{R}^\delta f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\delta + n)}{\Gamma(\delta + 1)\Gamma(n)} a_n z^n, \quad z \in \mathbb{D}.$$

It is easy to obtain from (1.5) that

$$\mathfrak{J}_\theta^\delta f(z) = z + \sum_{n=2}^{\infty} \frac{\theta^{n-1}\Gamma(\delta + n)}{\Gamma(\delta + 1)\left(\Gamma(n)\right)^2} e^{-\theta} a_n z^n, \quad z \in \mathbb{D},$$

where $\theta > 0$, $\delta \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

2. Main Results

We begin this section by defining the family $\Upsilon_\Sigma(\lambda, \alpha, \beta, \delta, \theta; h)$ as follows.

Definition 2.1. Assume that $\alpha \geq 0$, $\beta \geq 1$, $\delta \in \mathbb{N}_0$, $0 \leq \lambda \leq 1$, $\theta > 0$ and h is analytic in $\mathbb{D}$, $h(0) = 1$. The function $f \in \Sigma$ is in the family $\Upsilon_\Sigma(\alpha, \beta, \delta, \lambda, \theta; h)$ if it fulfills the subordinations:

$$(1-\lambda)\frac{z^{1-\alpha}\left(\mathfrak{J}_\theta^\delta f(z)\right)'}{\left(\mathfrak{J}_\theta^\delta f(z)\right)^{1-\alpha}} + \lambda \frac{z\left(\left(\mathfrak{J}_\theta^\delta f(z)\right)'\right)^\beta}{\mathfrak{J}_\theta^\delta f(z)} \prec h(z)$$

and

$$(1-\lambda)\frac{w^{1-\alpha}\left(\mathfrak{J}_\theta^\delta f^{-1}(w)\right)'}{\left(\mathfrak{J}_\theta^\delta f^{-1}(w)\right)^{1-\alpha}} + \lambda \frac{w\left(\left(\mathfrak{J}_\theta^\delta f^{-1}(w)\right)'\right)^\beta}{\mathfrak{J}_\theta^\delta f^{-1}(w)} \prec 1 + e_1 z + e_2 z^2 + \cdots,$$

where f^{-1} is given by (1.2).

In particular, if we choose $\lambda = 1$ in Definition 2.1, the family $\Upsilon_\Sigma(\alpha, \beta, \delta, \lambda, \theta; h)$ reduces to the family $\mathcal{L}_\Sigma(\beta, \delta, \theta; h)$ of β-pseudo bi-starlike functions which satisfying the following subordinations:

$$\frac{z\left(\left(\mathfrak{J}_\theta^\delta f(z)\right)'\right)^\beta}{\mathfrak{J}_\theta^\delta f(z)} \prec h(z)$$

and

$$\frac{w\left(\left(\mathfrak{J}_{\theta}^{\delta}f^{-1}(w)\right)'\right)^{\beta}}{\mathfrak{J}_{\theta}^{\delta}f^{-1}(w)} \prec h(w).$$

If we choose $\lambda = 0$ in Definition 2.1, the family $\Upsilon_{\Sigma}(\alpha,\beta,\delta,\lambda,\theta;h)$ reduces to the family $\mathcal{B}_{\Sigma}(\alpha,\delta,\theta;h)$ of Bazilevič bi-univalent functions which satisfying the following subordinations:

$$\frac{z^{1-\alpha}\left(\mathfrak{J}_{\theta}^{\delta}f(z)\right)'}{\left(\mathfrak{J}_{\theta}^{\delta}f(z)\right)^{1-\alpha}} \prec h(z)$$

and

$$\frac{w^{1-\alpha}\left(\mathfrak{J}_{\theta}^{\delta}f^{-1}(w)\right)'}{\left(\mathfrak{J}_{\theta}^{\delta}f^{-1}(w)\right)^{1-\alpha}} \prec h(w).$$

If we choose $\lambda = \beta = 1$ in Definition 2.1, the family $\Upsilon_{\Sigma}(\alpha,\beta,\delta,\lambda,\theta;h)$ reduces to the family $\mathcal{S}_{\Sigma}(\delta,\theta;h)$ of bi-starlike functions which satisfying the following subordinations:

$$\frac{z\left(\mathfrak{J}_{\theta}^{\delta}f(z)\right)'}{\mathfrak{J}_{\theta}^{\delta}f(z)} \prec h(z)$$

and

$$\frac{w\left(\mathfrak{J}_{\theta}^{\delta}f^{-1}(w)\right)'}{\mathfrak{J}_{\theta}^{\delta}f^{-1}(w)} \prec h(w).$$

Theorem 2.1. *Assume that $\alpha \geq 0$, $\beta \geq 1$, $\delta \in \mathbb{N}_0$, $0 \leq \lambda \leq 1$ and $\theta > 0$. If $f \in \Sigma$ of the form* (1.1) *is in the class $\Upsilon_{\Sigma}(\alpha,\beta,\delta,\lambda,\theta;h)$, with $h(z) = 1 + e_1 z + e_2 z^2 + \cdots$, then*

$$|a_2| \leq \frac{|e_1|}{\left[(1-\lambda)(\alpha+1)+\lambda(2\beta-1)\right](\delta+1)\theta e^{-\theta}} = \frac{|e_1|}{A} \tag{2.1}$$

and

$$|a_3| \leq \min\left\{\max\left\{\left|\frac{e_1}{B}\right|, \left|\frac{e_2}{B} - \frac{Ce_1^2}{A^2B}\right|\right\}, \max\left\{\left|\frac{e_1}{B}\right|, \left|\frac{e_2}{B} - \frac{(2B+C)e_1^2}{A^2B}\right|\right\}\right\}, \tag{2.2}$$

where

$$\begin{aligned} A &= \left[(1-\lambda)(\alpha+1)+\lambda(2\beta-1)\right](\delta+1)\theta e^{-\theta}, \\ B &= \frac{1}{4}\left[(1-\lambda)(\alpha+2)+\lambda(3\beta-1)\right](\delta^2+3\delta+2)\theta^2 e^{-\theta}, \\ C &= \left[\frac{1}{2}(1-\lambda)(\alpha+2)(\alpha-1)+\lambda\left(2\beta(\beta-2)+1\right)\right](\delta+1)^2\theta^2 e^{-2\theta}. \end{aligned} \tag{2.3}$$

Proof. Suppose that $f \in \Upsilon_{\Sigma}(\alpha,\beta,\delta,\lambda,\theta;;e_1;e_2)$. Then there exist two holomorphic functions $\phi,\psi : \mathbb{D} \to \mathbb{D}$ given by

$$\phi(z) = r_1 z + r_2 z^2 + r_3 z^3 + \cdots, \quad z \in \mathbb{D}, \tag{2.4}$$

and

$$\psi(w) = s_1 w + s_2 w^2 + s_3 w^3 + \cdots, \quad w \in \mathbb{D}, \tag{2.5}$$

with $\phi(0) = \psi(0) = 0$, $|\phi(z)| < 1$, $|\psi(w)| < 1$, $z, w \in \mathbb{D}$ such that

$$(1-\lambda)\frac{z^{1-\alpha}\left(\mathfrak{J}_\theta^\delta f(z)\right)'}{\left(\mathfrak{J}_\theta^\delta f(z)\right)^{1-\alpha}} + \lambda\frac{z\left(\left(\mathfrak{J}_\theta^\delta f(z)\right)'\right)^\beta}{\mathfrak{J}_\theta^\delta f(z)} = 1 + e_1\phi(z) + e_2\phi^2(z) + \cdots \tag{2.6}$$

and

$$(1-\lambda)\frac{w^{1-\alpha}\left(\mathfrak{J}_\theta^\delta f^{-1}(w)\right)'}{\left(\mathfrak{J}_\theta^\delta f^{-1}(w)\right)^{1-\alpha}} + \lambda\frac{w\left(\left(\mathfrak{J}_\theta^\delta f^{-1}(w)\right)'\right)^\beta}{\mathfrak{J}_\theta^\delta f^{-1}(w)} = 1 + e_1\psi(w) + e_2\psi^2(w) + \cdots. \tag{2.7}$$

Combining (2.4), (2.5), (2.6) and (2.7), yield

$$(1-\lambda)\frac{z^{1-\alpha}\left(\mathfrak{J}_\theta^\delta f(z)\right)'}{\left(\mathfrak{J}_\theta^\delta f(z)\right)^{1-\alpha}} + \lambda\frac{z\left(\left(\mathfrak{J}_\theta^\delta f(z)\right)'\right)^\beta}{\mathfrak{J}_\theta^\delta f(z)} = 1 + e_1 r_1 z + \left[e_1 r_2 + e_2(x) r_1^2\right] z^2 + \cdots \tag{2.8}$$

and

$$(1-\lambda)\frac{w^{1-\alpha}\left(\mathfrak{J}_\theta^\delta f^{-1}(w)\right)'}{\left(\mathfrak{J}_\theta^\delta f^{-1}(w)\right)^{1-\alpha}} + \lambda\frac{w\left(\left(\mathfrak{J}_\theta^\delta f^{-1}(w)\right)'\right)^\beta}{\mathfrak{J}_\theta^\delta f^{-1}(w)} = 1 + e_1 s_1 w + \left[e_1 s_2 + e_2 s_1^2\right] w^2 + \cdots. \tag{2.9}$$

It is quite well-known that if $|\phi(z)| < 1$ and $|\psi(w)| < 1$, $z, w \in \mathbb{D}$, we get

$$|r_j| \leq 1 \quad \text{and} \quad |s_j| \leq 1, \quad j \in \mathbb{N}. \tag{2.10}$$

In the light of (2.8) and (2.9), after simplifying, we find that

$$\left[(1-\lambda)(\alpha+1) + \lambda(2\beta-1)\right](\delta+1)\theta e^{-\theta} a_2 = e_1 r_1, \tag{2.11}$$

$$\begin{aligned}&\frac{1}{4}\left[(1-\lambda)(\alpha+2) + \lambda(3\beta-1)\right](\delta^2+3\delta+2)\theta^2 e^{-\theta} a_3 \\ &+ \left[\frac{1}{2}(1-\lambda)(\alpha+2)(\alpha-1) + \lambda\left(2\beta(\beta-2)+1\right)\right](\delta+1)^2\theta^2 e^{-2\theta} a_2^2\end{aligned}$$

$$= e_1 r_2 + e_2 r_1^2, \tag{2.12}$$

$$-\left[(1-\lambda)(\alpha+1) + \lambda(2\beta-1)\right](\delta+1)\theta e^{-\theta} a_2 = e_1 s_1 \tag{2.13}$$

and

$$\frac{1}{4}\left[(1-\lambda)(\alpha+2)+\lambda(3\beta-1)\right](\delta^2+3\delta+2)\theta^2e^{-\theta}\left(2a_2^2-a_3\right)$$
$$+\left[\frac{1}{2}(1-\lambda)(\alpha+2)(\alpha-1)+\lambda\left(2\beta(\beta-2)+1\right)\right](\delta+1)^2\theta^2e^{-2\theta}a_2^2$$
$$=e_1s_2+e_2s_1^2. \tag{2.14}$$

Inequality (2.1) follows from (2.11) and (2.13). If we apply notation (2.3), then (2.11) and (2.12) become

$$Aa_2=e_1r_1, \quad Ba_3+Ca_2^2=e_1r_2+e_2r_1^2. \tag{2.15}$$

This gives

$$\frac{B}{e_1}a_3=r_2+\left(\frac{e_2}{e_1}-\frac{Ce_1}{A^2}\right)r_2^2, \tag{2.16}$$

and on using the known sharp result [7, page 10]:

$$|r_2-\mu r_1^2|\le\max\left\{1,|\mu|\right\}, \tag{2.17}$$

for all $\mu\in\mathbb{C}$, we obtain

$$\left|\frac{B}{e_1}\right||a_3|\le\max\left\{1,\left|\frac{e_2}{e_1}-\frac{Ce_1}{A^2}\right|\right\}. \tag{2.18}$$

In the same way, (2.13) and (2.14) become

$$-Aa_2=e_1s_1, \quad B(2a_2^2-a_3)+Ca_2^2=e_1s_2+e_2s_1^2. \tag{2.19}$$

This gives

$$-\frac{B}{e_1}a_3=s_2+\left(\frac{e_2}{e_1}-\frac{(2B+C)e_1}{A^2}\right)s_2^2. \tag{2.20}$$

Applying (2.17), we obtain

$$\left|\frac{B}{e_1}\right||a_3|\le\max\left\{1,\left|\frac{e_2}{e_1}-\frac{(2B+C)e_1}{A^2}\right|\right\}. \tag{2.21}$$

Inequality (2.2) follows from (2.18) and (2.21). □

If we take the generating function (1.4) of the (M,N)-Lucas polynomials $L_{M,N,k}(x)$ as $h(z)+1$, then from (1.3), we have $e_1=M(x)$ and $e_2=M^2(x)+2N(x)$ and Theorem 2.1 becomes the following corollary.

Corollary 2.1. *If $f\in\Sigma$ of the form* (1.1) *is in the class $\Upsilon_\Sigma(\alpha,\beta,\delta,\lambda,\theta;T_{M(x),N(x)}-1)$, then*

$$|a_2|\le\frac{|M(x)|}{\left[(1-\lambda)(\alpha+1)+\lambda(2\beta-1)\right](\delta+1)\theta e^{-\theta}}$$

and

$$|a_3| \le \min\left\{\max\left\{\left|\frac{M(x)}{B}\right|, \left|\frac{M^2(x)+2N(x)}{B} - \frac{CM^2(x)}{A^2B}\right|\right\},\right.$$
$$\left.\max\left\{\left|\frac{M(x)}{B}\right|, \left|\frac{M^2(x)+2N(x)}{B} - \frac{(2B+C)M^2(x)}{A^2B}\right|\right\}\right\},$$

for all $\alpha, \beta, \delta, \lambda, \theta, x$ *such that* $\alpha \ge 0$, $\beta \ge 1$, $\delta \in \mathbb{N}_0$, $0 \le \lambda \le 1$, $\theta > 0$ *and* $x \in \mathbb{R}$, *where* A, B, C *are given by* (2.3) *and* $T_{M(x),N(x)}$ *is given by* (1.4).

In the next theorem, we discuss a bound for $|a_3 - \eta a_2^2|$ called "the Fekete-Szegö problem".

Theorem 2.2. *If* $f \in \Sigma$ *of the form* (1.1) *is in the class* $\Upsilon_\Sigma(\alpha, \beta, \delta, \lambda, \theta; h)$, *then*

(2.22)
$$\left|a_3 - \eta a_2^2\right|$$
$$\le \frac{|e_1|}{B}\min\left\{\max\left\{1, \left|\frac{e_2}{e_1} - \frac{(C-\eta B)e_1}{A^2}\right|\right\}, \max\left\{1, \left|\frac{e_2}{e_1} - \frac{(2B+C-\eta B)e_1}{A^2}\right|\right\}\right\},$$

for all $\alpha, \beta, \delta, \lambda, \theta, \eta$ *such that* $\alpha \ge 0$, $\beta \ge 1$, $\delta \in \mathbb{N}_0$, $0 \le \lambda \le 1$, $\theta > 0$ *and* $\eta \in \mathbb{C}$, *where* A, B, C *are given by* (2.3).

Proof. We apply the notations from the proof of Theorem 2.1. From (2.15) and from (2.16), we have

$$a_3 - \eta a_2^2 = \frac{e_1}{B}\left(r_2 + \left(\frac{e_2}{e_1} - \frac{(C-\eta B)e_1}{A^2}\right) r_1^2\right) \tag{2.23}$$

and on using the known sharp result $|r_2 - \mu r_1^2| \le \max\{1, |\mu|\}$, we get

$$|a_3 - \eta a_2^2| \le \frac{|e_1|}{B}\max\left\{1, \left|\frac{e_2}{e_1} - \frac{(C-\eta B)e_1}{A^2}\right|\right\}. \tag{2.24}$$

In the same way, from (2.19) and from (2.20), we have

$$a_3 - \eta a_2^2 = -\frac{e_1}{B}\left(s_2 + \left(\frac{e_2}{e_1} - \frac{(2B+C-\eta B)e_1}{A^2}\right) s_1^2\right) \tag{2.25}$$

and on using $|s_2 - \mu s_1^2| \le \max\{1, |\mu|\}$, we get

$$|a_3 - \eta a_2^2| \le \frac{|e_1|}{B}\max\left\{1, \left|\frac{e_2}{e_1} - \frac{(2B+C-\eta B)e_1}{A^2}\right|\right\}. \tag{2.26}$$

Inequality (2.22) follows from (2.24) and (2.26). □

Corollary 2.2. *If $f \in \Sigma$ of the form* (1.1) *is in the class $\Upsilon_\Sigma(\alpha, \beta, \delta, \lambda, \theta; T_{M(x),N(x)} - 1)$, then*

$$\left|a_3 - \eta a_2^2\right| \leq \frac{|M(x)|}{B} \min\left\{\max\left\{1, \left|\frac{M^2(x) + 2N(x)}{M(x)} - \frac{(C - \eta B)M(x)}{A^2}\right|\right\},\right.$$
$$\left.\max\left\{1, \left|\frac{M^2(x) + 2N(x)}{M(x)} - \frac{(2B + C - \eta B)M(x)}{A^2}\right|\right\}\right\},$$

for all $\alpha, \beta, \delta, \lambda, \theta, \eta, x$ such that $\alpha \geq 0$, $\beta \geq 1$, $\delta \in \mathbb{N}_0$, $0 \leq \lambda \leq 1$, $\theta > 0$, $\eta \in \mathbb{C}$ and $x \in \mathbb{R}$, where A, B, C are given by (2.3) *and $T_{M(x),N(x)}$ is given by* (1.4).

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