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A FIXED POINT THEOREM FOR MAPPINGS SATISFYING CYCLICAL CONTRACTIVE CONDITIONS IN $(3,2)$-W-SYMMETRIZABLE SPACES

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Abstract. In this paper we are concerned with $(3,2)$-symmetrics and $(3,2)$-W-symmetrizable spaces. First we give the basic definitions, the notation, some examples and elementary results about these spaces, then we prove the existence of a fixed point for self mappings satisfying cyclical contractive conditions in $(3,2)$-W-symmetrizable spaces.

1. Introduction

The geometric properties of the metric spaces, their axiomatic classification and generalizations have been considered in a lot of papers: [1, 5, 11, 13–16, 18–20].

The notion of an (n,m,ρ)-metric, $n>m$, as a generalization of the usual notion of a pseudometric (the case $n=2$, $m=1$), and the notion of an $(n+1)$-metric (as in [14] and [11]) was introduced in [6]. Some connections between the topologies induced by a $(3,1,\rho)$-metric and topologies induced by a pseudo-o-metric, o-metric and symmetric (as in [19]) are given in [7]. Other characterizations of $(3,j,\rho)$-metrizable topological spaces, $j\in\{1,2\}$ are given in [3, 4, 8, 9].

Fixed points theory plays a basic role in applications of many branches of mathematics. The Banach fixed point theorem [2] is a very simple and powerful theorem with a wide range of applications. Several extensions and generalizations of this result have appeared in the literature. Through the years this theorem has been generalized and extended by many authors in various ways and directions. In [12] Kirk, Srinivasan

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and Veeramani introduced the notion of cyclical representation and characterized the Banach's contraction mapping principle in context of a mapping which satisfy a cyclical contractive condition.

Here we consider only $(3, 2)$-W-metrizable spaces. The purpose of this paper is to prove, using a new type of implicit relation, a fixed point theorem for mappings which satisfy a cyclical contractive condition.

2. Preliminaries

We give the basic definitions for $(3, 2, \rho)$-metric spaces and $(3, 2)$-metric spaces, as in [3].

Let $M \neq \emptyset$ and $M^{(3)} = M^3/\alpha$, where α is the equivalence relation on M^3 defined by:

$$(x, y, z)\alpha(u, v, w) \Leftrightarrow \pi(u, v, w) = (x, y, z),$$

where π is a permutation. We will use the same notation (x, y, z) for the elements in $M^{(3)}$ keeping in mind that $(x, y, z) = (u, v, w)$ in $M^{(3)}$ if and only if (x, y, z) is a permutation of (u, v, w).

Let $d : M^{(3)} \to \mathbb{R}_0^+$. We state three conditions for such map:

$(M0)$ $d(x, x, x) = 0$, for any $x \in M$;
$(M1)$ $d(x, y, z) \leq d(x, a, b) + d(y, a, b) + d(z, a, b)$, for any $x, y, z, a, b \in M$;
(Ms) $d(x, x, y) = d(x, y, y)$, for any $x, y \in M$.

Let ρ be a subset of $M^{(3)}$. We consider the following two conditions for such a set:

$(E0)$ $(x, x, x) \in \rho$, for all $x \in M$;
$(E1)$ $(x, a, b), (a, y, b), (a, b, z) \in \rho$ implies $(x, y, z) \in \rho$, for any $x, y, z, a, b \in M$.

Definition 2.1. If ρ satisfies $(E0)$ and $(E1)$, we say that ρ is a $(3, 2)$-equivalence.

Example 2.1. The set $\Delta = \{(x, x, x) \mid x \in M\}$ is a $(3, 2)$-equivalence on M.

Example 2.2. The set $\rho_d = \{(x, y, z) \mid (x, y, z) \in M^{(3)}, d(x, y, z) = 0\}$, where d satisfies $(M0)$ and $(M1)$ is a $(3, 2)$-equivalence.

Definition 2.2. Let $d : M^{(3)} \to \mathbb{R}_0^+$ and $\rho = \rho_d$ are as above.

i) If d satisfies $(M0)$ and $(M1)$, then we say that d is a $(3, 2, \rho)$-metric on M and the pair (M, d) is said to be a $(3, 2, \rho)$-metric space.

ii) If d satisfies $(M0)$,$(M1)$ and (Ms), then we say that d is a $(3, 2, \rho)$-symmetric on M, and the pair (M, d) is said to be a $(3, 2, \rho)$-symmetric space.

If $\rho = \Delta = \{(x, x, x) \mid x \in M\}$, then we write $(3, 2)$ instead of $(3, 2, \Delta)$.

Example 2.3. Let M be a nonempty set. The map $d : M^{(3)} \to \mathbb{R}_0^+$ defined by:

$$d(x, y, z) = \begin{cases} 0, & x = y = z, \\ 1, & \text{otherwise}, \end{cases}$$

is a $(3, 2)$-metric on M (the discrete 3-metric).

Proposition 2.1. *If d is a $(3, 2, \rho)$-metric on M, then*

(i) $d(x,x,y) \leq 2d(x,a,b) + d(y,a,b)$;
(ii) $d(x,x,y) \leq 2d(x,y,y)$;
(iii) $d(x,x,y) \leq 2d(x,z,z) + d(y,z,z)$,

for any $x, y, z, a, b \in M$.

Proof. Follows directly from Definition 2.2. □

Definition 2.3. Let d be a $(3,2,\rho)$-metric on M, $x, y \in M$ and $\epsilon > 0$. We define the following ϵ-balls as subsets of M:

i) $B(x,y,\epsilon) = \{z \mid z \in M, d(x,y,z) < \epsilon\}$ - with center at (x,y) and radius ϵ;

ii) $B(x,\epsilon) = \{z \mid z \in M$, there is a $v \in M$ such that $d(x,z,v) < \epsilon\}$ - with center at x and radius ϵ.

Proposition 2.2. *For any* $(3,2,\rho)$*-metric* d *on* M *and for any* $x \in M$, $\epsilon > 0$, $B(x,x,\epsilon) \subseteq B(x,\epsilon)$.

Proof. Follows directly from the previous definition. □

Definition 2.4. For a $(3,2,\rho)$-metric d on M and $U \subseteq M$, we define the topology $\tau(W,d)$ on M by: $U \in \tau(W,d)$ if and only if for any $x \in U$, there is an $\epsilon > 0$ such that $B(x,\epsilon) \subseteq U$.

Definition 2.5. We say that a topological space (M,τ) is $(3,2)$-W-metrizable if there is a $(3,2)$-metric d on M such that $\tau = \tau(W,d)$.

Definition 2.6. We say that a topological space (M,τ) is $(3,2)$-W-symmetrizable if there is a $(3,2)$-symmetric d on M such that $\tau = \tau(W,d)$.

Proposition 2.3. *For any* $(3,2,\rho)$*-metric* d *and any sequence* $(x_n)_{n=1}^{+\infty}$*, the following conditions are equivalent:*

$(C1)$ $d(x_n,x_m,x_p) \to 0$ *as* $n,m,p \to +\infty$ *and*
$(C2)$ $d(x_n,x_m,x_m) \to 0$ *as* $n,m \to +\infty$.

Proof. Let d satisfy $(C1)$. For any $n, m \in \mathbb{N}$ we choose $p, q > \max\ \{m,n\}$. By the previous proposition we obtain

$$d(x_n,x_m,x_m) \leq d(x_n,x_p,x_q) + 2d(x_m,x_p,x_q).$$

Thus, $d(x_n,x_m,x_m) \to 0$, as $n,m \to +\infty$.

Let d satisfy the condition $(C2)$. For any $n,m,p \in \mathbb{N}$ we choose $q > \max\{m,n,p\}$ and we obtain

$$d(x_n,x_m,x_p) \leq d(x_n,x_q,x_q) + d(x_m,x_q,x_q) + d(x_p,x_q,x_q).$$

Thus, $d(x_n,x_m,x_p) \to 0$ as $n,m,p \to +\infty$. □

Definition 2.7. A sequence $(x_n)_{n=1}^{+\infty}$ in $(3,2,\rho)$-metric space (M,d) is called $(3,2)$-Cauchy if it satisfies $(C1)$ or $(C2)$.

In the following we use notations and results from [10].

Definition 2.8 ([10])**.** We say that a sequence $(x_n)_{n=1}^{+\infty}$ in a $(3,2,\rho)$-metric space (M,d):

(i) 1-converges to $x \in M$ if $d(x,x,x_n) \to 0$ as $n \to +\infty$;
(ii) 2-converges to $x \in M$ if $d(x,x_n,x_n) \to 0$ as $n \to +\infty$;
(iii) 3-converges to $x \in M$ if $d(x,x_n,x_m) \to 0$ as $n,m \to +\infty$.

Theorem 2.1 ([10])**.** *For any sequence $(x_n)_{n=1}^{+\infty}$ in $(3,2,\rho)$-metric space (M,d) the following conditions are equivalent:*

(i) $(x_n)_{n=1}^{+\infty}$ 1-*converges to* $x \in M$;
(ii) $(x_n)_{n=1}^{+\infty}$ 2-*converges to* $x \in M$;
(iii) $(x_n)_{n=1}^{+\infty}$ 3-*converges to* $x \in M$.

Definition 2.9. We say that a sequence $(x_n)_{n=1}^{+\infty}$ in a $(3,2,\rho)$-metric space (M,d) is $(3,2)$-convergent if it satisfies any of the conditions in the previous theorem.

Lemma 2.1. *Let $x,y \in M$ and $(x_n)_{n=1}^{+\infty}$, $(y_n)_{n=1}^{+\infty}$ be sequences in M. For any $(3,2,\rho)$-symmetric d on M, if $d(x_n,x,x) \to 0$ and $d(y_n,y,y) \to 0$ as $n \to +\infty$, then $d(x_n,y,y) \to d(x,y,y)$ and $d(x_n,y_n,y_n) \to d(x,y,y)$ as $n \to +\infty$.*

Proof. From

$$d(x_n,y,y) = d(x_n,x_n,y) \le 2d(x_n,x,x) + d(y,x,x) = 2d(x_n,x,x) + d(x,y,y),$$

we obtain

$$d(x_n,y,y) - d(x,y,y) \le 2d(x_n,x,x). \tag{2.1}$$

From $d(x,y,y) = d(x,x,y) \le 2d(x,x_n,x_n) + d(y,x_n,x_n) = 2d(x_n,x,x) + d(x_n,y,y)$, we obtain

$$d(x_n,y,y) - d(x,y,y) \ge -2d(x_n,x,x). \tag{2.2}$$

From (2.1) and (2.2) it follows that

$$|d(x_n,y,y) - d(x,y,y)| \le 2d(x_n,x,x),$$

from where $d(x_n,y,y) \to d(x,y,y)$ as $n \to +\infty$.

From $d(x_n,y_n,y_n) = d(x_n,x_n,y_n) \le 2d(x_n,x,x) + d(y_n,x,x) = 2d(x_n,x,x) + d(y_n,y_n,x) \le 2d(x_n,x,x) + 2d(y_n,y,y) + d(x,y,y)$ we obtain

$$d(x_n,y_n,y_n) - d(x,y,y) \le 2(d(x_n,x,x) + d(y_n,y,y)). \tag{2.3}$$

From $d(x,y,y) = d(x,x,y) \le 2d(x,x_n,x_n) + d(y,x_n,x_n) = 2d(x_n,x,x) + d(x_n,y,y) \le 2d(x_n,x,x) + 2d(y,y_n,y_n) + d(x_n,y_n,y_n)$ we obtain

$$d(x_n,y_n,y_n) - d(x,y,y) \ge -2(d(x_n,x,x) + d(y_n,y,y)). \tag{2.4}$$

From (2.3) and (2.4) it follows that

$$|d(x_n,y_n,y_n) - d(x,y,y)| \le 2(d(x_n,x,x) + d(y_n,y,y)),$$

from where $d(x_n,y_n,y_n) \to d(x,y,y)$ as $n \to +\infty$. □

Lemma 2.2. *Let (M, τ)-be a $(3, 2)$-W-metrizable space via $(3, 2)$-metric d. Let $A \subseteq M, x \in M$ and $(x_n)_{n=1}^{+\infty}$ be a sequence in A. If $d(x_n, x, x) \to 0$ as $n \to +\infty$, then $x \in \overline{A}$.*

Proof. Let $(x_n)_{n=1}^{+\infty}$ be a sequence in A such that $d(x_n, x, x) \to 0$ as $n \to +\infty$. Let $U \in \tau$ and $x \in U$. Then there is an $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$. There is an $n_0 \in \mathbb{N}$ such that for $n \geq n_0, x_n \in B(x, x, \epsilon)$. So, $x_n \in B(x, x, \epsilon) \subseteq B(x, \epsilon) \subseteq U$ for $n \geq n_0$. Thus, $U \cap A \neq \emptyset$, i.e., $x \in \overline{A}$. □

Definition 2.10. Let (M, τ)-be a $(3, 2)$-W-metrizable space via $(3, 2)$-metric d. We say that (M, τ) is $(3, 2)$-complete if any $(3, 2)$-Cauchy sequence in M is $(3, 2)$-convergent (with respect to the $(3, 2)$-metric d).

3. Main results

Definition 3.1 ([12])**.** Let $\{A_i\}_{i=1}^{p}$ be a family of nonempty sets and $\mathcal{A} = \cup_{i=1}^{p} A_i$. We say that a mapping $f : \mathcal{A} \to \mathcal{A}$ is p-cyclic if $f(A_i) \subseteq A_{i+1}$, $i = 1, 2, 3, \ldots, p$, where $A_{p+1} = A_1$.

Definition 3.2. Let $\mathcal{F}$ denote the set of all lower semi-continuous functions $F : (\mathbb{R}^+)^6 \to \mathbb{R}$ satisfying the following conditions:

(F1) for all $x, y \in \mathbb{R}^+$, $x < y$, $F(a, b, c, d, e, x) \geq F(a, b, c, d, e, y)$ (non-increasing on the 6^{th} coordinate);

(F2) there is an $h \in [0, 1)$ such that for all $u, v \geq 0$, $F(u, v, v, u, 0, 2u + v) \leq 0$ implies $u \leq hv$;

(F3) $F(t, t, 0, 0, t, t) > 0$ for $t > 0$.

Example 3.1. The function $F(a, b, c, d, e, f) = a - xb - y \max\{c, d, e, f\}$, where $x, y \geq 0$ and $x + 3y < 1$ is an element of $\mathcal{F}$.

$(F1)$ Obviously true.

$(F2)$ Let $u, v \geq 0$ and $F(u, v, v, u, 0, 2u + v) = u - xv - y \max\{u, v, 2u + v\} \leq 0$. Then if $u > v$ we obtain that $u[1 - (x + 3y)] \leq 0$, which is a contradiction. Hence, $u \leq v$, which implies $u \leq hv$, where $0 \leq h = x + 3y < 1$.

$(F3)$ $F(t, t, 0, 0, t, t) = t[1 - (x + 3y)] > 0$, for all $t > 0$.

Theorem 3.1. *Let (M, τ) be a $(3, 2)$-complete $(3, 2)$-W-symmetrizable space via $(3, 2)$-symmetric d and $\{A_i\}_{i=1}^{p}$ be a family of nonempty closed subsets of M. Let $\mathcal{A} = \cup_{i=1}^{p} A_i$ and let $f : \mathcal{A} \to \mathcal{A}$ be a p-cyclic mapping such that for all $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, p$ and $F \in \mathcal{F}$*

$$F \begin{pmatrix} d(fx, fy, fy), d(x, y, y), d(x, fx, fx), \\ d(y, fy, fy), d(y, fx, fx), d(x, fy, fy) \end{pmatrix} \leq 0. \tag{3.1}$$

Then f has a unique fixed point in $\cap_{i=1}^{p} A_i$.

Proof. Let x_0 be an arbitrary point of A_1. We define $x_n = fx_{n-1}$, $n = 1, 2, \ldots$ From Definition 3.1 and (3.1), for $x_0 \in A_1$ and $x_1 \in A_2$, we have $x_{p-1} = fx_{p-2} \in A_p$,

$x_p = fx_{p-1} \in A_{p+1} = A_1$, $x_{p+1} = fx_p \in A_2$ and

$$F\begin{pmatrix} d(fx_0, fx_1, fx_1), d(x_0, x_1, x_1), d(x_0, fx_0, fx_0), \\ d(x_1, fx_1, fx_1), d(x_1, fx_0, fx_0), d(x_0, fx_1, fx_1) \end{pmatrix} \leq 0,$$

i.e., $F(d(x_1, x_2, x_2), d(x_0, x_1, x_1), d(x_0, x_1, x_1), d(x_1, x_2, x_2), 0, d(x_0, x_2, x_2)) \leq 0$. Since d is $(3, 2)$-symmetric, we have

$$d(x_0, x_2, x_2) \leq d(x_0, x_1, x_1) + 2d(x_2, x_1, x_1) = d(x_0, x_1, x_1) + 2d(x_1, x_2, x_2). \tag{3.2}$$

From (3.2) and (F1) we obtain

$$F\begin{pmatrix} d(x_1, x_2, x_2), d(x_0, x_1, x_1), d(x_0, x_1, x_1), \\ d(x_1, x_2, x_2), 0, d(x_0, x_1, x_1) + 2d(x_1, x_2, x_2) \end{pmatrix} \leq 0.$$

By (F2) we obtain

$$d(x_1, x_2, x_2) \leq hd(x_0, x_1, x_1).$$

Hence, we have

$$d(x_n, x_{n+1}, x_{n+1}) \leq hd(x_{n-1}, x_n, x_n) \leq \cdots \leq h^n d(x_0, x_1, x_1).$$

Then, for all $m, n \in \mathbb{N}$, $m > n$,

$$\begin{aligned} d(x_n, x_m, x_m) &\leq d(x_n, x_{n+1}, x_{n+1}) + 2d(x_m, x_{n+1}, x_{n+1}) \\ &\leq d(x_n, x_{n+1}, x_{n+1}) + 4d(x_{n+1}, x_{n+2}, x_{n+2}) + 2d(x_m, x_{n+2}, x_{n+2}) \\ &\leq d(x_n, x_{n+1}, x_{n+1}) + 4d(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + 4d(x_{n+2}, x_{n+3}, x_{n+3}) + 2d(x_m, x_{n+3}, x_{n+3}) \\ &\leq d(x_n, x_{n+1}, x_{n+1}) + 4d(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + 4d(x_{n+2}, x_{n+3}, x_{n+3}) + \cdots + 4d(x_{m-1}, x_m, x_m) \\ &\leq 4(h^n + h^{n+1} + \cdots + h^{m-1})d(x_0, x_1, x_1) \\ &\leq 4\frac{h^n}{1-h}d(x_0, x_1, x_1). \end{aligned}$$

Thus, $d(x_n, x_m, x_m) \to 0$ as $n, m \to +\infty$, i.e., $(x_n)_{n=1}^{+\infty}$ is $(3, 2)$-Cauchy sequence. Since (M, τ) is $(3, 2)$-complete, there is $z \in M$, such that $d(x_n, z, z) \to 0$ as $n \to +\infty$. Then the sequences $(x_{np})_{n=0}^{+\infty}$,$(x_{np+1})_{n=0}^{+\infty}, \ldots, (x_{np+p-1})_{n=0}^{+\infty}$ converge to z. Since $x_{np+i-1} \in A_i$, $i = 1, 2, \ldots, p$, and the family $\{A_i\}_{i=1}^{p}$ is a family of nonempty closed subsets of M, by Lemma 2.2 we get $z \in \cap_{i=1}^{p} A_i$.

Next we will prove that z is a fixed point of f. If we set $x = x_n$ and $y = z$ at the inequality (3.1), we obtain

$$F\begin{pmatrix} d(fx_n, fz, fz), d(x_n, z, z), d(x_n, fx_n, fx_n), \\ d(z, fz, fz), d(z, fx_n, fx_n), d(x_n, fz, fz) \end{pmatrix} \leq 0,$$

i.e.,

$$F\begin{pmatrix} d(x_{n+1}, fz, fz), d(x_n, z, z), d(x_n, x_{n+1}, x_{n+1}), \\ d(z, fz, fz), d(z, x_{n+1}, x_{n+1}), d(x_n, fz, fz) \end{pmatrix} \leq 0. \tag{3.3}$$

It is obvious that $d(x_n, x_{n+1}, x_{n+1}) \to 0$ and $d(z, x_{n+1}, x_{n+1}) \to 0$ as $n \to +\infty$. From lemma 2.1 it follows that $d(x_{n+1}, fz, fz) \to d(z, fz, fz)$ and $d(x_n, fz, fz) \to d(z, fz, fz)$ as $n \to +\infty$. If we use all these combined with the fact that F is lower semi-continuous function, and let $n \to +\infty$ in (3.3), we obtain that

$$F(d(z, fz, fz), 0, 0, d(z, fz, fz), 0, d(z, fz, fz)) \leq 0. \tag{3.4}$$

From (3.4) and the condition (F1) we get

$$F(d(z, fz, fz), 0, 0, d(z, fz, fz), 0, 2d(z, fz, fz)) \leq 0.$$

And by (F2) we obtain that $d(z, fz, fz) = 0$, i.e., $fz = z$.

Next we will prove the uniqueness of point z. Suppose that there is another fixed point $z' \in \cap_{i=1}^{p} A_i$. If we set $x = z$ and $y = z'$ at the inequality (3.1), we obtain

$$F\begin{pmatrix} d(fz, fz', fz'), d(z, z', z'), d(z, fz, fz), \\ d(z', fz', fz'), d(z', fz, fz), d(z, fz', fz') \end{pmatrix} \leq 0,$$

i.e.

$$F(d(z, z', z'), d(z, z', z'), 0, 0, d(z', z, z), d(z, z', z')) \leq 0.$$

Since d is a $(3,2)$-symmetric, $d(z', z, z) = d(z, z', z')$. Hence,

$$F(d(z, z', z'), d(z, z', z'), 0, 0, d(z, z', z'), d(z, z', z')) \leq 0.$$

From (F3) it follows that $d(z, z', z') = 0$. Thus, $z = z'$, i.e., z is the unique fixed point of f such that $z' \in \cap_{i=1}^{p} A_i$. $\square$

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