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ALMOST MULTI-DIAGONAL DETERMINANTS

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Abstract. We found motivation for this paper in the conjectures about multi-diagonal determinants published in a few recent papers. Especially, we were interested in the case with a few non-zero elements in the lower left corner or/and in the upper right corner. Our research with changeable free elements lead us to the systems of partial differential equations. Also, we include some generalizations of the problems and conjectures.

1. Introduction

These determinants are of the theoretical and applicable interest. We can emphasize the computational problems related to the such matrices and their determinants as: the calculation of spectra, permanent, characteristic polynomial, inverse matrix, power, and decomposition of a matrix. They appear in the numerical methods for the differential equations. It is known that the three diagonal determinants are very important in the number theory and the theory of orthogonal polynomials and the five diagonal determinants in the statistics.

An almost (nearly) five constant diagonal determinant of ordinary order was considered in the paper [6], and the numerical methods for its numerical computing were developed. Similar problem was considered in the paper [7, 8].

Recently, [1] in 2020. Conjectures 6.1. and 6.2. about the almost four constant diagonal unit determinants were formulated. They caused a lot of attention and were proven a few months later in [9].

But, they initialized other considerations in that direction.

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In the paper [4], the two sided almost constant multi-diagonal determinants were studied.

Papers about multi-diagonal matrices with equally spaced diagonals appeared soon. In the papers [10, 11], the multi-diagonal determinants with rare nonzero elements were considered.

This paper is organized as follows. In the Section 1, it is given the survey of the papers which deal with the multi-diagonal determinants and nearby multi-diagonal determinants. The preliminaries, i.e., definitions and known theorems were exposed in the Section 2. The last section is fulfilled with original contributions to the almost multi-diagonal determinants and their reduction to the systems of partial differential equations. We did not see any trial with such approach as we did in the Section 3. We believe that this point of view can be of interest for all which are investigating in this area.

2. Multi-Diagonal Determinants

In the paper [3], there is the following definition.

Definition 2.1. A square matrix $P_n(r,s) = [p_{i,j}]_{i,j=0}^{n-1}$ is (r,s)-*banded matrix* if

$$p_{i,j} = 0, \quad \text{for all } (i,j) : i-j > r \text{ or } j-i > s, \quad s, r \in \mathbb{N} : r+s < n. \tag{2.1}$$

The bandwidth of an (r,s)-banded matrix is $r+s+1$. In the expanded form, it can be written as

$$P_n(r,s) = \begin{bmatrix} p_{0,0} & p_{0,1} & \cdots & & p_{0,s} & 0 & \cdots & 0 \\ p_{1,0} & p_{1,1} & & & & p_{1,s+1} & & 0 \\ \vdots & & \ddots & & & \ddots & & \\ p_{r,0} & & & p_{r,r} & & & & \\ 0 & p_{r+1,1} & & & & & & \\ \vdots & & \ddots & & & \ddots & & 0 \\ & & & & & & & p_{n-s-1,n-1} \\ 0 & & & & & & \ddots & \vdots \\ 0 & & & & & p_{n-1,n-r-1} & \cdots & p_{n-1,n-1} \end{bmatrix}.$$

Let us remind that a rational function $f(x_1,\dots,x_n)$ is *homogeneous* of degree k if

$$f(tx_1,\dots,tx_n) = t^k f(x_1,\dots,x_n), \quad \text{for all } (x_1,\dots,x_n) \in \mathbb{R}^n.$$

Lemma 2.1 ([16]). *Let $P_n(r,s) = [p_{i,j}]_{i,j=0}^{n-1}$ be an (r,s)-banded matrix with the principal minors π_k. Then, for every $n > \delta = \binom{r+s}{r}$, the sequence $\{\pi_k\}$ satisfies a nontrivial homogeneous linear recurrence relation of the form*

$$\pi_n = \sum_{k=1}^{\delta} R_k \pi_{n-k}, \tag{2.2}$$

where R_k is a homogeneous rational function of degree k with entries

$$\{a_{n-i,n-j}\}_{0\le i\le \delta-1;-s\le j\le r+\delta-1}.$$

In the continuation we will deal with the following matrices.

Definition 2.2. A square matrix $P_n(r,s;A)=[p_{i,j}]_{i,j=0}^{n-1}$ is (r,s)-*constant diagonal matrix* if it is (r,s)-*banded matrix* and

$$p_{i,i+j}=a_j,\quad j=-r,-r+1,\dots,s;i=0,1,\dots,n-1.$$

Consider the constant five-diagonal, i.e., $(2,2)$-banded determinants:

$$\pi_n=\pi_n(2,2;\mathbf{A_5})=\begin{vmatrix} a_0 & a_1 & a_2 & 0 & \cdots & 0 & 0 & 0 & \\ a_{-1} & a_0 & a_1 & a_2 & & 0 & 0 & 0 & \\ a_{-2} & a_{-1} & a_0 & a_1 & & 0 & 0 & 0 & \\ 0 & a_{-2} & a_{-1} & a_0 & & 0 & 0 & 0 & \\ \vdots & & & & \ddots & & & & \\ 0 & 0 & 0 & & & a_0 & a_1 & a_2 & 0 \\ 0 & 0 & 0 & & & a_{-1} & a_0 & a_1 & a_2 \\ 0 & 0 & 0 & & & a_{-2} & a_{-1} & a_0 & a_1 \\ 0 & 0 & 0 & & & 0 & a_{-2} & a_{-1} & a_0 \end{vmatrix}_{n\times n},$$

where

$$\mathbf{A_5}=\{a_{-2},a_{-1},a_0;a_1,a_2\}.$$

Lemma 2.2 ([15])**.** *The sequence $\{\pi_n\}$, where $\pi_n=\pi_n(2,2;\mathbf{A_5})$, satisfies the seventh-term recurrence relation*

(2.3)
$$\begin{aligned}\pi_n=&a_0\pi_{n-1}+(a_2a_{-2}-a_1a_{-1})\pi_{n-2}+(a_2a_{-1}^2+a_1^2a_{-2}-2a_0a_2a_{-2})\pi_{n-3}\\ &+a_2a_{-2}(a_2a_{-2}-a_1a_{-1})\pi_{n-4}+a_0(a_2a_{-2})^2\pi_{n-5}-(a_2a_{-2})^3\pi_{n-6},\quad n=5,6,\dots\end{aligned}$$

Example 2.1. The three unit diagonal determinants $D_{3,n}=\pi_n(1,1;\{1,1,1\})$ satisfy the three-term recurrence relation

$$D_{3,n}=D_{3,n-1}-D_{3,n-2},\quad n\ge 5, \tag{2.4}$$

with initial values

$$D_{3,0}=1,\quad D_{3,1}=0. \tag{2.5}$$

The general solution of this difference equation and the initial values (2.5) give us the explicit form of the determinant $D_{3,n}$ with

$$D_{3,n}=\cos\left(\frac{n\pi}{3}\right)+\frac{1}{\sqrt{3}}\sin\left(\frac{n\pi}{3}\right).$$

Even more, because of the presence of the cosine and sine function which have the periods, the determinants $D_{3,n}$ have the periodicity $T=6$, and the values:

$$D_{3,6n}=D_{3,6n+1}=1,\quad D_{3,6n+2}=0,\quad D_{3,6n+3}=D_{3,6n+4}=-1,\quad D_{3,6n+5}=0,$$

for $n=0,1,\dots$

Example 2.2. Let $\mathbf{I_4} = \{1, 1, 1, 1\}$. The four unit diagonal determinants

$$D_{4,n} = \pi_n(2, 1; \mathbf{I_4}) = |d_{i,j}|_{n\times n}: \quad d_{i,j} = \begin{cases} 1, & \text{if } |i-j| \le 1, \\ 1, & \text{if } i = j+2, \\ 0, & \text{others}, \end{cases} \tag{2.6}$$

satisfy the four-term recurrence relation

$$D_{4,n} = D_{4,n-1} - D_{4,n-2} + D_{4,n-3}.$$

Its general solution is

$$D_{4,n} = C_1 + C_2 \cos\frac{n\pi}{2} + C_3 \sin\frac{n\pi}{2}.$$

Using the initial values $D_{4,1} = 1$, $D_{4,2} = D_{4,3} = 0$, we find

$$D_{4,n} = \frac{1}{2}\left(1 + \cos\frac{n\pi}{2} + \sin\frac{n\pi}{2}\right).$$

Hence, its value is

$$D_{4,n} = \frac{1 + (-1)^{\lfloor n/2 \rfloor}}{2}, \quad n \in \mathbb{N}, \tag{2.7}$$

i.e.,

$$D_{4,n} = \begin{cases} 1, & \text{if } n \equiv 0 \pmod 4 \vee n \equiv 1 \pmod 4, \\ 0, & \text{if } n \equiv 2 \pmod 4 \vee n \equiv 3 \pmod 4. \end{cases}$$

Remark 2.1. Notice that we will get the same value for the non-symmetric unit diagonal upper or lower with respect to the main diagonal. But, in some further considerations, it will be important for conclusions.

Example 2.3. The five unit diagonal determinants $D_{5,n} = \pi_n(2, 2; \{\mathbf{1}\})$ satisfy the seven-term recurrence relation

$$D_{5,n} = D_{5,n-1} + D_{5,n-5} - D_{5,n-6}, \quad n \ge 5, \tag{2.8}$$

with initial values

$$D_{5,0} = D_{5,1} = 1, \quad D_{5,k} = 0, \quad k = 2, 3, 4, \quad D_{5,5} = 1. \tag{2.9}$$

The general solution of this difference equation is

$$D_{5,n} = C_1 + C_2 n + C_3 \cos\frac{4n\pi}{5} - C_4 \sin\frac{4n\pi}{5} + C_5 \cos\frac{2n\pi}{5} + C_6 \sin\frac{2n\pi}{5}.$$

Including the initial values (2.9), we find the explicit form of the determinant $D_{5,n}$ with

$$C_1 = \frac{2}{5}, \quad C_2 = 0, \quad C_3 = \frac{3-\sqrt{5}}{10},$$

$$C_4 = -\frac{1}{5}\sqrt{\frac{5-\sqrt{5}}{2}}, \quad C_5 = \frac{3+\sqrt{5}}{10}, \quad C_6 = \frac{1}{5}\sqrt{\frac{5+\sqrt{5}}{2}}.$$

Even more, the determinants $D_{5,n}$ have the periodicity $T = 5$ and the values:

$$D_{5,5n} = D_{5,5n+1} = 1, \quad D_{5,5n+k} = 0, \quad k = 2, 3, 4; n \in \mathbb{N}. \tag{2.10}$$

The computation of the exact values of the determinants $D_{k,n}$ for a lot of k's and n's, shows that we can establish the following conjecture.

Conjecture 2.1. *The determinants $\{D_{2k,n}\}_{n\in\mathbb{N}}$ have the periodicity $T = 2k$. The determinants $\{D_{2k+1,n}\}_{n\in\mathbb{N}}$ have the periodicity $T = 2k+1$ or $T = 4k+2$.*

Remark 2.2. A useful method for computing multi-diagonal determinants is, if it is possible, to decompose them into the product of lower and upper triangular matrix.

Remark 2.3. Many papers reals with the multi-diagonal determinants with the special numbers. For example, the role of the Fibonacci numbers in the nature and science induce that a lot attention is ascribed them. Numerous papers deal with their properties and representations (see [14]). They appear like values of special determinant sequences what was shown in the papers [2, 13] and [12].

Let $A(t)$ be a functional matrix

$$A(t) = [a_{i,j}(t)]_{n\times n}\,. \tag{2.11}$$

If we denote by $\hat{a}_k(t)$ the k^{th} row, we can write

$$\hat{a}_k(t) = \begin{bmatrix} a_{k,1}(t) & a_{k,2}(t) & \cdots & a_{k,n}(t) \end{bmatrix}, \quad A(t) = \begin{bmatrix} \hat{a}_1(t) \\ \hat{a}_2(t) \\ \vdots \\ \hat{a}_n(t) \end{bmatrix}. \tag{2.12}$$

The k^{th} derivative of the matrix $A(t)$ is

$$A^{(k)}(t) = \left[a_{i,j}^{(k)}(t)\right]_{n\times n}, \quad k \in \mathbb{N},$$

with assumption that all derivatives $a_{i,j}^{(k)}(t)$ exist.

Lemma 2.3 (Jacobi formula)**.** *The derivative of the determinant* (2.11) *can be expressed in the form*

$$D_t \det A(t) = \sum_{k=1}^{n} \mathcal{T}_k(A; t), \tag{2.13}$$

where

$$\mathcal{T}_1(A;t) = \begin{vmatrix} D_t\hat{a}_1(t) \\ \hat{a}_2(t) \\ \vdots \\ \hat{a}_{n-1}(t) \\ \hat{a}_n(t) \end{vmatrix}, \quad \mathcal{T}_k(A;t) = \begin{vmatrix} \hat{a}_1(t) \\ \vdots \\ \hat{a}_{k-1}(t) \\ \hat{D}_t\hat{a}_k(t) \\ \hat{a}_{k+1}(t) \\ \vdots \\ \hat{a}_{n-1}(t) \\ \hat{a}_n(t) \end{vmatrix}, \quad k = 2, \dots, n. \tag{2.14}$$

In more general form, we can find it in [5]:

$$D\det(A)(X) = \operatorname{tr}(\operatorname{adj}(A)X),$$

i.e.,

$$D\det(A)(X) = \sum_{i,j} \det M_{i,j} x_{i,j},$$

where $M_{i,j}$ is (i,j)-cofactor of A.

Denote by

$$\nabla_{k,n} = D_{k,n} - D_{k.n-1}. \tag{2.15}$$

3. Some Almost Multi-Diagonal Determinants

There are determinants which have at least an element out of multi-diagonals. The Lagrange expansion was applied for some easier cases in a few papers (see, for example [4] and [9]). But, it requires a lot of computation and a lot of difficulties appear.

Here, we will use Jacobi formula for differentiation of determinants (2.11) for finding their closed form values.

Theorem 3.1. *The almost three unit diagonal determinant*

$$A_{3,n} = \begin{vmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & y & x \\ 1 & 1 & 1 & 0 & & 0 & 0 & z \\ 0 & 1 & 1 & 0 & & 0 & 0 & 0 \\ \vdots & & & \ddots & & & \vdots & \\ & & & & & & & \\ 0 & 0 & 0 & & & 1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & & 1 & 1 & 1 \\ 0 & 0 & 0 & & & 0 & 1 & 1 \end{vmatrix}_n : \quad a_{i,j} = \begin{cases} 1, & \text{if } |i-j| \le 1, \\ x, & \text{if } i = 1 \wedge j = n, \\ y, & \text{if } i = 1 \wedge j = n-1, \\ z, & \text{if } i = 2 \wedge j = n, \\ 0, & \text{others,} \end{cases}$$

has the value

$$A_{3,n} = (-1)^{n+1}(x - y - z) + D_{3,n}. \tag{3.1}$$

Proof. Applying the Jacobi formula for determinants (2.13), we get the system of partial differential equations

$$\frac{\partial A_{3,n}}{\partial x} = (-1)^{n+1}, \quad \frac{\partial A_{3,n}}{\partial y} = (-1)^{n}, \quad \frac{\partial A_{3,n}}{\partial z} = (-1)^{n}.$$

Integrating the first equation, we find

$$A_{3,n} = (-1)^{n+1}x + \varphi(y,z).$$

Hence, $\frac{\partial A_{3,n}}{\partial y} = \frac{\partial \varphi}{\partial y} = (-1)^n$ implies $\varphi = (-1)^n y + \psi(z)$. Now, we have

$$A_{3,n} = (-1)^{n+1}x + (-1)^n y + \psi(z).$$

By differentiation via z, we obtain $\frac{\partial A_{3,n}}{\partial z} = \frac{\partial \psi}{\partial z} = (-1)^n$ implies $\psi = (-1)^n z + C(n)$. Finally, we have

$$A_{3,n} = (-1)^{n+1}(x - y - z) + C_n.$$

Knowing that $A_{3,n}(0,0,0) = D_{3,n}$, we get the statement. □

Theorem 3.2. *The upper almost four unit diagonal determinant*

$$A_{4,n} = |a_{i,j}|_{n\times n}: \quad a_{i,j} = \begin{cases} 1, & \text{if } |i-j| \le 1 \wedge \ i = j+2, \\ x, & \text{if } i = 1 \wedge j = n, \\ y, & \text{if } i = 1 \wedge j = n-1, \\ z, & \text{if } i = 2 \wedge j = n, \\ 0, & \text{others}, \end{cases} \tag{3.2}$$

has the value

$$A_{4,n} = D_{4,n} + yz + (-1)^n \Big(-D_{4,n-1}x + \left(\nabla_{4,n-3} - \nabla_{4,n-4} + \nabla_{4,n-5}\right)(y+z) \Big).$$

Proof. Derivative of a determinant is the sum of determinants provided by successive deriving the rows in the given determinant. Hence,

$$\begin{aligned} \frac{\partial A_{4,n}}{\partial x} &= (-1)^{n-1} D_{4,n-1}, \\ \frac{\partial A_{4,n}}{\partial y} &= z + (-1)^n \left(\nabla_{4,n-3} - \nabla_{4,n-4} + \nabla_{4,n-5}\right), \\ \frac{\partial A_{4,n}}{\partial z} &= y + (-1)^n \left(\nabla_{4,n-3} - \nabla_{4,n-4} + \nabla_{4,n-5}\right). \end{aligned}$$

Here, we have the system of three partial linear differential equations with unknown function $A_{4,n}(x,y,z)$. By integrating the first one, we get

$$A_{4,n} = (-1)^{n-1} D_{4,n-1}x + \varphi(y,z),$$

where $\varphi(y,z)$ is an arbitrary differentiable real function. Differentiating $A_{4,n}$ by y, we find

$$\frac{\partial A_{4,n}}{\partial y} = \frac{\partial \varphi}{\partial y} = z + (-1)^n \left(\nabla_{4,n-3} - \nabla_{4,n-4} + \nabla_{4,n-5}\right),$$

wherefrom

$$\varphi = yz + (-1)^n \left(\nabla_{4,n-3} - \nabla_{4,n-4} + \nabla_{4,n-5}\right) y + \psi(z),$$

where $\psi(z)$ is an arbitrary differentiable real function. Hence,

$$A_{4,n} = (-1)^{n-1} D_{4,n-1} x + yz + (-1)^n \left(\nabla_{4,n-3} - \nabla_{4,n-4} + \nabla_{4,n-5}\right) y + \psi(z).$$

Finally, differentiating $A_{4,n}$ by z, we find

$$\frac{\partial A_{4,n}}{\partial z} = y + \psi'(z) = y + (-1)^n \left(\nabla_{4,n-3} - \nabla_{4,n-4} + \nabla_{4,n-5}\right),$$

wherefrom

$$\psi = (-1)^n \left(\nabla_{4,n-3} - \nabla_{4,n-4} + \nabla_{4,n-5}\right) z + C.$$

Knowing that $A_{4,n}(0,0,0) = D_{4,n}$, we get the statement. □

Remark 3.1. The statement of the theorem can be written in the from

$$A_{4,n} = \begin{cases} (1-y)(1-z), & \text{if } n \equiv 0 \pmod 4, \\ 1 + x + yz, & \text{if } n \equiv 1 \pmod 4, \\ -x + y + (1+y)z, & \text{if } n \equiv 2 \pmod 4, \\ yz, & \text{if } n \equiv 3 \pmod 4. \end{cases} \tag{3.3}$$

Remark 3.2. When $x = b$ and $y = z = a$, we confirm the main result in the paper [9].

In the similar way, we can prove the following theorems.

Theorem 3.3. *The almost five unit diagonal determinant*

$$A_{5,n} = |a_{i,j}|_{n\times n}: \quad a_{i,j} = \begin{cases} 1, & \text{if } |i-j| \le 2, \\ x, & \text{if } i = 1 \wedge j = n, \\ y, & \text{if } i = 1 \wedge j = n-1, \\ z, & \text{if } i = 2 \wedge j = n, \\ 0, & \text{others}, \end{cases} \tag{3.4}$$

has the value

$$A_{5,n} = \begin{cases} (1-y)(1-z), & \text{if } n \equiv 0 \pmod 5, \\ 1 + x + yz, & \text{if } n \equiv 1 \pmod 5, \\ -x + y + (1+y)z, & \text{if } n \equiv 2 \pmod 5, \\ yz, & \text{if } n \equiv 3 \pmod 5, \\ yz, & \text{if } n \equiv 4 \pmod 5. \end{cases}$$

Also, this method can be applied on the two sided almost multiple diagonal determinants considered in the paper [4].

Theorem 3.4. *The two sided almost five unit diagonal determinant*

$$A_n = \begin{vmatrix} 1 & 1 & 1 & 0 & \cdots & 0 & y & x \\ 1 & 1 & 1 & 1 & & 0 & 0 & z \\ 1 & 1 & 1 & 1 & & 0 & 0 & 0 \\ \vdots & & & \ddots & & & \vdots & \\ & & & & & & & \\ 0 & 0 & 0 & & & 1 & 1 & 1 \\ v & 0 & 0 & \cdots & & 1 & 1 & 1 \\ u & w & 0 & & & 1 & 1 & 1 \end{vmatrix} : \quad a_{i,j} = \begin{cases} 1, & \text{if } |i-j| \le 2, \\ x, & \text{if } i = 1 \wedge j = n, \\ y, & \text{if } i = 1 \wedge j = n-1, \\ z, & \text{if } i = 2 \wedge j = n, \\ u, & \text{if } i = n \wedge j = 1, \\ v, & \text{if } i = n-1 \wedge j = 1, \\ w, & \text{if } i = n \wedge j = 2, \\ 0, & \text{others,} \end{cases}$$

has the value

$$\begin{aligned} A_{5n} &= (1-y)(1-z)(1-v)(1-w), \\ A_{5n+1} &= 1 + x + yz + u + vw, \\ A_{5n+2} &= -x + y + (1+y)z - u + v + (1+v)w - xu + zv + yw, \\ A_{5n+3} &= yz + vw + (-u + v + w + vw)x + (u-w)z + y(u - v + uz), \\ A_{5n+4} &= yz + vw - vwx + vwz + y(vw + (-u + v + w + vw)z). \end{aligned}$$

Proof. Applying again the Jacobi formula for determinants (2.13), we get the system of partial differential equations. For example, deriving by x, and after that by u, we find

$$\frac{\partial^2 A_n}{\partial x \partial u} = -D_{5,n-2}.$$

We will miss the whole proof because of its largeness. □

4. Conclusions

We researched the closed form for the multiple diagonal determinants with at most three elements in the opposite corners. Although it seems easy to be done by the Lagrange expansion, this method requires finding the recurrence relation with large depth. We pointed to the Jacobi formula for the derivation of the determinants as useful tool. It will be of interest to continue this research, for example, to examine the influence of a nonzero element at random position outside of the multiple diagonals on the determinant value.

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References

[1] M. Andjelić and C. M. Fonseca, *Some determinantal considerations for pentadiagonal matrices*, Linear Multilinear Algebra (2020), 1–9. https://doi.org/10.1080/03081087.2019.1708845

[2] A. Cvetković, P. Rajković and M. Ivković, *Catalan numbers, the Hankel transform, and Fibonacci numbers*, J. Integer Seq. **5** (2002), Article ID 02.1.3., 8 pages.

[3] Z. Du, C. M. Fonseca and A. Pereira, *On determinantal recurrence relations of banded matrices*, Kuwait J. Sci. **49**(1) (2022), 1–9. https://10.48129/kjs.v49i1.11165

[4] K. Filipiak, A Markiewicz and A. Sawikowska, *Determinants of multidiagonal matrices*, Electron. J. Linear Algebra **25** (2012), 102–118.

[5] P. Grover, *Derivatives of multilinear functions of matrices*, in: F. Nielsen, R. Bhatia, R. (Eds.), *Matrix Information Geometry*, Springer, Berlin, Heidelberg, 2013. https://doi.org/10.1007/978-3-642-30232-9_5

[6] J. Jia, Q. Kong and T. Sogabe, *A new algorithm for solving nearly penta-diagonal Toeplitz linear systems*, Comput. Math Appl. **63** (2012), 1238–1243. https://doi:10.1016/j.camwa.2011.12.044

[7] M. J. Karling, A. O. Lopes and S. R. C. Lopes, *Pentadiagonal matrices and an application to the centered $MA(1)$ stationary Gaussian process*, International Journal of Applied Mathematics and Statistics **61**(1) (2022), 1–22.

[8] C. Krattenthaler, *Advanced determinant calculus: A complement*, Linear Algebra Appl. **411** (2005), 68–166. https://doi:10.1016/j.laa.2005.06.042

[9] B. Kurmanbek, Y. Amanbek and Y. Erlangga, *A proof of Andjelić-Fonseca conjectures on the determinant of some Toeplitz matrices and their generalization*, Linear Multilinear Algebra (2020), 1–9. https://doi.org/10.1080/03081087.2020.1765959

[10] L. Losonczi, *Determinants of some pentadiagonal matrices*, Glas. Mat. **56**(2) (2021), 271–286. https://doi.org/10.3336/gm.56.2.05

[11] L. Losonczi, *Products and inverses of multidiagonal matrices with equally spaced diagonals*, Filomat **36**(3) (2022), 1021–1030. https://doi.org/10.2298/FIL2203021L

[12] A. Nalli and H. Civciv, *A generalization of tridiagonal matrix determinants, Fibonacci and Lucas numbers*, Chaos Solitons Fractals **40** (2009), 355–361. https://doi:10.1016/j.chaos.2007.07.069

[13] A. P. Stakhov, *Fibonacci matrices, a generalization of the "Cassini 3 formula", and a new coding theory*, Chaos Solitons Fractals **30**(1) (2006), 56–66. https://doi:10.1016/j.chaos.2005.12.054

[14] P. Stanimirović, J. Nikolov and I. Stanimirović, *A generalization of Fibonacci and Lucas matrices*, Discrete Appl. Math. **156** (2008), 2606–2619. https://doi:10.1016/j.dam.2007.09.028

[15] R. A. Sweet, *A recursive relation for the determinant of a pentadiagonal matrix*, Communications of the ACM **12**(6) (1969), 330–332. https://doi.org/10.1145/363011.363152

[16] H. Zakrajšek and M. Petkovšek, *Pascal-like determinants are recursive*, Adv. Appl. Math. **33** (2004), 431–450. https://doi:10.1016/j.aam.2003.09.004

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