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AN OPEN MAPPING THEOREM FOR ORDER-PRESERVING OPERATORS

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Abstract. In the main result of this paper we prove a version of the well-known open mapping theorem for weakly additive, order-preserving operators between ordered real vector spaces with an order unit. We also provide a few examples to illustrate obtained results.

1. Introduction and Preliminaries

The open mapping theorem (known also as the Banach-Schauder theorem) is one of most important theorems in functional analysis [4], [14, Theorem 2.11] and has a number of applications in complex analysis [15, Theorem 4.4], topology [7, 10, 11] and in other mathematical disciplines (see, for instance, [1–3, 5, 6, 8, 9, 12, 13, 16, 17]). In this note we prove a version of this theorem for operators between ordered real vector spaces with an order unit.

We begin with definitions of notions that will be used in the sequel.

An element 1_E of an ordered real vector space E is said to be an *order unit* in E if for each $x \in E$ there is a real number $\varepsilon > 0$ such that $\varepsilon 1_E \geq x$.

In this article "spac" means "ordered real vector space".

Recall that a subset L of a space E with an order unit 1_E is said to be an A^{1_E}-*subspace* of E if $0_E \in L$, and $x \in L$ implies that $x + c1_E \in L$ for all $c \in \mathbb{R}$.

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The *order topology* on an ordered real vector space E with an order unit 1_E is the topology whose base is the collection of balls (with center x and radius ε)

$$B(x,\varepsilon)=\{y\in E\colon \|y-x\|<\varepsilon\},\quad x\in E,\varepsilon>0,$$

where for $x\in E$

$$\|x\|=\inf\{\lambda>0\colon -\lambda 1_E\le x\le \lambda 1_E\}.$$

Recall that a mapping $f\colon X\to Y$ between topological spaces X and Y is said to be *open at* $x_0\in X$ if for each open neighbourhood U of x_0 there exists an open neighbourhood V of $f(x_0)$, which lies in $f(U)$. A mapping $f\colon X\to Y$ is said to be *open* if it is open at every point $x\in X$, or, equivalently, if for any open set U in X its image $f(U)$ is an open set in Y.

Let E, F be spaces with order unit. An operator $f\colon E\to F$ is said to be:

(1) *order-preserving* if for any pair x, $y\in E$ the inequality $x\le_E y$ implies $f(x)\le_F f(y)$;

(2) *weakly additive* if the equality $f(x+\lambda 1_E)=f(x)+\lambda f(1_E)$ holds, for every $x\in E$ and every $\lambda\in\mathbb{R}$;

(3) *normed*, if $f(1_E)=1_F$.

2. Results

We begin this section with some auxiliary results and examples.

Lemma 2.1. *Let E and F be spaces with order unit, $f:E\to F$ surjective, weakly additive order-preserving operator. If f is open at 0_E, then f is open over entire E.*

Proof. Let $x\in E$ be an arbitrary point and $B(x,\varepsilon)=x+B(0_E,\varepsilon)$ a neighbourhood of x. Since f is open in 0_E and $f(0_E)=0_F$, there is $\mu>0$ such that $B(0_F,\mu)\subset f(B(0_E,\varepsilon))$. We claim that

$$B(f(x),\mu)=f(x)+B(0_F,\mu)\subset f(B(x,\varepsilon)).$$

Let $y\in B(f(x),\mu)$. This means $y-f(x)\in B(0_F,\mu)\subset f(B(0_E,\varepsilon))$. It follows $y\in f(x)+f(B(0_E,\varepsilon))$, i.e., $y\in f(x+B(0_E,\varepsilon))=f(B(x,\varepsilon))$. Therefore, f is open in $x\in E$. □

Recall that a metric on a vector space X is said to be *invariant* if

$$d(x+z,y+z)=d(x,y),$$

for all x, y, $z\in X$.

Lemma 2.2. *The metric generated by the order norm on a space with order unit is invariant.*

Proof. Let E be a space with order unit 1_E, $x,y\in E$. According to the definition of order norm we have

$$d(x,y)=\|y-x\|=\inf\{\lambda>0\colon -\lambda 1_E\le y-x\le\lambda 1_E\}.$$

From here it follows $d(x+z,y+z)=d(x,y)$ for each vector $z\in E$. □

Recall that the graph of a mapping f of a set X into a set Y is the set of all pairs $(x, f(x))$ in the Cartesian product $X \times Y$. If X and Y are topological spaces, then in their product we will consider the usual product topology.

Let E and F be spaces with order unit. The product $E \times F$ becomes a space with order unit if one introduces on it coordinate-wise operations of addition and multiplication by a number:

$$\alpha(x_1,\, y_1) + \beta(x_2,\, y_2) = (\alpha x_1 + \beta x_2,\, \alpha y_1 + \beta y_2),$$

and coordinate-wise partial order:

$$(x_1,\, y_1) \le_{E\times F} (x_2,\, y_2) \Leftrightarrow ((x_1 \le_E x_2)\&(y_1 \le_F y_2)).$$

Further in this article, we will use inequality signs without any indices and will imply from the context in which set they are defined.

The order norm on $E \times F$ is defined by the rule

$$\|(x_1,\, y_1)\| = \inf\{\lambda > 0\colon\, -\lambda(1_E,\, 1_F) \le (x_1,\, y_1) \le \lambda(1_E,\, 1_F)\}.$$

Here $(1_E,\, 1_F)$ is an order unit in $E \times F$. So, instead of the couple $(1_E,\, 1_F)$ one can use the symbol $1_{E\times F}$.

Lemma 2.3. *Let E and F be spaces with order unit, 1_E an order unit in E, $f\colon\, E \to F$ weakly additive, order-preserving operator. Then the graph G of operator f is an $A^{1_{E\times f(E)}}$-subspace in the space $E \times f(E)$ with the order unit $1_{E\times f(E)}$.*

Proof. We have $0_{E\times F} \equiv (0_E, 0_F) \in G$, since $f(0_E) = 0_F$. Let $(x, y) \in G$ and $\lambda \in \mathbb{R}$. Then

$$\begin{aligned}(x, y) + \lambda 1_{E\times f(E)} &= (x, f(x)) + \lambda 1_{E\times f(E)} \left(x + \lambda 1_E, f(x) + \lambda 1_{f(E)}\right)\\ &= (x + \lambda 1_E, f(x + \lambda 1_E))\,,\end{aligned}$$

and consequently, $\left((x, y) + \lambda 1_{E\times f(E)}\right) \in G$. □

Corollary 2.1. *Let E and F be spaces with order unit, 1_E and 1_F, respectively, $f\colon\, E \to F$ a weakly additive, order-preserving, normed operator. Then the graph G of the operator f is $A^{1_{E\times F}}$-subspace of the space $E \times F$ with order unit $1_{E\times F}$.*

Remark 2.1. Note that in every topological vector space (in particular, in every space with an order unit) the only open subspace is the space itself. Unlike subspaces, A-subspaces of a space with an order unit can be open, closed, or everywhere dense.

Example 2.1. Consider the Euclidean plane $\mathbb{R}^2$ with the point-wise algebraic operations and the point-wise order. Then $\mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2\colon\, x_i \ge 0,\, i = 1, 2\}$ is a positive cone in $\mathbb{R}^2$. Arbitrary element of the set $\operatorname{Int}(\mathbb{R}^2_+) = \{(x_1,\, x_2) \in \mathbb{R}^2\colon\, x_i > 0,\, i = 1, 2\}$ can serve as an order unit. For precision, we fix $\mathbf{1} = (1, 1)$ as an order unit in $\mathbb{R}^2$. Then, as it is easy to check, the set

$$C = \left\{(x_1, x_2) \in \mathbb{R}^2\colon\, x_1 - 1 < x_2 < x_1 + 1\right\}$$

is an open (with respect to the order topology) A-subspace in $\mathbb{R}^2$, and $C \ne \mathbb{R}^2$.

Example 2.2. Let $(\mathbb{R}^2, \mathbf{1})$ be the space with an order unit built in Example 2.1. Then the set

$$D = \left\{(x_1, x_1 + r) \in \mathbb{R}^2 \colon r \in \mathbb{Q}\right\},$$

where $\mathbb{Q}$ is the set of rational numbers, is a dense A-subspace in $\mathbb{R}^2$.

Example 2.3. Let $(\mathbb{R}^2, \mathbf{1})$ be the space in Example 2.1. It is clear that the set $\Lambda = \{\lambda\mathbf{1} : \lambda \in \mathbb{R}\}$ is a closed A-subspace in $\mathbb{R}^2$.

Remark 2.2. Note that every weakly additive, order-preserving operator $f\colon E \to F$ is automatically continuous.

Proposition 2.1. *Let E and F be spaces with order unit, and $f\colon E \to F$ be a weakly additive, order-preserving operator. Then the image $f(E)$ is an $A^{1_{f(E)}}$-subspace in F.*

Proof. Since $f(0_E) = 0_F$, then $0_F \in f(E)$. Let $y \in f(E)$, $\lambda \in \mathbb{R}$. Then there exists a vector $x \in E$, such that $y = f(x)$. We have

$$y + \lambda 1_{f(E)} = f(x) + \lambda 1_{f(E)} = f(x + \lambda 1_E),$$

i.e., $y + \lambda 1_{f(E)} \in f(E)$. Thus, $f(E)$ is $A^{1_{f(E)}}$-subspace in F. $\square$

Finally, we formulate a version of the open mapping theorem for order-preserving operators.

Theorem 2.1. *Let E be a complete space with an order unit, F be a space with order unit and of the second category. If $f\colon E \to F$ is a surjective, weakly additive, order-preserving operator, then:*

(i) *the mapping f is open;*
(ii) *F is a complete space.*

Proof. (i) First we will show that $f(1_E)$ is an order unit in F.

Since E is complete, by the Baire category theorem we have $E = \bigcup_{m=1}^{\infty} mB(0_E, r)$ for every positive number r. Then one has $f(E) = F = \bigcup_{m=1}^{\infty} mf(B(0_E, r))$. Indeed, let $y \in F$. Since f is a surjective mapping, there exists $x \in E$ such that $y = f(x)$. There is such a positive integer m, that $-mr1_E < x < mr1_E$. Therefore, $-mrf(1_E) < f(x) < mrf(1_E)$, i.e., $y \in \bigcup_{m=1}^{\infty} mf(B(0_E, r))$.

So far we have $\operatorname{Int}(f(B(0_E, r))) \neq \emptyset$, i.e., the set $\operatorname{Int}(f(B(0_E, r)))$ is a neighbourhood of the zero of F. By definition of the order topology there exists σ such that $\sigma 1_F \in \operatorname{Int}(f(B(0_E, r))) \subset f(B(0_E, r))$. Hence, $-rf(1_E) < \sigma 1_F < rf(1_E)$, i.e., $f(1_E)$ is an order unit in F.

The arbitrariness of $r > 0$ guarantees that the operator f is open at O_E. But then, according to Lemma 2.1, the operator f is open at every point in E. So, the statement (i) is established.

(ii) Let $\{y_n\}$ be a Cauchy sequence in F, i.e., for every $\varepsilon > 0$ there exists n_ε such that for all $m \geq n_\varepsilon$ and $k \geq n_\varepsilon$ the double inequality

$$-\varepsilon 1_{f(E)} < y_m - y_k < \varepsilon 1_{f(E)}$$

holds. Without loss of generality, we can assume that for any positive integer n and any $m,\, k \geq n$ the following double inequality is fulfilled

$$-\tfrac{1}{n}1_{f(E)} < y_m - y_k < \tfrac{1}{n}1_{f(E)}. \tag{2.1}$$

Then $y_m - y_k \in B\left(0_F,\, \frac{1}{n}\right)$. According to the openness of the mapping f the set $f\left(B\left(0_E,\, \frac{1}{n}\right)\right)$ is an open neighbourhood of the zero in F. Moreover, we have

$$f\left(B\left(0_E,\, \tfrac{1}{n}\right)\right) = B\left(0_F,\, \tfrac{1}{n}\right). \tag{2.2}$$

Therefore, $y_m - y_k \in f\left(B\left(0_E,\, \frac{1}{n}\right)\right)$. It may turn out that for each pair m and k there exist a lot of pairs of vectors $x \in E$ and $x' \in E$, such that $f(x) = y_m$ and $f(x') = y_k$. As long as $y_m - y_k \in f\left(B\left(0_E,\, \frac{1}{n}\right)\right)$, then among such vector pairs must exist vectors $x \in E$ and $x' \in E$ with $f(x) = y_m$, $f(x') = y_k$ and $x - x' \in B\left(0_E,\, \frac{1}{n}\right)$.

For every positive integer n we denote by x_n any vector, which satisfies the following conditions:

1) $x_n \in f^{-1}(y_n)$;
2) for every $k \geq n$ there exists a vector $x \in f^{-1}(y_k)$ such that

$$x_n - x \in B\left(0_E,\, \tfrac{1}{n}\right).$$

Thus, we have built a sequence $\{x_n\}$ such that

$$f(x_n) = y_n, \quad n = 1, 2, \ldots, \tag{2.3}$$

on one side and, according to (2.1) and (2.2)

$$-\tfrac{1}{n}1_E < x_m - x_k < \tfrac{1}{n}1_E, \tag{2.4}$$

on the other side, for all n and for every pair of $m,\, k \geq n$.

By virtue of inequalities (2.4) we conclude, that $\{x_n\}$ is a Cauchy sequence in E. Since E is a complete space with an order unit, the sequence $\{x_n\}$ has to converge with respect to the order topology. Denote $x_0 = \lim\limits_{n\to\infty} x_n \in E$. Since f is a continuous mapping, then by (2.3) we have $f(x_0) = \lim\limits_{n\to\infty} y_n$. We put $y_0 = f(x_0)$. Then $y_0 = \lim\limits_{n\to\infty} y_n$. Thus, $\{y_n\}$ is a convergent sequence. Due to the arbitrariness of the chosen Cauchy sequence $\{y_n\}$, it follows that F is a complete space. $\square$

Remark 2.3. Note that the openness principle for weakly additive, order-preserving case cannot be formulated similarly to the linear case. In contrast of the linear case, the conditions f is weakly additive and order-preserving in Theorem 2.1 do not guarantee the surjectivity of the mapping f. On the other hand, the image $f(E)$ is not obliged to be open in F. Finally, if we do not require surjectivity in Lemma 2.1, then the openness of a weakly additive, order-preserving operator at zero does not provide its openness on the whole space.

Example 2.4. Let $(\mathbb{R}^2, \mathbf{1})$ be the space with the order unit built in Example 2.1. We put $\overline{S} = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1 - 1 \leq x_2 \leq x_1 + 1\}$. Define the mapping $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ by the rule

$$f(x_1, x_2) = \begin{cases} (x_1, x_1 - 1), & \text{if } x_2 \leq x_1 - 1, \\ (x_1, x_2), & \text{if } x_1 - 1 \leq x_2 \leq x_1 + 1, \\ (x_1, x_1 + 1), & \text{if } x_2 \geq x_1 + 1. \end{cases} \tag{2.5}$$

It is easy to check that f is a weakly additive mapping. We show that it is order-preserving. Since this property holds for the identity mappings, then f is order-preserving on $\overline{S}$. So, we have to check the first and the third cases in (2.5). But, the first case and the third case are checked similarly. That is why we will verify only the third case.

Let $x_2 \geq x_1 + 1$. Take any vector $(y_1, y_2) \in \mathbb{R}^2$ such that $(x_1, x_2) \leq (y_1, y_2)$. The last inequality is equivalent to $x_1 \leq y_1$ and $x_2 \leq y_2$.

The following three cases are possible.

1^0 $y_2 \geq y_1 + 1$. Then

$$f(x_1, x_2) = (x_1, x_1 + 1) \leq (\text{since } x_1 \leq y_1) \leq (y_1, y_1 + 1) = f(y_1, y_2).$$

2^0 $y_1 - 1 \leq y_2 \leq y_1 + 1$. We have $x_1 + 1 \leq y_2$. Therefore,

$$f(x_1, x_2) = (x_1, x_1 + 1) \leq (y_1, y_2) = f(y_1, y_2).$$

3^0 $y_2 \leq y_1 - 1$. But $x_1 + 1 \leq y_1 - 1$. Consequently,

$$f(x_1, x_2) = (x_1, x_1 + 1) \leq (y_1, y_1 - 1) = f(y_1, y_2).$$

So, f is order-preserving on $\mathbb{R}^2$.

For the operator f we have $f(\mathbb{R}^2) = \overline{S} \neq \mathbb{R}^2$, although the operator f is weakly additive, order-preserving, and the image $f(\mathbb{R}^2)$ is a set of the second category in $\mathbb{R}^2$. Clearly, $f(\mathbb{R}^2)$ is not open in $\mathbb{R}^2$. Moreover, it is easy to see that the mapping f is open at zero, but it is not open on $\mathbb{R}^2$. Indeed, for the open neighbourhood $B((2,4);1) = \{(x_1, x_2) \in \mathbb{R}^2 \colon 1 < x_1 < 3,\ 3 < x_2 < 5\}$ of $(2,4) \in \mathbb{R}^2$ its image $f(B((2,4);1)) = \{(x_1, x_1 + 1) \colon 1 < x_1 < 3\}$ is not open in $f(\mathbb{R}^2)$.

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