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CONTROLLED INTEGRAL FRAMES FOR HILBERT C^*-MODULES

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Abstract. The notion of controlled frames for Hilbert spaces were introduced by Balazs, Antoine and Grybos to improve the numerical efficiency of iterative algorithms for inverting the frame operator. Controlled frame theory has a great revolution in recent years. This theory have been extended from Hilbert spaces to Hilbert C^*-modules. In this paper we introduce and study the extension of this notion to integral frame for Hilbert C^*-modules. Also we give some characterizations between integral frame in Hilbert C^*-modules.

1. Introduction and preliminaries

The concept of frames in Hilbert spaces has been introduced by Duffin and Schaeffer [9] in 1952 to study some deep problems in nonharmonic Fourier series. After the fundamental paper [7] by Daubechies, Grossman and Meyer, frames theory began to be widely used, particularly in the more specialized context of wavelet frames and Gabor frames [12].

Hilbert C^*-module arose as generalization of the Hilbert space notion. The basic idea was to consider modules over C^*-algebras instead of linear spaces and to allow the inner product to take values in the C^*-algebras [17]. Continuous frames defined by Ali, Antoine and Gazeau [1]. Gabardo and Han in [11] called these kinds frames or frames associated with measurable spaces. For more details, the reader can refer to [4, 13–16, 20–32].

Key words and phrases. Integral frames, integral $*$-frame, controlled integral frames, controlled integral $*$-frame, C^*-algebra, Hilbert $\mathcal{A}$-modules.

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The goal of this article is the introduction and the study of the concept of controlled integral frames for Hilbert C^*-modules. Also we give some characterizations between integral frame in Hilbert C^*-modules.

In the following we briefly recall the definitions and basic properties of C^*-algebra and Hilbert $\mathcal{A}$-modules. Our references for C^*-algebras are [6, 8]. For a C^*-algebra $\mathcal{A}$, if $a \in \mathcal{A}$ is positive we write $a \geq 0$ and $\mathcal{A}^+$ denotes the set of positive elements of $\mathcal{A}$.

Definition 1.1 ([6])**.** Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{H}$ be a left $\mathcal{A}$-module, such that the linear structures of $\mathcal{A}$ and $\mathcal{H}$ are compatible. $\mathcal{H}$ is a pre-Hilbert $\mathcal{A}$-module if $\mathcal{H}$ is equipped with an $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle_{\mathcal{A}} : \mathcal{H} \times \mathcal{H} \to \mathcal{A}$, such that is sesquilinear, positive definite and respects the module action. In the other words

(i) $\langle x, x \rangle_{\mathcal{A}} \geq 0$ for all $x \in \mathcal{H}$, and $\langle x, x \rangle_{\mathcal{A}} = 0$ if and only if $x = 0$;
(ii) $\langle ax + y, z \rangle_{\mathcal{A}} = a\langle x, z \rangle_{\mathcal{A}} + \langle y, z \rangle_{\mathcal{A}}$ for all $a \in \mathcal{A}$ and $x, y, z \in \mathcal{H}$;
(iii) $\langle x, y \rangle_{\mathcal{A}} = \langle y, x \rangle_{\mathcal{A}}^*$ for all $x, y \in \mathcal{H}$.

For $x \in \mathcal{H}$, we define $||x|| = ||\langle x, x \rangle_{\mathcal{A}}||^{\frac{1}{2}}$. If $\mathcal{H}$ is complete with $|| \cdot ||$, it is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-modules over $\mathcal{A}$.

For every a in C^*-algebra $\mathcal{A}$, we have $|a| = (a^*a)^{\frac{1}{2}}$ and the $\mathcal{A}$-valued norm on $\mathcal{H}$ is defined by $|x| = \langle x, x \rangle_{\mathcal{A}}^{\frac{1}{2}}$ for all $x \in \mathcal{H}$.

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules, a map $T : \mathcal{H} \to \mathcal{K}$ is said to be adjointable if there exists a map $T^* : \mathcal{K} \to \mathcal{H}$ such that $\langle Tx, y \rangle_{\mathcal{A}} = \langle x, T^*y \rangle_{\mathcal{A}}$ for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$.

We reserve the notation $End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{K})$ for the set of all adjointable operators from $\mathcal{H}$ to $\mathcal{K}$ and $End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{H})$ is abbreviated to $End_{\mathcal{A}}^*(\mathcal{H})$.

The following lemmas will be used to prove our mains results.

Lemma 1.1 ([19])**.** *Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-modules. If $T \in End_{\mathcal{A}}^*(\mathcal{H})$, then*

$$\langle Tx, Tx \rangle_{\mathcal{A}} \leq \|T\|^2 \langle x, x \rangle_{\mathcal{A}}, \qquad x \in \mathcal{H}.$$

Lemma 1.2 ([3])**.** *Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules and $T \in End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{K})$. Then the following statements are equivalent:*

(i) *T is surjective;*
(ii) *T^* is bounded below associted to the norm, i.e., there is $m > 0$ such that $m\|x\| \leq \|T^*x\|$ for all $x \in \mathcal{K}$;*
(iii) *T^* is bounded below associted to the inner product, i.e., there is $m' > 0$ such that $m'\langle x, x \rangle_{\mathcal{A}} \leq \langle T^*x, T^*x \rangle_{\mathcal{A}}$ for all $x \in \mathcal{K}$.*

Lemma 1.3 ([2])**.** *Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules and $T \in End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{K})$.*

(i) *If T is injective and T has closed range, then the adjointable map T^*T is invertible and*
$$\|(T^*T)^{-1}\|^{-1} I_{\mathcal{H}} \leq T^*T \leq \|T\|^2 I_{\mathcal{H}}.$$
(ii) *If T is surjective, then the adjointable map TT^* is invertible and*
$$\|(TT^*)^{-1}\|^{-1} I_{\mathcal{K}} \leq TT^* \leq \|T\|^2 I_{\mathcal{K}}.$$

Lemma 1.4 ([33]). *Let (Ω, μ) be a measure spaces, X and Y are two Banach spaces, $\lambda : X \to Y$ be a bounded linear operator and $f : \Omega \to X$ is a measurable function, then*

$$\lambda\left(\int_\Omega f d\mu\right) = \int_\Omega (\lambda f) d\mu.$$

Theorem 1.1 ([5]). *Let X be a Banach spaces, $U : X \to X$ a bounded operator and $\|I - U\| < 1$. Then U is invertible.*

2. Controlled Integral Frames for Hilbert C^*-Modules

Let X be a Banach spaces, (Ω, μ) a measure space, and $f : \Omega \to X$ be a measurable function. Integral of Banach-valued function f has been defined by Bochner and others. Most properties of this integral are similar to those of the integral of real-valued functions (see [10, 33]). Since every C^*-algebra and Hilbert C^*-module are Banach spaces, we can use this integral and its properties.

Let (Ω, μ) be a measure space, $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert C^*-modules over a unital C^*-algebra and $\{\mathcal{H}_w\}_{w\in\Omega}$ is a family of submodules of $\mathcal{H}$. $End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_w)$ is the collection of all adjointable $\mathcal{A}$-linear maps from $\mathcal{H}$ into $\mathcal{H}_w$.

We define the following:

$$l^2(\Omega, \{\mathcal{H}_w\}_{\omega\in\Omega}) = \left\{x = \{x_w\}_{w\in\Omega} : x_w \in \mathcal{H}_w, \left\|\int_\Omega \langle x_w, x_w\rangle_{\mathcal{A}} d\mu(w)\right\| < \infty\right\}.$$

For any $x = \{x_w\}_{w\in\Omega}$ and $y = \{y_w\}_{w\in\Omega}$, the $\mathcal{A}$-valued inner product is defined by $\langle x, y\rangle_{\mathcal{A}} = \int_\Omega \langle x_w, y_w\rangle_{\mathcal{A}} d\mu(w)$ and the norm is defined by $\|x\| = \|\langle x, x\rangle_{\mathcal{A}}\|^{\frac{1}{2}}$.

In this case, the $l^2(\Omega, \{\mathcal{H}_w\}_{\omega\in\Omega})$ is a Hilbert C^*-modules (see [17]).

In what follows, let $GL^+(\mathcal{H})$ be the set of all positive bounded linear invertible operators on $\mathcal{H}$ with bounded inverse and let F be a function from Ω to $\mathcal{H}$.

The following definitions was introduced by Mohamed Rossafi, Frej Chouchene and Samir Kabbaj in the paper entitled *Integral frame in Hilbert C^*-module* (see arXiv preprint- arXiv:2005.09995v2 [math.FA] 30 Nov 2020).

Definition 2.1. Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-modules and (Ω, μ) be a measure space. A mapping $F : \Omega \to \mathcal{H}$ is called an integral frame associted to (Ω, μ) if

- for all $x \in \mathcal{H}$, $w \to \langle x, F_\omega\rangle_{\mathcal{A}}$ is a measurable function on Ω;
- there exists a pair of constants $0 < A, B$ such that

$$A\langle x, x\rangle_{\mathcal{A}} \leq \int_\Omega \langle x, F_\omega\rangle_{\mathcal{A}} \langle F_\omega, x\rangle_{\mathcal{A}} d\mu(w) \leq B\langle x, x\rangle_{\mathcal{A}}, \quad x \in \mathcal{H}.$$

Definition 2.2. Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module and (Ω, μ) be a measure space. A mapping $F : \Omega \to \mathcal{H}$ is called a $*$-integral frame associted to (Ω, μ) if

- for all $x \in \mathcal{H}$, $w \to \langle x, F_\omega\rangle_{\mathcal{A}}$ is a measurable function on Ω;
- there exist two non-zero elements A, B in $\mathcal{A}$ such that

$$A\langle x, x\rangle_{\mathcal{A}} A^* \leq \int_\Omega \langle x, F_\omega\rangle_{\mathcal{A}} \langle F_\omega, x\rangle_{\mathcal{A}} d\mu(w) \leq B\langle x, x\rangle_{\mathcal{A}} B^*, \quad x \in \mathcal{H}.$$

3. Main Results

Definition 3.1. Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-modules and (Ω, μ) be a measure space. A C-controlled integral frame in C^*-module $\mathcal{H}$ is a map $F : \Omega \to \mathcal{H}$ such that there exist $0 < A \leq B < \infty$ such that

$$A\langle x, x\rangle_{\mathcal{A}} \leq \int_{\Omega} \langle x, F_\omega\rangle_{\mathcal{A}} \langle CF_\omega, x\rangle_{\mathcal{A}} d\mu(w) \leq B\langle x, x\rangle_{\mathcal{A}}, \quad x \in \mathcal{H}. \tag{3.1}$$

The elements A and B are called the C-controlled integral frame bounds. If $A = B$, we call this a C-controlled integral tight frame. If $A = B = 1$, it's called a C-controlled integral parseval frame. If only the right hand inequality of (3.1) is satisfied, we call F a C-controlled integral Bessel mapping with bound B.

Example 3.1. Let $\mathcal{H} = \left\{ X = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \end{pmatrix} \mid a, b \in \mathbb{C} \right\}$ and $\mathcal{A} = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mid x, y \in \mathbb{C} \right\}$ which is a C^*-algebra. We define the inner product:

$$\begin{aligned} \mathcal{H} \times \mathcal{H} &\to \mathcal{A}, \\ (A, B) &\mapsto A(\overline{B})^t. \end{aligned}$$

This inner product makes $\mathcal{H}$ a C^*-module over $\mathcal{A}$. Let C be an operator defined by

$$C : \mathcal{H} \to \mathcal{H}, \qquad X \to \alpha X,$$

where α is a reel number strictly greater than zero. It's clair that $C \in Gl^+(\mathcal{H})$. Let $\Omega = [0, 1]$ endowed with the Lebesgue's measure. It's clear that it is a measure space.

We consider

$$F : [0, 1] \to \mathcal{H}, \qquad w \to F_w = \begin{pmatrix} w & 0 & 0 \\ 0 & 0 & \frac{w}{2} \end{pmatrix}.$$

In addition, for $X \in \mathcal{H}$, we have

$$\int_{\Omega} \langle X, F_w\rangle_{\mathcal{A}} \langle CF_w, X\rangle_{\mathcal{A}} d\mu(\omega) = \int_{\Omega} \alpha w^2 \begin{pmatrix} |a|^2 & 0 \\ 0 & \frac{|b|^2}{4} \end{pmatrix} d\mu(\omega) = \frac{\alpha}{3} \begin{pmatrix} |a|^2 & 0 \\ 0 & \frac{|b|^2}{4} \end{pmatrix}.$$

It's clear that

$$\frac{1}{4}\|X\|^2_{\mathcal{A}} \leq \begin{pmatrix} |a|^2 & 0 \\ 0 & \frac{|b|^2}{4} \end{pmatrix} \leq \begin{pmatrix} |a|^2 & 0 \\ 0 & |b|^2 \end{pmatrix} = \|X\|^2_{\mathcal{A}}.$$

Then we have

$$\frac{\alpha}{12}\|X\|^2_{\mathcal{A}} \leq \int_{\Omega} \langle X, F_w\rangle_{\mathcal{A}} \langle CF_w, X\rangle_{\mathcal{A}} d\mu(\omega) \leq \frac{\alpha}{3}\|X\|^2_{\mathcal{A}},$$

which show that F is a C-controlled integral frame for the C^*-module $\mathcal{H}$.

Definition 3.2. Let F be a C-controlled integral frame for $\mathcal{H}$ associted to (Ω, μ). We define the frame operator $S_C : \mathcal{H} \to \mathcal{H}$ for F by

$$S_C x = \int_\Omega \langle x, F_\omega\rangle_\mathcal{A} C F_\omega d\mu(\omega), \quad x \in \mathcal{H}.$$

Proposition 3.1. *The frame operator S_C is positive, selfadjoint, bounded and invertible.*

Proof. For all $x \in \mathcal{H}$, by Lemma 1.4, we have

$$\langle S_C x, x\rangle_\mathcal{A} = \left\langle \int_\Omega \langle x, F_\omega\rangle_\mathcal{A} C F_\omega d\mu(\omega), x \right\rangle_\mathcal{A} = \int_\Omega \langle x, F_\omega\rangle_\mathcal{A} \langle C F_\omega, x\rangle_\mathcal{A} d\mu(\omega).$$

By left hand of inequality (3.1), we have

$$0 \leq A\langle x, x\rangle_\mathcal{A} \leq \langle S_C x, x\rangle_\mathcal{A}.$$

Then S_C is a positive operator, also, it's sefladjoint. From (3.1), we have

$$A\langle x, x\rangle_\mathcal{A} \leq \langle S_C x, x\rangle_\mathcal{A} \leq B\langle x, x\rangle_\mathcal{A}, \quad x \in \mathcal{H}.$$

So,

$$A.I \leq S_C \leq B.I$$

Then S_C is a bounded operator. Moreover,

$$0 \leq I - B^{-1} S_C \leq \frac{B - A}{B}.I,$$

Consequently,

$$\|I - B^{-1} S_C\| = \sup_{x \in \mathcal{H}, \|x\| = 1} \|\langle (I - B^{-1} S_C)x, x\rangle_\mathcal{A}\| \leq \frac{B - A}{B} < 1.$$

The Theorem 1.1 shows that S_C is invertible. □

Corollary 3.1. *Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module and (Ω, μ) be a measure space. Let $F : \Omega \to \mathcal{H}$ be a mapping. Assume that S is the frame operator for F. Then the following statements are equivalent:*

(1) *F is an integral frame associted to (Ω, μ) with integral frame bounds A and B;*
(2) *we have $A.I \leq S \leq B.I$*

Proof. (1) $\Rightarrow$ (2) Let F be an integral frame associted to (Ω, μ) with integral frames bounds A and B, then

$$A\langle x, x\rangle_\mathcal{A} \leq \int_\Omega \langle x, F_\omega\rangle_\mathcal{A} \langle F_\omega, x\rangle_\mathcal{A} d\mu(w) \leq B\langle x, x\rangle_\mathcal{A}, \quad x \in \mathcal{H}.$$

Since

$$Sx = \int_\Omega \langle x, F_\omega\rangle_\mathcal{A} F_\omega d\mu(\omega),$$

we have

$$\langle Sx, x\rangle_\mathcal{A} = \left\langle \int_\Omega \langle x, F_\omega\rangle_\mathcal{A} F_\omega d\mu(\omega), x \right\rangle_\mathcal{A} = \int_\Omega \langle x, F_\omega\rangle_\mathcal{A} \langle F_\omega, x\rangle_\mathcal{A} d\mu(\omega),$$

then

$$\langle Ax, x\rangle_{\mathcal{A}} \leq \langle Sx, x\rangle_{\mathcal{A}} \leq \langle Bx, x\rangle_{\mathcal{A}}, \quad x \in \mathcal{H}.$$

So,

$$A.I \leq S \leq B.I.$$

(2) $\Rightarrow$ (1) Let $x \in \mathcal{H}$, then

$$\left\| \int_{\Omega} \langle x, F_{\omega}\rangle_{\mathcal{A}} \langle F_{\omega}, x\rangle_{\mathcal{A}} d\mu(w) \right\| = \|\langle Sx, x\rangle_{\mathcal{A}}\| \leq \|Sx\|\|x\| \leq B\|x\|^2. \tag{3.2}$$

Also,

$$\|\langle Sx, x\rangle_{\mathcal{A}}\| \geq \|\langle Ax, x\rangle_{\mathcal{A}}\| = A\|x\|^2. \tag{3.3}$$

By (3.2) and (3.3) we obtain

$$A\|x\|^2 \leq \left\| \int_{\Omega} \langle x, F_{\omega}\rangle_{\mathcal{A}} \langle F_{\omega}, x\rangle_{\mathcal{A}} d\mu(w) \right\| \leq B\|x\|^2,$$

which ends the proof. □

Theorem 3.1. *Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module, $C \in GL^+(\mathcal{H})$ and (Ω, μ) be a measure space and F be a mapping for Ω to $\mathcal{H}$. Then F is a C-controlled integral frame for $\mathcal{H}$ associted to (Ω, μ) if and only if there exist $0 < A \leq B < \infty$ such that*

$$A\|x\|^2 \leq \left\| \int_{\Omega} \langle x, F_{\omega}\rangle_{\mathcal{A}} \langle CF_{\omega}, x\rangle_{\mathcal{A}} d\mu(w) \right\| \leq B\|x\|^2 \quad x \in \mathcal{H}.$$

Proof. ($\Rightarrow$) obvious.

($\Leftarrow$) Supposes there exists $0 < A \leq B < \infty$, such that (3.1) holds. On one hand, for all $x \in \mathcal{H}$ we have

$$A\|x\|^2 \leq \left\| \int_{\Omega} \langle x, F_{\omega}\rangle_{\mathcal{A}} \langle CF_{\omega}, x\rangle_{\mathcal{A}} d\mu(\omega) \right\| = \|\langle S_C x, x\rangle_{\mathcal{A}}\| = \|\langle S_C^{\frac{1}{2}} x, S_C^{\frac{1}{2}} x\rangle_{\mathcal{A}}\| = \|S_C^{\frac{1}{2}} x\|^2.$$

By Lemma 1.2, there exists $0 < m$ such that

$$m\langle x, x\rangle_{\mathcal{A}} \leq \langle S_C^{\frac{1}{2}} x, S_C^{\frac{1}{2}} x\rangle_{\mathcal{A}} = \langle S_C x, x\rangle_{\mathcal{A}}. \tag{3.4}$$

On other hand, for all $x \in \mathcal{H}$ we have

$$B\|x\|^2 \geq \left\| \int_{\Omega} \langle x, F_{\omega}\rangle_{\mathcal{A}} \langle CF_{\omega}, x\rangle_{\mathcal{A}} d\mu(w) \right\|^2 = \|\langle S_C x, x\rangle_{\mathcal{A}}\| = \|\langle S_C^{\frac{1}{2}} x, S_C^{\frac{1}{2}} x\rangle_{\mathcal{A}}\| = \|S_C^{\frac{1}{2}} x\|^2.$$

By Lemma 1.2, there exist $0 < m'$ such that

$$\langle S_C^{\frac{1}{2}} x, S_C^{\frac{1}{2}} x\rangle_{\mathcal{A}} = \langle S_C x, x\rangle_{\mathcal{A}} \leq m'\langle x, x\rangle_{\mathcal{A}}. \tag{3.5}$$

From (3.4) and (3.5), we conclude that F is a C-controlled integral frame. □

Remark 3.1. If F is a mapping from Ω to $\mathcal{H}$, then F is an integral frame associted to (Ω, μ) if and only if there exist $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \| \int_{\Omega} \langle x, F_{\omega}\rangle_{\mathcal{A}} \langle F_{\omega}, x\rangle_{\mathcal{A}} d\mu(w)\| \leq B\|x\|^2, \quad x \in \mathcal{H}.$$

Corollary 3.2. *Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module and (Ω, μ) be a measure space. Let $F : \Omega \to \mathcal{H}$ be a mapping and $C \in GL^+(\mathcal{H})$. Then the following statements are equivalent.*

(1) *F is a C-controlled integral frame associted to (Ω, μ).*
(2) *We have $A.I \leq S_C \leq B.I$, where S_C is the frame operator for F, for A and B given.*

Proof. (1) $\Rightarrow$ (2) Let F be a C-controlled integral frame associted to (Ω, μ) with C-controlled integral frames bounds A and B, then

$$A\langle x, x\rangle_{\mathcal{A}} \leq \int_\Omega \langle x, F_\omega\rangle_{\mathcal{A}} \langle CF_\omega, x\rangle_{\mathcal{A}} d\mu(w) \leq B\langle x, x\rangle_{\mathcal{A}}, \quad x \in \mathcal{H}.$$

Since,

$$S_C x = \int_\Omega \langle x, F_\omega\rangle_{\mathcal{A}} CF_\omega d\mu(\omega).$$

We have

$$\langle S_C x, x\rangle_{\mathcal{A}} = \left\langle \int_\Omega \langle x, F_\omega\rangle_{\mathcal{A}} CF_\omega d\mu(\omega), x \right\rangle_{\mathcal{A}} = \int_\Omega \langle x, F_\omega\rangle_{\mathcal{A}} \langle CF_\omega, x\rangle_{\mathcal{A}} d\mu(\omega),$$

then

$$\langle Ax, x\rangle_{\mathcal{A}} \leq \langle Sx, x\rangle_{\mathcal{A}} \leq \langle Bx, x\rangle_{\mathcal{A}}, \quad x \in \mathcal{H}.$$

So,

$$A.I \leq S \leq B.I.$$

(2) $\Rightarrow$ (1) Let $x \in \mathcal{H}$, then

$$\left\| \int_\Omega \langle x, F_\omega\rangle_{\mathcal{A}} \langle CF_\omega, x\rangle_{\mathcal{A}} d\mu(w) \right\| = \|\langle S_C x, x\rangle_{\mathcal{A}}\| \leq \|S_C x\| \|x\| \leq B\|x\|^2. \tag{3.6}$$

Also,

$$\|\langle S_C x, x\rangle_{\mathcal{A}}\| \geq \|\langle Ax, x\rangle_{\mathcal{A}}\| = A\|x\|^2. \tag{3.7}$$

By (3.6) and (3.7) we obtain

$$A\|x\|^2 \leq \left\| \int_\Omega \langle x, F_\omega\rangle_{\mathcal{A}} \langle CF_\omega, x\rangle_{\mathcal{A}} d\mu(w) \right\| \leq B\|x\|^2,$$

which ends the proof. □

Proposition 3.2. *Let $C \in GL^+(\mathcal{H})$ and F be a C-controlled integral frame for $\mathcal{H}$ associted to (Ω, μ) with bounds A and B. Then F is an integral frame for $\mathcal{H}$ associted to (Ω, μ) with bounds $A\|C^{\frac{1}{2}}\|^{-2}$ and $B\|C^{\frac{-1}{2}}\|^2$.*

Proof. Let F be a C-controlled integral frame for $\mathcal{H}$ associted to (Ω, μ) with bounds A and B.

On one hand we have

$$A\langle x, x\rangle_{\mathcal{A}} \leq \langle S_C x, x\rangle_{\mathcal{A}} = \langle CSx, x\rangle_{\mathcal{A}} = \langle C^{\frac{1}{2}} Sx, C^{\frac{1}{2}} x\rangle_{\mathcal{A}} \leq \|C^{\frac{1}{2}}\|^2 \langle Sx, x\rangle_{\mathcal{A}}.$$

So,

$$A\|C^{\frac{1}{2}}\|^{-2}\langle x,x\rangle_{\mathcal{A}} \leq \int_\Omega \langle x,F_\omega\rangle_{\mathcal{A}}\langle F_\omega,x\rangle_{\mathcal{A}} d\mu(w). \tag{3.8}$$

On other hand, for all $x\in\mathcal{H}$, we have

$$\begin{aligned}
\int_\Omega \langle x,F_\omega\rangle_{\mathcal{A}}\langle F_\omega,x\rangle_{\mathcal{A}} d\mu(w) &= \langle Sx,x\rangle_{\mathcal{A}}\\
&= \langle C^{-1}CSx,x\rangle_{\mathcal{A}}\\
&= \langle (C^{-1}CS)^{\frac{1}{2}}x,(C^{-1}CS)^{\frac{1}{2}}x\rangle_{\mathcal{A}}\\
&\leq \|C^{\frac{-1}{2}}\|^2\langle (CS)^{\frac{1}{2}}x,(CS)^{\frac{1}{2}}x\rangle_{\mathcal{A}}\\
&= \|C^{\frac{-1}{2}}\|^2\langle (S_C)^{\frac{1}{2}}x,(S_C)^{\frac{1}{2}}x\rangle_{\mathcal{A}}\\
&= \|C^{\frac{-1}{2}}\|^2\langle S_Cx,x\rangle_{\mathcal{A}}\\
&\leq \|C^{\frac{-1}{2}}\|^2B\langle x,x\rangle_{\mathcal{A}}.
\end{aligned}$$

Then

$$\int_\Omega \langle x,F_\omega\rangle_{\mathcal{A}}\langle F_\omega,x\rangle_{\mathcal{A}} d\mu(w) \leq \|C^{\frac{-1}{2}}\|^2B\langle x,x\rangle_{\mathcal{A}}. \tag{3.9}$$

From (3.8) and (3.9) we conclude that F is an integral frame $\mathcal{H}$ associted to (Ω,μ) with bounds $A\|C^{\frac{1}{2}}\|^{-2}$ and $B\|C^{\frac{-1}{2}}\|^2$. ☐

Proposition 3.3. *Let $C\in GL^+(\mathcal{H})$ and F be an integral frame for $\mathcal{H}$ associted to (Ω,μ) with bounds A and B. Then F is a C-controlled integral frame for $\mathcal{H}$ associted to (Ω,μ) with bounds $A\|C^{\frac{-1}{2}}\|^2$ and $B\|C^{\frac{1}{2}}\|^2$.*

Proof. Let F be an integral frame for $\mathcal{H}$ associted to (Ω,μ) with bounds A and B. Then for all $x\in\mathcal{H}$, we have

$$\begin{aligned}
A\langle x,x\rangle_{\mathcal{A}} &\leq \langle Sx,x\rangle_{\mathcal{A}}\\
&= \langle C^{-1}CSx,x\rangle_{\mathcal{A}}\\
&= \langle (C^{-1}CS)^{\frac{1}{2}}x,(C^{-1}CS)^{\frac{1}{2}}x\rangle_{\mathcal{A}}\\
&\leq \|C^{\frac{-1}{2}}\|^2\langle (CS)^{\frac{1}{2}}x,(CS)^{\frac{1}{2}}x\rangle_{\mathcal{A}}\\
&= \|C^{\frac{-1}{2}}\|^2\langle (S_C)^{\frac{1}{2}}x,(S_C)^{\frac{1}{2}}x\rangle_{\mathcal{A}}\\
&= \|C^{\frac{-1}{2}}\|^2\langle S_Cx,x\rangle_{\mathcal{A}}.
\end{aligned}$$

So,

$$A\|C^{\frac{-1}{2}}\|^{-2}\langle x,x\rangle_{\mathcal{A}} \leq \langle S_Cx,x\rangle_{\mathcal{A}}.$$

Hence, for all $x\in\mathcal{H}$, we have

$$\langle S_Cx,x\rangle_{\mathcal{A}} = \langle CSx,x\rangle_{\mathcal{A}} = \langle C^{\frac{1}{2}}Sx,C^{\frac{1}{2}}x\rangle_{\mathcal{A}} \leq \|C^{\frac{1}{2}}\|^2\langle Sx,x\rangle_{\mathcal{A}} \leq \|C^{\frac{1}{2}}\|^2B\langle x,x\rangle_{\mathcal{A}}.$$

Therefore we conclude that F is a C-controlled integral frame for $\mathcal{H}$ associted to (Ω,μ) with bounds $A\|C^{\frac{-1}{2}}\|^{-2}$ and $B\|C^{\frac{1}{2}}\|^2$. ☐

Theorem 3.2. *Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module and (Ω, μ) be a measure space. Let F be a C-controlled integral frame for $\mathcal{H}$ associted to (Ω, μ) with the frame operator S_C and bounds A and B. Let $K \in End^*_{\mathcal{A}}(\mathcal{H})$ be a surjective operator such that $KC = CK$. Then KF is a C-controlled integral frame for $\mathcal{H}$ with the operator frame KS_CK^*.*

Proof. Let F be a C-controlled integral frame for $\mathcal{H}$ associted to (Ω, μ), then

$$A\langle K^*x, K^*x\rangle_{\mathcal{A}} \leq \int_\Omega \langle K^*x, F_\omega\rangle_{\mathcal{A}}\langle CF_\omega, K^*x\rangle_{\mathcal{A}} d\mu(w) \leq B\langle K^*x, K^*x\rangle_{\mathcal{A}}, \quad x \in \mathcal{H}.$$

By Lemma 1.1 and Lemma 1.3, we obtain

$$A\|(KK^*)^{-1}\|^{-1}\langle x, x\rangle_{\mathcal{A}} \leq \int_\Omega \langle x, KF_\omega\rangle_{\mathcal{A}}\langle CKF_\omega, x\rangle_{\mathcal{A}} d\mu(w) \leq B\|K^*\|^2\langle x, x\rangle_{\mathcal{A}}, \quad x \in \mathcal{H},$$

which shows that KF is a C-controlled integral frame.

Moreover, by Lemma 1.4, we have

$$KS_CK^*x = K\int_\Omega \langle K^*x, F_\omega\rangle_{\mathcal{A}} CF_\omega d\mu(\omega) = \int_\Omega \langle x, KF_\omega\rangle_{\mathcal{A}} CKF_\omega d\mu(\omega),$$

which ends the proof. □

4. Controlled $*$-Integral Frames

Definition 4.1. Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module and (Ω, μ) be a measure space. A C-controlled $*$-integral frame in $\mathcal{A}$-module $\mathcal{H}$ is a map $F : \Omega \to \mathcal{H}$ such that there exist two strictly nonzero elements A,B in $\mathcal{A}$ such that

$$A\langle x, x\rangle_{\mathcal{A}} A^* \leq \int_\Omega \langle x, F_\omega\rangle_{\mathcal{A}}\langle CF_\omega, x\rangle_{\mathcal{A}} d\mu(w) \leq B\langle x, x\rangle_{\mathcal{A}} B^*, \quad x \in \mathcal{H}. \tag{4.1}$$

The elements A and B are called the C-controlled $*$-integral frame bounds. If $A = B$, we call this a C-controlled $*$-integral tight frame. If $A = B = 1$, it's called a C-controlled $*$-integral parseval frame. If only the right hand inequality of (4.1) is satisfied, we call F a C-controlled $*$-integral Bessel mapping with bound B.

Example 4.1. Let $\mathcal{H} = \mathcal{A} = \{(a_n)_{n\in\mathbb{N}} \subset \mathbb{C} \mid \sum_{n\geq 0} |a_n| < \infty\}$. Endowed with the product and the inner product defined as follow.

$$\begin{array}{lcl} \mathcal{A} \times \mathcal{A} & \to & \mathcal{A}, \\ ((a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}) & \mapsto & (a_n)_{n\in\mathbb{N}}.(b_n)_{n\in\mathbb{N}} = (a_nb_n)_{n\in\mathbb{N}}, \end{array}$$

and

$$\begin{array}{lcl} \mathcal{H} \times \mathcal{H} & \to & \mathcal{A}, \\ ((a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}) & \mapsto & \langle (a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}\rangle_{\mathcal{A}} = (a_n\overline{b_n})_{n\in\mathbb{N}}. \end{array}$$

Let $\Omega = [0, +\infty[$ endowed with the Lebesgue's measure which is a measure space

$$F : [0, +\infty[\to \mathcal{H}, \qquad w \mapsto F_w = (F_n^w)_{n\in\mathbb{N}},$$

where

$$F_n^w = \frac{1}{n+1}, \quad \text{if} \quad n = [w], \quad \text{and} \quad F_n^w = 0, \quad \text{elsewhere},$$

where $[w]$ is the whole part of w.

On the other hand, we consider the measure space (Ω,μ), where μ is the Lebesgue measure restricted to $[0,+\infty[$, and the operator

$$C:\mathcal{H}\to\mathcal{H},$$
$$(a_n)_{n\in\mathbb{N}}\to(\alpha a_n)_{n\in\mathbb{N}},$$

where α is a strictly positive real number.

It's clear that C is an invertible and both operators and C and C^{-1} are bounded. So,

$$\begin{aligned}
&\int_\Omega\langle(a_n)_{n\in\mathbb{N}},F_w\rangle_{\mathcal{A}}\langle CF_w,(a_n)_{n\in\mathbb{N}}\rangle_{\mathcal{A}}d\mu(w)\\
=&\int_0^{+\infty}\left(0,0,\dots,\frac{a_{[w]}}{[w]+1},0,\dots\right)\alpha\left(0,0,\dots,\overline{\frac{a_{[w]}}{[w]+1}},0,\dots\right)d\mu(w)\\
=&\alpha\sum_{p=0}^{+\infty}\int_p^{p+1}\left(0,0,\dots,\frac{|a_{[w]}|^2}{([w]+1)^2},0,\dots\right)d\mu(w)\\
=&\alpha\sum_{p=0}^{+\infty}\left(0,0,\dots,\frac{|a_{[p]}|^2}{(p+1)^2},0,\dots\right)\\
=&\alpha\left(\frac{|a_n|^2}{(n+1)^2}\right)_{n\in\mathbb{N}}\\
=&\sqrt{\alpha}\left(1,\frac{1}{2},\frac{1}{3},\dots,\frac{1}{n},\dots\right)\langle(a_n)_{n\in\mathbb{N}},(a_n)_{n\in\mathbb{N}}\rangle_{\mathcal{A}}\sqrt{\alpha}\left(1,\frac{1}{2},\frac{1}{3},\dots,\frac{1}{n},\dots\right),
\end{aligned}$$

which shows that F is a C-controlled $*$-integral tight frame for $\mathcal{H}$ with bound $A=\sqrt{\alpha}\left(1,\frac{1}{2},\frac{1}{3},\dots,\frac{1}{n},\dots\right)\in\mathcal{A}$.

Definition 4.2. Let F be a C-controlled $*$-integral frame for $\mathcal{H}$ associted to (Ω,μ). We define the frame operator $S_C:\mathcal{H}\to\mathcal{H}$ for F by

$$S_Cx=\int_\Omega\langle x,F_\omega\rangle_{\mathcal{A}}CF_\omega d\mu(\omega),\quad x\in\mathcal{H}.$$

Proposition 4.1. *The frame operator S_C is positive, selfadjoint, bounded and invertible.*

Proof. For all $x\in\mathcal{H}$, by Lemma 1.4, we have

$$\langle S_Cx,x\rangle_{\mathcal{A}}=\left\langle\int_\Omega\langle x,F_\omega\rangle_{\mathcal{A}}CF_\omega d\mu(\omega),x\right\rangle_{\mathcal{A}}=\int_\Omega\langle x,F_\omega\rangle_{\mathcal{A}}\langle CF_\omega,x\rangle_{\mathcal{A}}d\mu(\omega).$$

By left hand of inequality (4.1), we deduce that S_C is a positive operator, also, it's sefladjoint. From (4.1), we have

$$A\langle x,x\rangle_{\mathcal{A}}A^*\le\langle S_Cx,x\rangle_{\mathcal{A}}\le B\langle x,x\rangle_{\mathcal{A}}B^*,\quad x\in\mathcal{H}.$$

The Theorem 2.5 in [18] shows that S_C is invertible. □

Proposition 4.2. *Let $C \in GL^+(\mathcal{H})$ and F be a C-controlled $*$-integral frame for $\mathcal{H}$ associted to (Ω, μ) with bounds A and B. Then F is a $*$-integral frame $\mathcal{H}$ associted to (Ω, μ) with bounds $\|C^{\frac{1}{2}}\|^{-1}A$ and $\|C^{\frac{-1}{2}}\|B$.*

Proof. Let F be a C-controlled $*$-integral frame for $\mathcal{H}$ associted to (Ω, μ), with bounds A and B.

On one hand we have

$$A\langle x, x\rangle_{\mathcal{A}} A^* \leq \langle S_C x, x\rangle_{\mathcal{A}} = \langle CSx, x\rangle_{\mathcal{A}} = \langle C^{\frac{1}{2}} Sx, C^{\frac{1}{2}} x\rangle_{\mathcal{A}} \leq \|C^{\frac{1}{2}}\|^2 \langle Sx, x\rangle_{\mathcal{A}}.$$

So,

$$(\|C^{\frac{1}{2}}\|^{-1}A)\langle x, x\rangle_{\mathcal{A}}(\|C^{\frac{1}{2}}\|^{-1}A)^* \leq \int_\Omega \langle x, F_\omega\rangle_{\mathcal{A}} \langle F_\omega, x\rangle_{\mathcal{A}} d\mu(w). \tag{4.2}$$

On other hand, for all $x \in \mathcal{H}$, we have

$$\begin{aligned}
\int_\Omega \langle x, F_\omega\rangle_{\mathcal{A}} \langle F_\omega, x\rangle_{\mathcal{A}} d\mu(w) &= \langle Sx, x\rangle_{\mathcal{A}} \\
&= \langle C^{-1}CSx, x\rangle_{\mathcal{A}} \\
&= \langle (C^{-1}CS)^{\frac{1}{2}} x, (C^{-1}CS)^{\frac{1}{2}} x\rangle_{\mathcal{A}} \\
&\leq \|C^{\frac{-1}{2}}\|^2 \langle (CS)^{\frac{1}{2}} x, (CS)^{\frac{1}{2}} x\rangle_{\mathcal{A}} \\
&= \|C^{\frac{-1}{2}}\|^2 \langle (S_C)^{\frac{1}{2}} x, (S_C)^{\frac{1}{2}} x\rangle_{\mathcal{A}} \\
&= \|C^{\frac{-1}{2}}\|^2 \langle S_C x, x\rangle_{\mathcal{A}} \\
&\leq \|C^{\frac{-1}{2}}\|^2 B\langle x, x\rangle_{\mathcal{A}} B^*.
\end{aligned}$$

Then

$$\int_\Omega \langle x, F_\omega\rangle_{\mathcal{A}} \langle F_\omega, x\rangle_{\mathcal{A}} d\mu(w) \leq (\|C^{\frac{-1}{2}}\|B)\langle x, x\rangle_{\mathcal{A}}(\|C^{\frac{-1}{2}}\|B)^*. \tag{4.3}$$

From (4.2) and (4.3) we conclude that F is a $*$-integral frame $\mathcal{H}$ associted to (Ω, μ) with bounds $A\|C^{\frac{1}{2}}\|^{-2}$ and $B\|C^{\frac{-1}{2}}\|^2$. □

Proposition 4.3. *Let $C \in GL^+(\mathcal{H})$ and F be an $*$-integral frame for $\mathcal{H}$ associted to (Ω, μ) with bounds A and B. Then F is a C-controlled $*$-integral frame for $\mathcal{H}$ associted to (Ω, μ) with bounds $\|C^{\frac{-1}{2}}\|^{-1}A$ and $\|C^{\frac{1}{2}}\|B$.*

Proof. Let F be an integral frame for $\mathcal{H}$ associted to (Ω, μ) with bounds A and B. Then for all $x \in \mathcal{H}$, we have

$$\begin{aligned}
A\langle x, x\rangle_{\mathcal{A}} A^* &\leq \langle Sx, x\rangle_{\mathcal{A}} \\
&= \langle C^{-1}CSx, x\rangle_{\mathcal{A}} \\
&= \langle (C^{-1}CS)^{\frac{1}{2}} x, (C^{-1}CS)^{\frac{1}{2}} x\rangle_{\mathcal{A}} \\
&\leq \|C^{\frac{-1}{2}}\|^2 \langle (CS)^{\frac{1}{2}} x, (CS)^{\frac{1}{2}} x\rangle_{\mathcal{A}} \\
&= \|C^{\frac{-1}{2}}\|^2 \langle (S_C)^{\frac{1}{2}} x, (S_C)^{\frac{1}{2}} x\rangle_{\mathcal{A}}
\end{aligned}$$

$$= \|C^{\frac{-1}{2}}\|^2 \langle S_C x, x\rangle_{\mathcal{A}}.$$

So,

$$(\|C^{\frac{-1}{2}}\|^{-1}A)\langle x, x\rangle_{\mathcal{A}}(\|C^{\frac{-1}{2}}\|^{-1}A)^* \leq \langle S_C x, x\rangle_{\mathcal{A}}.$$

Hence, for all $x \in \mathcal{H}$,

$$\begin{aligned}\langle S_C x, x\rangle_{\mathcal{A}} &= \langle CSx, x\rangle_{\mathcal{A}} \\ &= \langle C^{\frac{1}{2}}Sx, C^{\frac{1}{2}}x\rangle_{\mathcal{A}} \\ &\leq \|C^{\frac{1}{2}}\|^2\langle Sx, x\rangle_{\mathcal{A}} \\ &\leq \|C^{\frac{1}{2}}\|^2 B\langle x, x\rangle_{\mathcal{A}}B^* \\ &= (\|C^{\frac{1}{2}}\|B)\langle x, x\rangle_{\mathcal{A}}(\|C^{\frac{1}{2}}\|B)^*.\end{aligned}$$

Therefore, we conclude that F is a C-controlled $*$-integral frame $\mathcal{H}$ associted to (Ω, μ) with bounds $\|C^{\frac{-1}{2}}\|^{-1}A$ and $\|C^{\frac{1}{2}}\|B$. □

Theorem 4.1. *Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module and (Ω, μ) be a measure space. Let F a C-controlled $*$-integral frame for $\mathcal{H}$ associted to (Ω, μ) with the frame operator S_C and bounds A and B. Let $K \in End^*_{\mathcal{A}}(\mathcal{H})$ a surjective operator such that $KC = CK$. Then KF is a C-controlled $*$-integral frame for $\mathcal{H}$ with the operator frame KS_CK^*.*

Proof. By (4.1), we have

$$A\langle K^*x, K^*x\rangle_{\mathcal{A}}A^* \leq \int_\Omega \langle K^*x, F_\omega\rangle_{\mathcal{A}}\langle CF_\omega, K^*x\rangle_{\mathcal{A}} d\mu(w) \leq B\langle K^*x, K^*x\rangle_{\mathcal{A}}B^*, \quad x \in \mathcal{H}.$$

By Lemma 1.1 and Lemma 1.3, we obtain

$$A\|(KK^*)^{-1}\|^{-1}\langle x, x\rangle_{\mathcal{A}}A^* \leq \int_\Omega \langle x, KF_\omega\rangle_{\mathcal{A}}\langle CKF_\omega, x\rangle_{\mathcal{A}} d\mu(w) \leq B\|K^*\|^2\langle x, x\rangle_{\mathcal{A}}B^*,$$

which shows that KF is a C-controlled $*$-integral operator. Moreover, by Lemma 1.4, we have

$$KS_CK^*x = K\int_\Omega \langle K^*x, F_\omega\rangle_{\mathcal{A}} CF_\omega d\mu(\omega) = \int_\Omega \langle x, KF_\omega\rangle_{\mathcal{A}} CKF_\omega d\mu(\omega),$$

which ends the proof. □

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