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ON A DETERMINANTAL FORMULA FOR DERANGEMENT NUMBERS

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Abstract. The aim of this note is to provide succinct proofs for a recent formula of the derangement numbers in terms of the determinant of a tridiagonal matrix.

1. Preliminaries

The nth derangement number $!n$, also known as subfactorial of n, is the number of permutations on n elements, such that no element appears in its original position, i.e., is a permutation that has no fixed points.

Derangement numbers were first combinatorially studied by the French mathematician and Fellow of the Royal Society, Pierre Rémond de Montmort in his celebrated book *Essay d'analyse sur les jeux de hazard* published in 1708.

The two well-known recurrence relations

$$!n = (n-1)(!(n-1)+!(n-2)), \quad \text{for } n \geqslant 2, \tag{1.1}$$

and

$$!n = n\,(!(n-1)) + (-1)^n, \quad \text{for } n \geqslant 1, \tag{1.2}$$

with $!0 = 1$ and $!1 = 0$, were established and proved by Euler. They can be written in the explicit forms

$$!n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} i!,$$

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which coincide with the permanent of the all ones matrix minus the identity matrix, all of order n [4].

The arithmetic properties of the sequence of derangements are very interesting, as we can find in [5]. There, they are studied in terms of the periodicity modulo a positive integer, p-adic valuations, and prime divisors. We can also find attractive relations to other number sequences. For example, in [11], for any prime number p co-prime with a positive integer m, we have

$$\sum_{0<k<p} \frac{B_k}{(-m)^k} \equiv (-1)^{m-1}\, !(m-1) \pmod p,$$

where B_k denotes the kth Bell number.

Among the most relevant generalizations we have the so-called r-derangement numbers [12], when some of the elements are restricted to be in distinct cycles in the cycle decomposition. For more details on this matter, recent formulas, and interpretations, the reader is referred to [1, 6, 10].

The first terms of this sequence are

$$1\,,0\,,1\,,2\,,9\,,44\,,265\,,1854\,,14833\,,133496\,,1334961\,,14684570$$

and it was coined by The On-Line Encyclopedia of Integer Sequences [9] as the sequence A000166.

Another interesting representation of the derangement numbers is in terms of the determinant of a certain family tridiagonal matrices. Kittappa [3] and Janjić [2] showed independently two similar formulas:

$$!(n+1) = \begin{vmatrix} 2 & -1 & & & \\ 3 & 3 & -1 & & \\ & 4 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & n & n \end{vmatrix}, \tag{1.3}$$

for $n \geqslant 2$, and

$$!(n+1) = \begin{vmatrix} 1 & -1 & & & & \\ 1 & 1 & -1 & & & \\ & 3 & 3 & -1 & & \\ & & 4 & \ddots & \ddots & \\ & & & \ddots & \ddots & -1 \\ & & & & n & n \end{vmatrix}, \tag{1.4}$$

for any positive integer n, respectively. Subtracting to the second row the first one, in (1.4), it is a straightforward exercise to check that both representations are exactly the same. Moreover they trivially satisfy (1.1)–(1.2).

In two recent replicated papers [7, 8], Qi, Wang, and Guo claim the discovery of a new representation for the derangement numbers in terms of the determinant of a new tridiagonal matrix. The aim of this short note is to show that this can be proven using elementary matrix theory and the above well-known representations.

2. Derangement Numbers and Tridiagonal Matrices

In [7, 8] it is simultaneously claimed the discovery of a new representation for $!n$ in terms of the determinant of the tridiagonal matrix of order $n+1$, namely,

$$!n = -\begin{vmatrix} -1 & -1 & & & & & \\ 0 & 0 & -1 & & & & \\ & 1 & 1 & -1 & & & \\ & & 2 & 2 & -1 & & \\ & & & 3 & \ddots & \ddots & \\ & & & & \ddots & \ddots & -1 \\ & & & & & n-1 & n-1 \end{vmatrix}, \tag{2.1}$$

for any nonnegative integer. The proof is intricate and based on the higher derivatives of the generating function of $!n$.

However, using the elementary operations on rows R_i and columns C_i

$$R_1 \leftarrow -R_1\,, \quad C_2 \leftarrow C_2 - C_1\,, \quad R_4 \leftarrow R_4 + 2R_2\,, \quad C_4 \leftarrow C_4 + C_2\,,$$

it follows that (2.1) equals

$$\left|\begin{array}{c|cc|cccc} 1 & & & & & & \\ \hline & 0 & -1 & & & & \\ & 1 & 1 & & & & \\ \hline & & & 2 & -1 & & \\ & & & 3 & \ddots & \ddots & \\ & & & & \ddots & \ddots & -1 \\ & & & & & n-1 & n-1 \end{array}\right|$$

and this determinant is exactly (1.3).

Yet, there is also another way to check (2.1). For, expanding of the determinant along last row (or column) we immediately get (1.3). The conclusion now follows from the fact that for $n = 0$ and $n = 1$ the determinant (2.1) is, respectively, 1 and 0.

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