KRAGUJEVAC JOURNAL OF MATHEMATICS
VOLUME 47(6) (2023), PAGES 947–964.

CONSTRUCTION OF SIMULTANEOUS COSPECTRAL GRAPHS FOR ADJACENCY, LAPLACIAN AND NORMALIZED LAPLACIAN MATRICES

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ABSTRACT. In this paper we construct several classes of non-regular graphs which are co-spectral with respect to all the three matrices, namely, adjacency, Laplacian and normalized Laplacian, and hence we answer a question asked by Butler [2]. We make these constructions starting with two pairs $(G_1,\ H_1)$ and $(G_2,\ H_2)$ of A-cospectral regular graphs, then considering the subdivision graphs $S(G_i)$ and R-graphs $\mathcal{R}(H_i)$, $i = 1, 2$, and finally making some kind of partial joins between $S(G_1)$ and $\mathcal{R}(G_2)$ and $S(H_1)$ and $\mathcal{R}(H_2)$. Moreover, we determine the number of spanning trees and the Kirchhoff index of the newly constructed graphs.

1. INTRODUCTION

Cospectral graphs are non-isomorphic graphs which share the same eigenvalues of the same matrices associated with them. Several cospectral graphs are known for adjacency, combinatorial Laplacian and normalized Laplacian matrices separately. In 2010, Butler [2] asked that "Is there an example of two non-regular graphs which are cospectral with respect to the adjacency, combinatorial Laplacian and normalized Laplacian at the same time?" Normally regular graphs are always cospectral for all the matrices mentioned in the question. Here we construct some non-regular cospectral graphs for all the three matrices and hence give an answer to the above question of Butler. To present the results of the paper we need some definitions and terminology as follow. All graphs considered in the paper are simple and undirected. For any graph G, we take $V(G)$ and $E(G)$ as the vertex set and edge set of G

Key words and phrases. Adjacency matrix, Laplacian matrix, normalized Laplacian matrix, cospectral graphs.
2010 *Mathematics Subject Classification.* Primary: 05C50.
DOI 10.46793/KgJMat2306.947D
Received: September 07, 2020.
Accepted: January 28, 2021.

respectively. The *adjacency matrix* of graph G, denoted by $A(G)$, is a square matrix whose rows and columns are indexed by vertices of graph G, and $(u,v)^{\text{th}}$ entry is 1 if and only if vertex u is adjacent to vertex v and 0 otherwise. If $D(G)$ is the diagonal matrix of vertex degrees in G, then the *Laplacian matrix* $L(G)$ is defined as $L(G) = D(G) - A(G)$ and the *normalized Laplacian matrix* $\mathcal{L}(G)$ of G is defined as $\mathcal{L}(G) = I - D(G)^{-1/2}A(G)D(G)^{-1/2}$ with the convention that $D(G)^{-1}(u,u) = 0$ if degree of u is zero. For a given square matrix M of size n, we denote the characteristic polynomial $\det(xI_n - M)$ by $f_M(x)$. The eigenvalues of $A(G)$, $L(G)$ and $\mathcal{L}(G)$ are denoted by $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$, $0 = \mu_1(G) \leq \mu_2(G) \leq \cdots \leq \mu_n(G)$, and $0 = \delta_1(G) \leq \delta_2(G) \leq \cdots \leq \delta_n(G) \leq 2$ respectively, where n is the number of vertices of G. The multiset of eigenvalues of $A(G)$ (respectively $L(G)$, $\mathcal{L}(G)$) is called the *adjacency* (respectively *Laplacian*, *normalized Laplacian*) *spectrum* of G, and denoted by A-spectrum (respectively L-spectrum, $\mathcal{L}$-spectrum). Two graphs are said to be A-cospectral (respectively L-cospectral, $\mathcal{L}$-cospectral) if they have the same A-spectrum (respectively L-spectrum, $\mathcal{L}$-spectrum).

The adjacency, Laplacian and normalized Laplacian spectra of different kinds of graphs have been computed by several researchers [4, 7, 11, 12]. The *subdivision graph* $S(G)$ [6] of a graph G is obtained by inserting a new vertex into every edge of G. The *R-graph* $\mathcal{R}(G)$ [5] of a graph G is the graph obtained from G by introducing a new vertex u_e for each $e \in E(G)$ and making u_e adjacent to both the end vertices of e. The set of such new vertices is denoted by $I(G)$, i.e., $I(G) = V(S(G))\backslash V(G) = V(\mathcal{R}(G))\backslash V(G)$. The partial joins of subdivision graph and R-graph which are considered in the paper are given in the definition below.

Definition 1.1. Let G_1 and G_2 be two vertex-disjoint graphs with number of vertices n_1 and n_2, and edges m_1 and m_2, respectively. Then the following hold.

(i) The *subdivision-vertex-R-vertex join* of G_1 and G_2, denoted by $S(G_1)\ddot{\nabla}\mathcal{R}(G_2)$, is the graph obtained from $S(G_1)$ and $\mathcal{R}(G_2)$ by joining each vertex of $V(G_1)$ with every vertex of $V(G_2)$. The graph $S(G_1)\ddot{\nabla}\mathcal{R}(G_2)$ has $n_1 + n_2 + m_1 + m_2$ vertices and $2m_1 + n_1n_2 + 3m_2$ edges.

(ii) The *subdivision-edge-R-edge join* of G_1 and G_2, denoted by $S(G_1)\overline{\overline{\nabla}}\mathcal{R}(G_2)$, is the graph obtained from $S(G_1)$ and $\mathcal{R}(G_2)$ by joining each vertex of $I(G_1)$ with every vertex of $I(G_2)$. The graph $S(G_1)\overline{\overline{\nabla}}\mathcal{R}(G_2)$ has $n_1 + n_2 + m_1 + m_2$ vertices and $m_1(2 + m_2) + 3m_2$ edges.

(iii) The *subdivision-edge-R-vertex join* of G_1 and G_2, denoted by $S(G_1)\dot{\overline{\nabla}}\mathcal{R}(G_2)$, is the graph obtained from $S(G_1)$ and $\mathcal{R}(G_2)$ by joining each vertex of $I(G_1)$ with every vertex of $V(G_2)$. The graph $S(G_1)\dot{\overline{\nabla}}\mathcal{R}(G_2)$ has $n_1 + n_2 + m_1 + m_2$ vertices and $m_1(2 + n_2) + 3m_2$ edges.

(iv) The *subdivision-vertex-R-edge join* of G_1 and G_2, denoted by $S(G_1)\dot{\nabla}\mathcal{R}(G_2)$, is the graph obtained from $S(G_1)$ and $\mathcal{R}(G_2)$ by joining each vertex of $V(G_1)$ with every vertex of $I(G_2)$. The graph $S(G_1)\dot{\nabla}\mathcal{R}(G_2)$ has $n_1 + n_2 + m_1 + m_2$ vertices and $2m_1 + m_2(3 + n_1)$ edges.

Example 1.1. Let us consider two graphs $G_1 = P_4$ and $G_2 = P_3$. The set of dark vertices of G_1 and G_2 are $I(G_1)$ and $I(G_2)$, respectively.

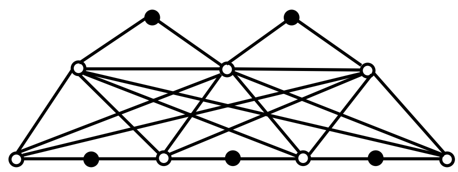

FIGURE 1. Subdivision-vertex-R-vertex join of P_4 and P_3

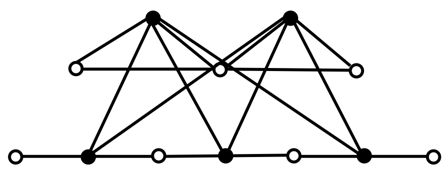

FIGURE 2. Subdivision-edge-R-edge join of P_4 and P_3

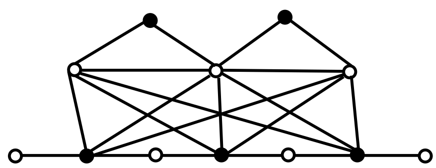

FIGURE 3. Subdivision-edge-R-vertex join of P_4 and P_3

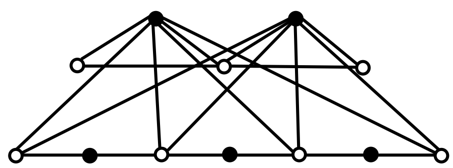

FIGURE 4. Subdivision-vertex-R-edge join of P_4 and P_3

In the following lemma we find the degrees of vertices in the above constructed graphs.

Lemma 1.1. (i) *The degree of any vertex v in $S(G_1)\ddot{\vee}\mathcal{R}(G_2)$ is given by*

$$d_{S(G_1)\ddot{\vee}\mathcal{R}(G_2)}(v)=\begin{cases} n_2+d_{G_1}(v), & \text{if } v\in V(G_1),\\ 2, & \text{if } v\in I(G_1)\bigcup I(G_2),\\ n_1+2d_{G_2}(v), & \text{if } v\in V(G_2).\end{cases}$$

(ii) *The degree of any vertex v in $S(G_1)\overline{\overline{\vee}}\mathcal{R}(G_2)$ is given by*

$$d_{S(G_1)\overline{\overline{\vee}}\mathcal{R}(G_2)}(v)=\begin{cases} d_{G_1}(v), & \text{if } v\in V(G_1),\\ 2+m_2, & \text{if } v\in I(G_1),\\ 2d_{G_2}(v), & \text{if } v\in V(G_2),\\ 2+m_1, & \text{if } v\in I(G_2).\end{cases}$$

(iii) *The degree of any vertex v in $S(G_1)\overline{\dot{\vee}}\mathcal{R}(G_2)$ is given by*

$$d_{S(G_1)\overline{\dot{\vee}}\mathcal{R}(G_2)}(v)=\begin{cases} d_{G_1}(v), & \text{if } v\in V(G_1),\\ 2+n_2, & \text{if } v\in I(G_1),\\ 2d_{G_2}(v)+m_1, & \text{if } v\in V(G_2),\\ 2, & \text{if } v\in I(G_2).\end{cases}$$

(iv) *The degree of any vertex v in $S(G_1)\dot{\overline{\vee}}\mathcal{R}(G_2)$ is given by*

$$d_{S(G_1)\dot{\overline{\vee}}\mathcal{R}(G_2)}(v)=\begin{cases} d_{G_1}(v)+m_2, & \text{if } v\in V(G_1),\\ 2, & \text{if } v\in I(G_1),\\ 2d_{G_2}(v), & \text{if } v\in V(G_2),\\ 2+n_1, & \text{if } v\in I(G_2).\end{cases}$$

For two matrices A and B, of same size $m\times n$, the *Hadamard product* $A\bullet B$ of A and B is a matrix of the same size $m\times n$ with entries given by $(A\bullet B)_{ij}=(A)_{ij}\cdot(B)_{ij}$ (that is entrywise multiplication). Hadamard product is commutative, that is $A\bullet B=B\bullet A$.

Notation. Throughout the paper, for any positive integers k, n_1 and n_2, I_k denotes the identity matrix of size k, $J_{n_1\times n_2}$ denotes $n_1\times n_2$ matrix whose all entries are 1, $\mathbf{1}_n$ stands for the column vector of size n with all entries equal to 1, $K_{n\times n}$ denotes an $n\times n$ matrix whose all entries are the same. In other words, $K_{n\times n}=\alpha J_{n\times n}$, for a real number α. For any positive integers s and t, $O_{s\times t}$ denotes the zero matrix of size $s\times t$.

To prove our results we need some basics as given below.

Lemma 1.2 (Schur Complement [6])**.** *Suppose that the order of all four matrices M, N, P and Q satisfy the rules of operations on matrices. Then we have*

$$\begin{vmatrix} M & N\\ P & Q\end{vmatrix}=\begin{cases} |Q||M-NQ^{-1}P|, & \text{if } Q \text{ is a non-singular square matrix},\\ |M||Q-PM^{-1}N|, & \text{if } M \text{ is a non-singular square matrix}.\end{cases}$$

Lemma 1.3 ([6])**.** *For a square matrix A of size n and a scalar α,*

$$\det(A+\alpha J_{n\times n})=\det(A)+\alpha\mathbf{1}_n^T\operatorname{adj}(A)\mathbf{1}_n,$$

where $\operatorname{adj}(A)$ *is the adjugate matrix of A.*

Lemma 1.4. *For any real numbers $c, d > 0$, we have*

$$(cI_n - dJ_{n\times n})^{-1} = \frac{1}{c}I_n + \frac{d}{c(c-nd)}J_{n\times n}.$$

Proof.

$$\begin{aligned}(cI_n - dJ_{n\times n})^{-1} =& \frac{\operatorname{adj}(cI_n - dJ_{n\times n})}{\det(cI_n - dJ_{n\times n})} = \frac{c^{n-2}(c-nd)I_n + c^{n-2}dJ_{n\times n}}{c^{n-1}(c-nd)} \\ =& \frac{1}{c}I_n + \frac{d}{c(c-nd)}J_{n\times n}.\end{aligned}$$

□

For a graph G on n vertices and m edges, the *vertex-edge incidence matrix* [8] $R(G)$ of G is a matrix of size $n \times m$, with entry $r_{ij} = 1$ if the i^{th} vertex is incident to the j^{th} edge, and 0 otherwise. The *line graph* [8] of a graph G is the graph $\mathbf{L}_G$, whose vertices are the edges of G and two of these are adjacent in $\mathbf{L}_G$ if and only if they are incident on a common vertex in G.

The following is an well known result, may be found in [6].

Lemma 1.5. *Let G be an r-regular graph. Then*
(i) $R(G)^TR(G) = A(\mathbf{L}_G) + 2I_m$ and $R(G)R(G)^T = A(G) + rI_n$;
(ii) the eigenvalues of $A(\mathbf{L}_G)$ are the eigenvalues of $A(G)+(r-2)I_n$ and -2 repeated $m-n$ times.

Notation. The M-coronal of an $n\times n$ matrix M, denoted by $\Gamma_M(x)$, is defined [3,13] as the sum of the entries of the matrix $(xI_n-M)^{-1}$, that is, $\Gamma_M(x) = \mathbf{1}_n^T(xI_n-M)^{-1}\mathbf{1}_n$.

Lemma 1.6 ([3]). *If M is an $n \times n$ matrix with each row sum equal to a constant t, then $\Gamma_M(x) = \frac{n}{x-t}$.*

Butler [2] constructed non-regular bipartite graphs which are cospectral with respect to both the adjacency and normalized Laplacian matrices, and then asked for existence of non-regular graphs which are cospectral with respect to all the three matrices, namely, adjacency, Laplacian and normalized Laplacian. In this paper we construct several classes of such graphs taking help of the operations subdivision-vertex-R-vertex join, subdivision-edge-R-edge join, subdivision-edge-R-vertex join and subdivision-vertex-R-edge join. We also find the number of spanning trees and Kirchhoff index for all the partial join of subdivision graph and R-graph constructed here.

2. Adjacency, Laplacian and Normalized Laplacian Spectra of the Graphs

In this section we consider regular graphs G_i on n_i vertices, m_i edges, and with degree of regularity r_i, $i = 1, 2$. To obtain the required matrices we label the vertices of the graphs in the following way. Let $V(G_1) = \{v_1, \ldots, v_{n_1}\}$, $I(G_1) = \{e_1, \ldots, e_{m_1}\}$, $V(G_2) = \{u_1, \ldots, u_{n_2}\}$, $I(G_2) = \{f_1, \ldots, f_{m_2}\}$. Then $V(G_1) \cup I(G_1) \cup V(G_2) \cup$

$I(G_2)$ is a partition for all $V(S(G_1)\ddot{\vee}\mathcal{R}(G_2))$, $V(S(G_1)\overline{\overline{\vee}}\mathcal{R}(G_2))$, $V(S(G_1)\overline{\dot{\vee}}\mathcal{R}(G_2))$ and $V(S(G_1)\dot{\vee}\mathcal{R}(G_2))$.

Lemma 2.1. *For $i = 1, 2$, let G_i be a graph with n_i vertices and m_i edges. Then we have the following:*

$$(i)\ A(S(G_1)\ddot{\vee}\mathcal{R}(G_2)) = \begin{pmatrix} O_{n_1} & R(G_1) & J_{n_1\times n_2} & O_{n_1\times m_2} \\ R(G_1)^T & O_{m_1} & O_{m_1\times n_2} & O_{m_1\times m_2} \\ J_{n_2\times n_1} & O_{n_2\times m_1} & A(G_2) & R(G_2) \\ O_{m_2\times n_1} & O_{m_2\times m_1} & R(G_2)^T & O_{m_2} \end{pmatrix};$$

$$(ii)\ A(S(G_1)\overline{\overline{\vee}}\mathcal{R}(G_2)) = \begin{pmatrix} O_{n_1} & R(G_1) & O_{n_1\times n_2} & O_{n_1\times m_2} \\ R(G_1)^T & O_{m_1} & O_{m_1\times n_2} & J_{m_1\times m_2} \\ O_{n_2\times n_1} & O_{n_2\times m_1} & A(G_2) & R(G_2) \\ O_{m_2\times n_1} & J_{m_2\times m_1} & R(G_2)^T & O_{m_2} \end{pmatrix};$$

$$(iii)\ A(S(G_1)\overline{\dot{\vee}}\mathcal{R}(G_2)) = \begin{pmatrix} O_{n_1} & R(G_1) & O_{n_1\times n_2} & O_{n_1\times m_2} \\ R(G_1)^T & O_{m_1} & J_{m_1\times n_2} & O_{m_1\times m_2} \\ O_{n_2\times n_1} & J_{n_2\times m_1} & A(G_2) & R(G_2) \\ O_{m_2\times n_1} & O_{m_2\times m_1} & R(G_2)^T & O_{m_2} \end{pmatrix};$$

$$(iv)\ A(S(G_1)\dot{\vee}\mathcal{R}(G_2)) = \begin{pmatrix} O_{n_1} & R(G_1) & O_{n_1\times n_2} & J_{n_1\times m_2} \\ R(G_1)^T & O_{m_1} & O_{m_1\times n_2} & O_{m_1\times m_2} \\ O_{n_2\times n_1} & O_{n_2\times m_1} & A(G_2) & R(G_2) \\ J_{m_2\times n_1} & O_{m_2\times m_1} & R(G_2)^T & O_{m_2} \end{pmatrix}.$$

Theorem 2.1. *For $i = 1, 2$, let G_i be an r_i-regular graph with n_i vertices and m_i edges. Then the adjacency spectrum of $S(G_1)\ddot{\vee}\mathcal{R}(G_2)$ consists of:*

(i) the eigenvalue $\pm\sqrt{r_1 + \lambda_i(G_1)}$ for every eigenvalue $\lambda_i(G_1)$, $i = 2, 3, \ldots, n_1$, of $A(G_1)$;

(ii) roots of the equation $x^2 - \lambda_j(G_2)x - r_2 - \lambda_j(G_2) = 0$ for every eigenvalue $\lambda_j(G_2)$, $j = 2, 3, \ldots, n_2$, of $A(G_2)$;

(iii) the eigenvalue 0 with multiplicity $m_1 + m_2 - n_1 - n_2$;

(iv) four roots of the equation $x^4 - r_2x^3 - (2r_1 + n_1n_2 + 2r_2)x^2 + 2r_1r_2x + 4r_1r_2 = 0$.

Proof. The adjacency characteristic polynomial of $S(G_1)\ddot{\vee}\mathcal{R}(G_2)$ is

$$f_{A(S(G_1)\ddot{\vee}\mathcal{R}(G_2))}(x) = \det \begin{pmatrix} xI_{n_1} & -R(G_1) & -J_{n_1\times n_2} & O_{n_1\times m_2} \\ -R(G_1)^T & xI_{m_1} & O_{m_1\times n_2} & O_{m_1\times m_2} \\ -J_{n_2\times n_1} & O_{n_2\times m_1} & xI_{n_2} - A(G_2) & -R(G_2) \\ O_{m_2\times n_1} & O_{m_2\times m_1} & -R(G_2)^T & xI_{m_2} \end{pmatrix} = x^{m_2}\det(S),$$

where

$$S = \begin{pmatrix} xI_{n_1} & -R(G_1) & -J_{n_1\times n_2} \\ -R(G_1)^T & xI_{m_1} & O_{m_1\times n_2} \\ -J_{n_2\times n_1} & O_{n_2\times m_1} & xI_{n_2} - A(G_2) \end{pmatrix}$$

$$- \begin{pmatrix} O_{n_1\times m_2} \\ O_{m_1\times m_2} \\ -R(G_2) \end{pmatrix} \frac{1}{x} \begin{pmatrix} O_{m_2\times n_1} & O_{m_2\times m_1} & -R(G_2)^T \end{pmatrix}$$

$$= \begin{pmatrix} xI_{n_1} & -R(G_1) & -J_{n_1\times n_2} \\ -R(G_1)^T & xI_{m_1} & O_{m_1\times n_2} \\ -J_{n_2\times n_1} & O_{n_2\times m_1} & xI_{n_2} - A(G_2) - \frac{1}{x}R(G_2)R(G_2)^T \end{pmatrix}.$$

Hence,

$$\begin{aligned} \det(S) &= \det\left(xI_{n_2} - A(G_2) - \frac{1}{x}R(G_2)R(G_2)^T\right)\det(W) \\ &= \prod_{j=1}^{n_2}\left(x - \lambda_j(G_2) - \frac{r_2}{x} - \frac{\lambda_j(G_2)}{x}\right)\det(W), \end{aligned}$$

where

$$\begin{aligned} W &= \begin{pmatrix} xI_{n_1} & -R(G_1) \\ -R(G_1)^T & xI_{m_1} \end{pmatrix} \\ &\quad - \begin{pmatrix} -J_{n_1\times n_2} \\ O_{m_1\times n_2} \end{pmatrix}\left(xI_{n_2} - A(G_2) - \frac{1}{x}R(G_2)R(G_2)^T\right)^{-1}\begin{pmatrix} -J_{n_2\times n_1} & O_{n_2\times m_1} \end{pmatrix} \\ &= \begin{pmatrix} xI_{n_1} - \Gamma_{A(G_2)+\frac{1}{x}R(G_2)R(G_2)^T}(x)J_{n_1\times n_1} & -R(G_1) \\ -R(G_1)^T & xI_{m_1} \end{pmatrix}. \end{aligned}$$

Then

$$\begin{aligned} \det(W) =& x^{m_1}\det\left(xI_{n_1} - \Gamma_{A(G_2)+\frac{1}{x}R(G_2)R(G_2)^T}(x)J_{n_1\times n_1} - \frac{1}{x}R(G_1)R(G_1)^T\right) \\ =& x^{m_1}\Bigg[\det\left(xI_{n_1} - \frac{1}{x}R(G_1)R(G_1)^T\right) \\ &- \Gamma_{A(G_2)+\frac{1}{x}R(G_2)R(G_2)^T}(x)\mathbf{1}_{n_1}^T\,\mathrm{adj}\left(xI_{n_1} - \frac{1}{x}R(G_1)R(G_1)^T\right)\mathbf{1}_{n_1}\Bigg] \\ =& x^{m_1}\det\left(xI_{n_1} - \frac{1}{x}R(G_1)R(G_1)^T\right) \\ &\times\left[1 - \Gamma_{A(G_2)+\frac{1}{x}R(G_2)R(G_2)^T}(x)\mathbf{1}_{n_1}^T\left(xI_{n_1} - \frac{1}{x}R(G_1)R(G_1)^T\right)^{-1}\mathbf{1}_{n_1}\right] \\ =& x^{m_1}\prod_{i=1}^{n_1}\left(x - \frac{r_1}{x} - \frac{\lambda_i(G_1)}{x}\right)\left[1 - \Gamma_{A(G_2)+\frac{1}{x}R(G_2)R(G_2)^T}(x)\Gamma_{\frac{1}{x}R(G_1)R(G_1)^T}(x)\right] \\ =& x^{m_1}\prod_{i=1}^{n_1}\left(x - \frac{r_1}{x} - \frac{\lambda_i(G_1)}{x}\right)\left[1 - \frac{n_2}{x - r_2 - \frac{2r_2}{x}}\frac{n_1}{x - \frac{2r_1}{x}}\right]. \end{aligned}$$

Therefore,

$$f_{A(S(G_1)\ddot{\vee}\mathcal{R}(G_2))}(x) = x^{m_1}x^{m_2}\prod_{i=1}^{n_1}\left(x - \frac{r_1}{x} - \frac{\lambda_i(G_1)}{x}\right)\prod_{j=1}^{n_2}\left(x - \lambda_j(G_2) - \frac{r_2}{x} - \frac{\lambda_j(G_2)}{x}\right)$$

$$\times \left[1 - \frac{n_2}{x - r_2 - \frac{2r_2}{x}} \frac{n_1}{x - \frac{2r_1}{x}}\right]$$
$$= x^{m_1 - n_1} x^{m_2 - n_2} \prod_{i=2}^{n_1} \{x^2 - r_1 - \lambda_i(G_1)\}$$
$$\times \prod_{j=2}^{n_2} \{x^2 - \lambda_j(G_2)x - r_2 - \lambda_j(G_2)\}$$
$$\times \{x^4 - r_2 x^3 - (2r_1 + n_1 n_2 + 2r_2)x^2 + 2r_1 r_2 x + 4r_1 r_2\},$$

and the result follows immediately. □

Theorem 2.2. *For $i = 1, 2$, let G_i be an r_i-regular graph with n_i vertices and m_i edges. Then the adjacency spectrum of $S(G_1)\overline{\overline{\nabla}}\mathcal{R}(G_2)$ consists of:*

(i) the eigenvalue $\pm\sqrt{r_1 + \lambda_i(G_1)}$ for every eigenvalue $\lambda_i(G_1)$, $i = 2, 3, \ldots, n_1$, of $A(G_1)$;

(ii) roots of the equation $x^2 - \lambda_j(G_2)x - r_2 - \lambda_j(G_2) = 0$ for every eigenvalue $\lambda_j(G_2)$, $j = 2, 3, \ldots, n_2$, of $A(G_2)$;

(iii) the eigenvalue 0 with multiplicity $m_1 + m_2 - n_1 - n_2$;

(iv) four roots of the equation $x^4 - r_2 x^3 - (2r_1 + m_1 m_2 + 2r_2)x^2 + (2r_1 r_2 + m_1 m_2 r_2)x + 4r_1 r_2 = 0$.

Proof. The adjacency characteristic polynomial of $S(G_1)\overline{\overline{\nabla}}\mathcal{R}(G_2)$ is

$$f_{A(S(G_1)\overline{\overline{\nabla}}\mathcal{R}(G_2))}(x) = \det\begin{pmatrix} xI_{n_1} & -R(G_1) & O_{n_1\times n_2} & O_{n_1\times m_2} \\ -R(G_1)^T & xI_{m_1} & O_{m_1\times n_2} & -J_{m_1\times m_2} \\ O_{n_2\times n_1} & O_{n_2\times m_1} & xI_{n_2} - A(G_2) & -R(G_2) \\ O_{m_2\times n_1} & -J_{m_2\times m_1} & -R(G_2)^T & xI_{m_2} \end{pmatrix} = x^{n_1}\det(S),$$

where

$$S = \begin{pmatrix} xI_{m_1} & O_{m_1\times n_2} & -J_{m_1\times m_2} \\ O_{n_2\times m_1} & xI_{n_2} - A(G_2) & -R(G_2) \\ -J_{m_2\times m_1} & -R(G_2)^T & xI_{m_2} \end{pmatrix}$$
$$- \begin{pmatrix} -R(G_1)^T \\ O_{n_2\times n_1} \\ O_{m_2\times n_1} \end{pmatrix} \frac{1}{x} \begin{pmatrix} -R(G_1) & O_{n_1\times n_2} & O_{n_1\times m_2} \end{pmatrix}$$
$$= \begin{pmatrix} xI_{m_1} - \frac{1}{x}R(G_1)^T R(G_1) & O_{m_1\times n_2} & -J_{m_1\times m_2} \\ O_{n_2\times m_1} & xI_{n_2} - A(G_2) & -R(G_2) \\ -J_{m_2\times m_1} & -R(G_2)^T & xI_{m_2} \end{pmatrix}.$$

Hence,

$$\det(S) = \det\left(xI_{m_1} - \frac{1}{x}R(G_1)^T R(G_1)\right)\det(W)$$
$$= \det\left(xI_{m_1} - \frac{1}{x}(A(\mathbf{L}_{G_1}) + 2I_{m_1})\right)\det(W)$$

$$= x^{m_1-n_1}\prod_{i=1}^{n_1}\left(x-\frac{r_1}{x}-\frac{\lambda_i(G_1)}{x}\right)\det(W),$$

where

$$\begin{aligned}
W =& \begin{pmatrix} xI_{n_2}-A(G_2) & -R(G_2)\\ -R(G_2)^T & xI_{m_2}\end{pmatrix}\\
&-\begin{pmatrix} O_{n_2\times m_1}\\ -J_{m_2\times m_1}\end{pmatrix}\left(xI_{m_1}-\frac{1}{x}R(G_1)^TR(G_1)\right)^{-1}\begin{pmatrix} O_{m_1\times n_2} & -J_{m_1\times m_2}\end{pmatrix}\\
=& \begin{pmatrix} xI_{n_2}-A(G_2) & -R(G_2)\\ -R(G_2)^T & xI_{m_2}-\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x)J_{m_2\times m_2}\end{pmatrix}.
\end{aligned}$$

Then

$$\begin{aligned}
\det(S) =& \det(xI_{m_2}-\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x)J_{m_2\times m_2})\\
&\times\det(xI_{n_2}-A(G_2)-R(G_2)(xI_{m_2}-\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x)J_{m_2\times m_2})^{-1}R(G_2)^T)\\
=& x^{m_2}\left(1-\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x)\frac{m_2}{x}\right)\det\Bigg[xI_{n_2}-A(G_2)\\
&-R(G_2)\left\{\frac{1}{x}I_{m_2}+\frac{\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x)}{x(x-m_2\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x))}J_{m_2\times m_2}\right\}R(G_2)^T\Bigg]\\
=& x^{m_2}\left(1-\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x)\frac{m_2}{x}\right)\det\Bigg(xI_{n_2}-A(G_2)\\
&-\frac{1}{x}R(G_2)R(G_2)^T-\frac{\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x)}{x(x-m_2\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x))}R(G_2)J_{m_2\times m_2}R(G_2)^T\Bigg)\\
=& x^{m_2}\left(1-\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x)\frac{m_2}{x}\right)\det\Bigg(xI_{n_2}-A(G_2)\\
&-\frac{1}{x}R(G_2)R(G_2)^T-r_2^2\frac{\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x)}{x(x-m_2\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x))}J_{n_2\times n_2}\Bigg)\\
=& x^{m_2}\left(1-\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x)\frac{m_2}{x}\right)\Bigg[\det\left(xI_{n_2}-A(G_2)-\frac{1}{x}R(G_2)R(G_2)^T\right)\\
&\times\frac{r_2^2\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x)}{x(x-m_2\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x))}\mathbf{1}_{n_2}^T\operatorname{adj}\left(xI_{n_2}-A(G_2)-\frac{1}{x}R(G_2)R(G_2)^T\right)\mathbf{1}_{n_2}\Bigg]\\
=& x^{m_2}\left(1-\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x)\frac{m_2}{x}\right)\det\left(xI_{n_2}-A(G_2)-\frac{1}{x}R(G_2)R(G_2)^T\right)\\
&\times\Bigg[1-\frac{r_2^2\Gamma_{\frac{1}{x}R(G_1)^TR(G_1))}(x)}{x(x-m_2\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x))}\mathbf{1}_{n_2}^T\\
&\times\left(xI_{n_2}-A(G_2)-\frac{1}{x}R(G_2)R(G_2)^T\right)^{-1}\mathbf{1}_{n_2}\Bigg]
\end{aligned}$$

$$
\begin{aligned}
=&x^{m_2}\Big(1-\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x)\frac{m_2}{x}\Big)\det\Big(xI_{n_2}-A(G_2)-\frac{1}{x}(r_2I_{n_2}+A(G_2))\Big)\\
&\times\left[1-\frac{r_2^2\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x)\Gamma_{A(G_2)+\frac{1}{x}R(G_2)R(G_2)^T}(x)}{x(x-m_2\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x))}\right]\\
=&x^{m_2}\Big(1-\frac{m_1m_2}{x(x-\frac{2r_1}{x})}\Big)\prod_{j=1}^{n_2}\Big\{x-\lambda_j(G_2)-\frac{1}{x}(r_2+\lambda_j(G_2))\Big\}\\
&\times\left[1-\frac{r_2^2m_1n_2}{x(x-\frac{2r_1}{x})(x-\frac{m_1m_2}{x-\frac{2r_1}{x}})(x-r_2-\frac{2r_2}{x})}\right].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
f_{A(S(G_1)\overline{\overline{\nabla}}\mathcal{R}(G_2))}(x)=&x^{n_1}x^{m_1-n_1}x^{m_2}\left(1-\frac{m_1m_2}{x(x-\frac{2r_1}{x})}\right)\prod_{i=1}^{n_1}\left(x-\frac{r_1}{x}-\frac{\lambda_i(G_1)}{x}\right)\\
&\times\prod_{j=1}^{n_2}\Big\{x-\lambda_j(G_2)-\frac{1}{x}(r_2+\lambda_j(G_2))\Big\}\\
&\times\left[1-\frac{r_2^2m_1n_2}{x(x-\frac{2r_1}{x})(x-\frac{m_1m_2}{x-\frac{2r_1}{x}})(x-r_2-\frac{2r_2}{x})}\right]\\
=&x^{m_1-n_1}x^{m_2-n_2}\prod_{i=2}^{n_1}\{x^2-r_1-\lambda_i(G_1)\}\\
&\times\prod_{j=2}^{n_2}\{x^2-\lambda_j(G_2)x-r_2-\lambda_j(G_2)\}\\
&\times\{x^4-r_2x^3-(2r_1+m_1m_2+2r_2)x^2\\
&+(2r_1r_2+m_1m_2r_2)x+4r_1r_2\},
\end{aligned}
$$

and hence the result follows. ☐

Theorem 2.3. *For $i=1,2$, let G_i be an r_i-regular graph with n_i vertices and m_i edges. Then the adjacency spectrum of $S(G_1)\bar{\nabla}\mathcal{R}(G_2)$ consists of:*

(i) the eigenvalue $\pm\sqrt{r_1+\lambda_i(G_1)}$ for every eigenvalue $\lambda_i(G_1)$, $i=2,3,\dots,n_1$, of $A(G_1)$;

(ii) roots of the equation $x^2-\lambda_j(G_2)x-r_2-\lambda_j(G_2)=0$ for every eigenvalue $\lambda_j(G_2)$, $j=2,3,\dots,n_2$, of $A(G_2)$;

(iii) the eigenvalue 0 with multiplicity $m_1+m_2-n_1-n_2$;

(iv) four roots of the equation $x^4-r_2x^3-(2r_1+m_1n_2+2r_2)x^2+2r_1r_2x+4r_1r_2=0$.

Proof. The proof is similar to that of proof of Theorem 2.2. ☐

Theorem 2.4. *For $i=1,2$, let G_i be an r_i-regular graph with n_i vertices and m_i edges. Then the adjacency spectrum of $S(G_1)\dot{\nabla}\mathcal{R}(G_2)$ consists of:*

(i) the eigenvalue $\pm\sqrt{r_1+\lambda_i(G_1)}$ *for every eigenvalue* $\lambda_i(G_1)$, $i=2,3,\ldots,n_1$, *of* $A(G_1)$;

(ii) roots of the equation $x^2-\lambda_j(G_2)x-r_2-\lambda_j(G_2)=0$ *for every eigenvalue* $\lambda_j(G_2)$, $j=2,3,\ldots,n_2$, *of* $A(G_2)$;

(iii) the eigenvalue 0 *with multiplicity* $m_1+m_2-n_1-n_2$;

(iv) four roots of the equation $x^4-r_2x^3-(2r_1+m_1n_2+2r_2)x^2+(2r_1r_2+r_2n_1m_2)x+4r_1r_2=0$.

Proof. The proof is similar to that of proof of Theorem 2.1. □

In the similar way as above we obtain Laplacian and normalized Laplacian spectra of the partial join graphs, which are given below.

Lemma 2.2. *We have the following Laplacian matrices:*

$$(i)\ L(S(G_1)\ddot{\nabla}\mathcal{R}(G_2))=\begin{pmatrix}(r_1+n_2)I_{n_1} & -R(G_1) & -J_{n_1\times n_2} & O_{n_1\times m_2}\\ -R(G_1)^T & 2I_{m_1} & O_{m_1\times n_2} & O_{m_1\times m_2}\\ -J_{n_2\times n_1} & O_{n_2\times m_1} & (r_2+n_1)I_{n_2}+L(G_2) & -R(G_2)\\ O_{m_2\times n_1} & O_{m_2\times m_1} & -R(G_2)^T & 2I_{m_2}\end{pmatrix};$$

$$(ii)\ L(S(G_1)\overline{\overline{\nabla}}R(G_2))=\begin{pmatrix}r_1I_{n_1} & -R(G_1) & O_{n_1\times n_2} & O_{n_1\times m_2}\\ -R(G_1)^T & (2+m_2)I_{m_1} & O_{m_1\times n_2} & -J_{m_1\times m_2}\\ O_{n_2\times n_1} & O_{n_2\times m_1} & r_2I_{n_2}+L(G_2) & -R(G_2)\\ O_{m_2\times n_1} & -J_{m_2\times m_1} & -R(G_2)^T & (2+m_1)I_{m_2}\end{pmatrix};$$

$$(iii)\ L(S(G_1)\overline{\ddot{\nabla}}\mathcal{R}(G_2))=\begin{pmatrix}r_1I_{n_1} & -R(G_1) & O_{n_1\times n_2} & O_{n_1\times m_2}\\ -R(G_1)^T & (2+n_2)I_{m_1} & -J_{m_1\times n_2} & O_{m_1\times m_2}\\ O_{n_2\times n_1} & -J_{n_2\times m_1} & (r_2+m_1)I_{n_2}+L(G_2) & -R(G_2)\\ O_{m_2\times n_1} & O_{m_2\times m_1} & -R(G_2)^T & 2I_{m_2}\end{pmatrix};$$

$$(iv)\ L(S(G_1)\dot{\nabla}\mathcal{R}(G_2))=\begin{pmatrix}(r_1+m_2)I_{n_1} & -R(G_1) & O_{n_1\times n_2} & -J_{n_1\times m_2}\\ -R(G_1)^T & 2I_{m_1} & O_{m_1\times n_2} & O_{m_1\times m_2}\\ O_{n_2\times n_1} & O_{n_2\times m_1} & r_2I_{n_2}+L(G_2) & -R(G_2)\\ -J_{m_2\times n_1} & O_{m_2\times m_1} & -R(G_2)^T & (2+n_1)I_{m_2}\end{pmatrix}.$$

Theorem 2.5. *For* $i=1,2$, *let* G_i *be an* r_i*-regular graph with* n_i *vertices and* m_i *edges. Then the Laplacian spectrum of* $S(G_1)\ddot{\nabla}\mathcal{R}(G_2)$ *consists of:*

(i) roots of the equation $x^2-(2+r_1+n_2)x+2n_2+\mu_i(G_1)=0$ *for every eigenvalue* $\mu_i(G_1)$,$i=2,3,\ldots,n_1$, *of* $L(G_1)$;

(ii) roots of the equation $x^2-(2+r_2+n_1+\mu_j(G_2))x+2n_1+3\mu_j(G_2)=0$ *for every eigenvalue* $\mu_j(G_2)$, $j=2,3,\ldots,n_2$, *of* $L(G_2)$;

(iii) the eigenvalue 2 *with multiplicity* $m_1+m_2-n_1-n_2$;

(iv) four roots of the equation $x^4-(4+r_1+r_2+n_1+n_2)x^3+(4+4n_1+4n_2+2r_1+2r_2+r_1r_2+r_1n_1+r_2n_2)x^2-2(2n_1+2n_2+r_1n_1+r_2n_2)x=0$.

Theorem 2.6. *For* $i=1,2$, *let* G_i *be an* r_i*-regular graph with* n_i *vertices and* m_i *edges. Then the Laplacian spectrum of* $S(G_1)\overline{\overline{\nabla}}\mathcal{R}(G_2)$ *consists of:*

(i) roots of the equation $x^2-(2+r_1+m_2)x+r_1m_2+\mu_i(G_1)=0$ *for every eigenvalue* $\mu_i(G_1)$, $i=2,3,\ldots,n_1$, *of* $L(G_1)$;

(ii) roots of the equation $x^2-(2+r_2+m_1+\mu_j(G_2))x+r_2m_1+3\mu_j(G_2)+m_1\mu_j(G_2)=0$ *for every eigenvalue* $\mu_j(G_2)$, $j=2,3,\ldots,n_2$, *of* $L(G_2)$;

(iii) the eigenvalue $2+m_2$ *with multiplicity* m_1-n_1;

(iv) the eigenvalue $2+m_1$ *with multiplicity* m_2-n_2;

(v) four roots of the equation $x^4-(4+r_1+r_2+m_1+m_2)x^3+(4+2r_1+2r_2+r_1r_2+r_1m_1+r_2m_2+2m_1+2m_2+r_1m_2+r_2m_1)x^2-(2r_1m_2+2r_2m_1+r_1r_2m_1+r_1r_2m_2)x=0$.

Theorem 2.7. *For* $i=1,2$, *let* G_i *be an* r_i*-regular graph with* n_i *vertices and* m_i *edges. Then the Laplacian spectrum of* $S(G_1)\ddot{\vee}\mathcal{R}(G_2)$ *consists of:*

(i) roots of the equation $x^2-(2+r_1+n_2)x+r_1n_2+\mu_i(G_1)=0$ *for every eigenvalue* $\mu_i(G_1)$, $i=2,3,\ldots,n_1$, *of* $L(G_1)$;

(ii) roots of the equation $x^2-(2+r_2+m_1+\mu_j(G_2))x+2m_1+3\mu_j(G_2)=0$ *for every eigenvalue* $\mu_j(G_2)$, $j=2,3,\ldots,n_2$, *of* $L(G_2)$;

(iii) the eigenvalue $2+n_2$ *with multiplicity* m_1-n_1;

(iv) the eigenvalue 2 *with multiplicity* m_2-n_2;

(v) four roots of the equation $x^4-(4+r_1+r_2+m_1+n_2)x^3+(4+2r_1+2r_2+4m_1+2n_2+r_1r_2+r_1m_1+r_1n_2+r_2n_2)x^2-(4m_1+2r_1m_1+2r_1n_2+r_1r_2n_2)x=0$.

Theorem 2.8. *For* $i=1,2$, *let* G_i *be an* r_i*-regular graph with* n_i *vertices and* m_i *edges. Then the Laplacian spectrum of* $S(G_1)\dot{\vee}\mathcal{R}(G_2)$ *consists of:*

(i) roots of the equation $x^2-(2+r_1+m_2)x+2m_2+\mu_i(G_1)=0$ *for every eigenvalue* $\mu_i(G_1)$, $i=2,3,\ldots,n_1$ *of* $L(G_1)$;

(ii) roots of the equation $x^2-(2+r_2+n_1+\mu_j(G_2))x+r_2n_1+3\mu_j(G_2)+n_1\mu_j(G_2)=0$ *for every eigenvalue* $\mu_j(G_2)$, $j=2,3,\ldots,n_2$, *of* $L(G_2)$;

(iii) the eigenvalue 2 *with multiplicity* m_1-n_1;

(iv) the eigenvalue $2+n_1$ *with multiplicity* m_2-n_2;

(v) four roots of the equation $x^4-(4+r_1+r_2+m_2+n_1)x^3+(4+2r_1+2r_2+4m_2+2n_1+r_1r_2+r_2m_2+r_1n_1+r_2n_1)x^2-(4m_2+2r_2m_2+2r_2n_1+r_1r_2n_1)x=0$.

Lemma 2.3. *We have the following normalized Laplacian matrices:*

(i)

$$\mathcal{L}(S(G_1)\ddot{\vee}\mathcal{R}(G_2))=\begin{pmatrix} I_{n_1} & -cR(G_1) & -K_{n_1\times n_2} & O_{n_1\times m_2}\\ -cR(G_1)^T & I_{m_1} & O_{m_1\times n_2} & O_{m_1\times m_2}\\ -K_{n_2\times n_1} & O_{n_2\times m_1} & \mathcal{L}(G_2)\bullet B(G_2) & -dR(G_2)\\ O_{m_2\times n_1} & O_{m_2\times m_1} & -dR(G_2)^T & I_{m_2}\end{pmatrix},$$

where $K_{n_1\times n_2}$ *is the matrix of size* $n_1\times n_2$ *with all entries equal to* $\frac{1}{\sqrt{(r_1+n_2)(2r_2+n_1)}}$, $B(G_2)$ *is the* $n_2\times n_2$ *matrix whose all diagonal entries are* 1 *and off-diagonal entries are* $\frac{r_2}{2r_2+n_1}$, c *is the constant whose value is* $\frac{1}{\sqrt{2(r_1+n_2)}}$, d *is the constant whose value is* $\frac{1}{\sqrt{2(2r_2+n_1)}}$;

(ii)

$$\mathcal{L}(S(G_1)\overline{\overline{\vee}}\mathcal{R}(G_2)) = \begin{pmatrix} I_{n_1} & -cR(G_1) & O_{n_1\times n_2} & O_{n_1\times m_2} \\ -cR(G_1)^T & I_{m_1} & O_{m_1\times n_2} & -K_{m_1\times m_2} \\ O_{n_2\times n_1} & O_{n_2\times m_1} & \mathcal{L}(G_2)\bullet B(G_2) & -dR(G_2) \\ O_{m_2\times n_1} & -K_{m_2\times m_1} & -dR(G_2)^T & I_{m_2} \end{pmatrix},$$

where $K_{m_1\times m_2}$ is the matrix of size $m_1\times m_2$ with all entries equal to $\frac{1}{\sqrt{(2+m_2)(2+m_1)}}$, $B(G_2)$ is the $n_2\times n_2$ matrix whose all diagonal entries are 1 and off-diagonal entries are $\frac{r_2}{2r_2}$, c is the constant whose value is $\frac{1}{\sqrt{r_1(2+m_2)}}$, d is the constant whose value is $\frac{1}{\sqrt{2r_2(2+m_1)}}$;

(iii)

$$\mathcal{L}(S(G_1)\bar{\dot{\vee}}\mathcal{R}(G_2)) = \begin{pmatrix} I_{n_1} & -cR(G_1) & O_{n_1\times n_2} & O_{n_1\times m_2} \\ -cR(G_1)^T & I_{m_1} & -K_{m_1\times n_2} & O_{m_1\times m_2} \\ O_{n_2\times n_1} & -K_{n_2\times m_1} & \mathcal{L}(G_2)\bullet B(G_2) & -dR(G_2) \\ O_{m_2\times n_1} & O_{m_2\times m_1} & -dR(G_2)^T & I_{m_2} \end{pmatrix},$$

where $K_{m_1\times n_2}$ is the matrix of size $m_1\times n_2$ with all entries equal to $\frac{1}{\sqrt{(2+n_2)(2r_2+m_1)}}$, $B(G_2)$ is the $n_2\times n_2$ matrix whose all diagonal entries are 1 and off-diagonal entries are $\frac{r_2}{2r_2+m_1}$, c is the constant whose value is $\frac{1}{\sqrt{r_1(2+n_2)}}$, d is the constant whose value is $\frac{1}{\sqrt{2(2r_2+m_1)}}$;

(iv)

$$\mathcal{L}(S(G_1)\dot{\vee}\mathcal{R}(G_2)) = \begin{pmatrix} I_{n_1} & -cR(G_1) & O_{n_1\times n_2} & -K_{n_1\times m_2} \\ -cR(G_1)^T & I_{m_1} & O_{m_1\times n_2} & O_{m_1\times m_2} \\ O_{n_2\times n_1} & O_{n_2\times m_1} & \mathcal{L}(G_2)\bullet B(G_2) & -dR(G_2) \\ -K_{m_2\times n_1} & O_{m_2\times m_1} & -dR(G_2)^T & I_{m_2} \end{pmatrix},$$

where $K_{m_1\times n_2}$ is the matrix of size $m_1\times n_2$ with all entries equal to $\frac{1}{\sqrt{(2+n_1)(r_1+m_2)}}$, $B(G_2)$ is the $n_2\times n_2$ matrix whose all diagonal entries are 1 and off-diagonal entries are $\frac{r_2}{2r_2}$, c is the constant whose value is $\frac{1}{\sqrt{2(r_1+m_2)}}$, d is the constant whose value is $\frac{1}{\sqrt{2r_2(2+n_1)}}$.

Theorem 2.9. *For $i=1,2$, let G_i be an r_i-regular graph with n_i vertices and m_i edges. Then the normalized Laplacian spectrum of $S(G_1)\ddot{\vee}\mathcal{R}(G_2)$ consists of:*

(i) roots of the equation $2(r_1+n_2)x^2-4(r_1+n_2)x+2n_2+r_1\delta_i(G_1)=0$ for every eigenvalue $\delta_i(G_1)$, $i=2,3,\dots,n_1$, of $\mathcal{L}(G_1)$;

(ii) roots of the equation $2(2r_2+n_1)x^2-2(3r_2+2n_1+r_2\delta_j(G_2))x+2n_1+3r_2\delta_j(G_2)=0$ for every eigenvalue $\delta_j(G_2)$, $j=2,3,\dots,n_2$, of $\mathcal{L}(G_2)$;

(iii) the eigenvalue 1 with multiplicity $m_1+m_2-n_1-n_2$;

(iv) four roots of the equation $(2r_1r_2+r_1n_1+2r_2n_2+n_1n_2)x^4-(5r_1r_2+3r_1n_1+5r_2n_2+3n_1n_2)x^3+(3r_1r_2+3r_1n_1+5r_2n_2+3n_1n_2)x^2-(r_1n_1+3r_2n_2+n_1n_2)x=0$.

Theorem 2.10. *For $i = 1, 2$, let G_i be an r_i-regular graph with n_i vertices and m_i edges. Then the normalized Laplacian spectrum of $S(G_1)\overline{\overline{\vee}}\mathcal{R}(G_2)$ consists of:*

(i) *roots of the equation $(2 + m_2)x^2 - 2(2 + m_2)x + m_2 + \delta_i(G_1) = 0$ for every eigenvalue $\delta_i(G_1)$, $i = 2, 3, \ldots, n_1$, of $\mathcal{L}(G_1)$;*

(ii) *roots of the equation $2(2 + m_1)x^2 - (6 + 3m_1 + 2\delta_j(G_2) + m_1\delta_j(G_2))x + m_1 + 3\delta_j(G_2) + m_1\delta_j(G_2) = 0$ for every eigenvalue $\delta_j(G_2)$, $j = 2, 3, \ldots, n_2$, of $\mathcal{L}(G_2)$;*

(iii) *the eigenvalue 1 with multiplicity $m_1 + m_2 - n_1 - n_2$;*

(iv) *four roots of the equation $2(4 + 2m_1 + 2m_2 + m_1m_2)x^4 - 7(4 + 2m_1 + 2m_2 + m_1m_2)x^3 + (24 + 14m_1 + 16m_2 + 7m_1m_2)x^2 - 2(2m_1 + 3m_2 + m_1m_2)x = 0$.*

Theorem 2.11. *For $i = 1, 2$, let G_i be an r_i-regular graph with n_i vertices and m_i edges. Then the normalized Laplacian spectrum of $S(G_1)\dot{\overline{\vee}}\mathcal{R}(G_2)$ consists of:*

(i) *roots of the equation $(2+n_2)x^2-2(2+n_2)x+n_2+\delta_i(G_1) = 0$ for every eigenvalue $\delta_i(G_1)$, $i = 2, 3, \ldots, n_1$, of $\mathcal{L}(G_1)$;*

(ii) *roots of the equation $2(2r_2+m_1)x^2-2(3r_2+2m_1+r_2\delta_j(G_2))x+2m_1+3r_2\delta_j(G_2) = 0$ for every eigenvalue $\delta_j(G_2)$, $j = 2, 3, \ldots, n_2$, of $\mathcal{L}(G_2)$;*

(iii) *the eigenvalue 1 with multiplicity $m_1 + m_2 - n_1 - n_2$;*

(iv) *four roots of the equation $(4r_2+2r_2n_2+2m_1+m_1n_2)x^4-(10r_2+5r_2n_2+6m_1+3m_1n_2)x^3 + (6r_2 + 5r_2n_2 + 6m_1 + 3m_1n_2)x^2 - (3r_2n_2 + 2m_1 + m_1n_2)x = 0$.*

Theorem 2.12. *For $i = 1, 2$, let G_i be an r_i-regular graph with n_i vertices and m_i edges. Then the normalized Laplacian spectrum of $S(G_1)\dot{\overline{\vee}}\mathcal{R}(G_2)$ consists of:*

(i) *roots of the equation $2(r_1 + m_2)x^2 - 4(r_1 + m_2)x + 2m_2 + r_2\delta_i(G_1) = 0$ for every eigenvalue $\delta_i(G_1)$, $i = 2, 3, \ldots, n_1$, of $\mathcal{L}(G_1)$;*

(ii) *roots of the equation $2(2 + n_1)x^2 - (6 + 3n_2 + 2\delta_j(G_2) + n_1\delta_j(G_2))x + n_1 + 3\delta_j(G_2) + n_1\delta_j(G_2) = 0$ for every eigenvalue $\delta_j(G_2)$, $j = 2, 3, \ldots, n_2$, of $\mathcal{L}(G_2)$;*

(iii) *the eigenvalue 1 with multiplicity $m_1 + m_2 - n_1 - n_2$;*

(iv) *four roots of the equation $2(2r_1 + r_1n_1 + 2m_2 + m_2n_1)x^4 - 7(2r_1 + r_1n_1 + 2m_2 + m_2n_1)x^3 + (12r_1 + 7r_1n_1 + 16m_2 + 7m_2n_1)x^2 - 2(r_1n_1 + 3m_2 + m_2n_1)x = 0$.*

3. Simultaneous Cospectral Graphs

In this section we present the main result of the paper. We construct several classes of non-regular graphs which are cospectral with respect to all the three matrices, namely, adjacency, Laplacian and normalized Laplacian. For the construction of these graphs we consider two pairs of A-cospectral regular graphs, which are readily available in the literature, for example see [14]. Then we take partial join of subdivision graph and R-graph belong to different pairs.

The following lemma is immediate from the definition of Laplacian and normalized Laplacian matrices.

Lemma 3.1. (i) *If G is an r-regular graph, then $L(G) = rI_n - A(G)$ and $\mathcal{L}(G) = I_n - \frac{1}{r}A(G)$.*

(ii) *If G_1 and G_2 are A-cospectral regular graphs, then they are also cospectral with respect to the Laplacian and normalized Laplacian matrices.*

Observation. From all the theorems given in the previous section we observe that the adjacency, Laplacian and normalized Lpalacian spectra of all the partial join graphs $S(G_1)\ddot{\nabla}\mathcal{R}(G_2)$, $S(G_1)\overline{\overline{\nabla}}\mathcal{R}(G_2)$, $S(G_1)\overline{\dot{\nabla}}\mathcal{R}(G_2)$, and $S(G_1)\dot{\nabla}\mathcal{R}(G_2)$, depend only on the number of vertices, number of edges, degree of regularities, and the corresponding spectrum of G_1 and G_2. Furthermore, we note that, although G_1 and G_2 are regular graphs, $S(G_1)\ddot{\nabla}\mathcal{R}(G_2)$, $S(G_1)\overline{\overline{\nabla}}\mathcal{R}(G_2)$, $S(G_1)\overline{\dot{\nabla}}\mathcal{R}(G_2)$ and $S(G_1)\dot{\nabla}\mathcal{R}(G_2)$ are non-regular graphs.

The following theorem is the main result of the paper.

Theorem 3.1. *Let G_i, H_i, $i = 1, 2$ be regular graphs, where G_1 need not be different from H_1. If G_1 and H_1 are A-cospectral, and G_2 and H_2 are A-cospectral then $S(G_1)\ddot{\nabla}\mathcal{R}(G_2)$ (respectively, $S(G_1)\overline{\overline{\nabla}}\mathcal{R}(G_2)$, $S(G_1)\overline{\dot{\nabla}}\mathcal{R}(G_2)$, $S(G_1)\dot{\nabla}\mathcal{R}(G_2)$) and $S(H_1)\ddot{\nabla}\mathcal{R}(H_2)$ (respectively, $S(H_1)\overline{\overline{\nabla}}\mathcal{R}(H_2)$, $S(H_1)\overline{\dot{\nabla}}\mathcal{R}(H_2)$, $S(H_1)\dot{\nabla}\mathcal{R}(H_2)$) are simultaneously A-cospectral, L-cospectral and $\mathcal{L}$-cospectral.*

Proof. Follows from Lemma 3.1 and the above observation. □

4. Spanning Trees and Kirchhoff Indices

Applying the results on Laplacian and normalized Laplacian spectra given in Section 2, we find the number of spanning trees and Kirchhoff index of all the partial join graphs constructed in the paper.

Let $t(G)$ denote the number of *spanning trees* of G. It is well known [5] that if G is a connected graph on n vertices with Laplacian spectrum $0 = \mu_1(G) \le \mu_2(G) \le \cdots \le \mu_n(G)$, then $t(G) = \frac{\mu_2(G)\cdots\mu_n(G)}{n}$.

The *Kirchhoff index* of a graph G, denoted by $Kf(G)$, is defined as the sum of resistances between all pairs of vertices [1, 10] in G. For a connected graph G on n vertices, the Kirchhoff index [9] can be expressed as $Kf(G) = n\sum_{i=2}^{n}\frac{1}{\mu_i(G)}$.

Theorem 4.1. *For $i = 1, 2$, let G_i be an r_i-regular graph with n_i vertices and m_i edges. Then*

(i) $$t(S(G_1)\ddot{\nabla}\mathcal{R}(G_2)) = \frac{2^{m_1+m_2-n_1-n_2}\cdot 2(2n_1+2n_2+r_1n_1+r_2n_2)\cdot\prod_{i=2}^{n_1}(2n_2+\mu_i(G_1))\cdot\prod_{j=2}^{n_2}(2n_1+3\mu_j(G_2))}{n_1+n_2+m_1+m_2};$$

(ii)
$$\begin{aligned} t(S(G_1)\overline{\overline{\nabla}}\mathcal{R}(G_2)) =&(2+m_2)^{m_1-n_1}\cdot(2+m_1)^{m_2-n_2} \\ &\times\frac{(2r_1m_2+2r_2m_1+r_1r_2m_1+r_1r_2m_2)\cdot\prod_{i=2}^{n_1}(r_1m_2+\mu_i(G_1))\cdot\prod_{j=2}^{n_2}(r_2m_1+3\mu_j(G_2)+m_1\mu_j(G_2))}{n_1+n_2+m_1+m_2};\end{aligned}$$

(iii)
$$\begin{aligned} t(S(G_1)\overline{\dot{\nabla}}\mathcal{R}(G_2)) =&(2+n_2)^{m_1-n_1}\cdot 2^{m_2-n_2} \\ &\times\frac{(4m_1+2r_1m_1+2r_1n_2+r_1r_2n_2)\cdot\prod_{i=2}^{n_1}(r_1n_2+\mu_i(G_1))\cdot\prod_{j=2}^{n_2}(2m_1+3\mu_j(G_2))}{n_1+n_2+m_1+m_2};\end{aligned}$$

(iv)

$$t(S(G_1)\dot{\nabla}\mathcal{R}(G_2)) = 2^{m_1-n_1}\cdot(2+n_1)^{m_2-n_2} \times \frac{(4m_2+2r_2m_2+2r_2n_1+r_1r_2n_1)\cdot\prod_{i=2}^{n_1}(2m_2+\mu_i(G_1))\cdot\prod_{j=2}^{n_2}(r_2n_1+3\mu_j(G_2)+n_1\mu_j(G_2))}{n_1+n_2+m_1+m_2}.$$

Theorem 4.2. *For $i=1,2$, let G_i be an r_i-regular graph with n_i vertices and m_i edges. Then*

(i)

$$Kf(S(G_1)\ddot{\nabla}\mathcal{R}(G_2)) = (n_1+n_2+m_1+m_2)\Bigg(\frac{m_1+m_2-n_1-n_2}{2} + \frac{4+4n_1+4n_2+2r_1+2r_2+r_1r_2+r_1n_1+r_2n_2}{2(2n_1+2n_2+r_1n_1+r_2n_2)} + \sum_{i=2}^{n_1}\frac{2+r_1+n_2}{2n_2+\mu_i(G_1)} + \sum_{j=2}^{n_2}\frac{2+r_2+n_1+\mu_j(G_2)}{2n_1+3\mu_j(G_2)}\Bigg);$$

(ii)

$$Kf(S(G_1)\overline{\overline{\nabla}}\mathcal{R}(G_2)) = (n_1+n_2+m_1+m_2)\times\Bigg(\frac{m_1-n_1}{2+m_2} + \frac{m_2-n_2}{2+m_1} + \frac{4+2r_1+2r_2+r_1r_2+r_1m_1+r_2m_2+2m_1+2m_2+r_1m_2+r_2m_1}{2r_1m_2+2r_2m_1+r_1r_2m_1+r_1r_2m_2} + \sum_{i=2}^{n_1}\frac{2+r_1+m_2}{r_1m_2+\mu_i(G_1)} + \sum_{j=2}^{n_2}\frac{2+r_2+m_1+\mu_j(G_2)}{r_2m_1+3\mu_j(G_2)+m_1\mu_j(G_2)}\Bigg);$$

(iii)

$$Kf(S(G_1)\overline{\nabla}\mathcal{R}(G_2)) = (n_1+n_2+m_1+m_2) \times\Bigg(\frac{m_1-n_1}{2+n_2} + \frac{m_2-n_2}{2} + \frac{4+2r_1+2r_2+4m_1+2n_2+r_1r_2+r_1m_1+r_1n_2+r_2n_2}{4m_1+2r_1m_1+2r_1n_2+r_1r_2n_2} + \sum_{i=2}^{n_1}\frac{2+r_1+n_2}{r_1n_2+\mu_i(G_1)} + \sum_{j=2}^{n_2}\frac{2+r_2+m_1+\mu_j(G_2)}{2m_1+3\mu_j(G_2)}\Bigg);$$

(iv)

$$Kf(S(G_1)\dot{\nabla}\mathcal{R}(G_2)) = (n_1+n_2+m_1+m_2) \times\Bigg(\frac{m_1-n_1}{2} + \frac{m_2-n_2}{2+n_1}$$

$$+\frac{4+2r_1+2r_2+4m_2+2n_1+r_1r_2+r_2m_2+r_1n_1+r_2n_1}{4m_2+2r_2m_2+2r_2n_1+r_1r_2n_1}$$
$$+\sum_{i=2}^{n_1}\frac{2+r_1+m_2}{2m_2+\mu_i(G_1)}+\sum_{j=2}^{n_2}\frac{2+r_2+n_1+\mu_j(G_2)}{r_2n_1+3\mu_j(G_2)+n_1\mu_j(G_2)}\Bigg).$$

5. Concluding remarks

The main result of the paper is based on regular A-cospectral graphs and certain operations on a pair of these graphs so that the operated (or resultant) graphs are non-regular and having adjacency, Laplacian and normalized Laplacian spectra which depend on only the order, size, degree of regularity and spectrum of the original graphs. Thus one may search for some other graph operations to construct simultaneous cospectral graphs like in the paper.

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