Kragujevac Journal of Mathematics
Volume 47(5) (2023), Pages 801–815.

ON A GENERALIZED DRYGAS FUNCTIONAL EQUATION AND ITS APPROXIMATE SOLUTIONS IN 2-BANACH SPACES

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Abstract. In this paper, we introduce and solve the following generalized Drygas functional equation

$$f(x+ky)+f(x-ky)=2f(x)+k^2f(y)+k^2f(-y),$$

where $k\in\mathbb{N}$. Also, we discuss some stability and hyperstability results for the considered equation in 2-Banach spaces by using the fixed point approach.

1. Introduction and preliminaries

We begin this paper by some notations and symbols. We will denote the set of natural numbers by $\mathbb{N}$, the set of real numbers by $\mathbb{R}$, $\mathbb{R}_+=[0,\infty)$ and the set of all natural numbers greater than or equal to m will be denoted by $\mathbb{N}_m$, $m\in\mathbb{N}$. We write B^A to mean the family of all functions mapping from a nonempty set A into a nonempty set B.

S. Gähler [23, 24] introduced the basic concept of linear 2-normed spaces. He gave some important facts concerning 2-normed spaces and some preliminary results as follows.

Definition 1.1. Let X be a real linear space with $\dim X>1$ and $\|\cdot,\cdot\|:X\times X\to[0,\infty)$ be a function satisfying the following properties:

(a) $\|x,y\|=0$ if and only if x and y are linearly dependent;
(b) $\|x,y\|=\|y,x\|$;
(c) $\|\lambda x,y\|=|\lambda|\|x,y\|$;

Key words and phrases. Stability, hyperstability, Drygas functional equation, fixed point method, 2-Banach space.

2010 *Mathematics Subject Classification.* Primary: 39B52. Secondary: 54E50, 39B82, 47H10.
DOI 10.46793/KgJMat2305.801H
Received: August 18, 2020.
Accepted: December 02, 2020.

(d) $\|x, y+z\| \leq \|x, y\| + \|x, z\|$,

for all $x, y, z \in X$ and $\lambda \in \mathbb{R}$. Then the function $\|\cdot,\cdot\|$ is called *the* 2*-norm* on X and the pair $(X, \|\cdot,\cdot\|)$ is called *the linear* 2*-normed space*. Sometimes the condition (d) is called the *triangle inequality*.

Example 1.1. For $x = (x_1, x_2)$, $y = (y_1, y_2) \in X = \mathbb{R}^2$, the Euclidean 2-norm $\|x, y\|_{\mathbb{R}^2}$ is defined by

$$\|x, y\|_{\mathbb{R}^2} = |x_1 y_2 - x_2 y_1| .$$

Lemma 1.1. *Let $(X, \|\cdot,\cdot\|)$ be a* 2*-normed space. If $x \in X$ and $\|x, y\| = 0$ for all $y \in X$, then $x = 0$.*

Definition 1.2. A sequence $\{x_k\}$ in a 2-normed space X is called a *convergent sequence* if there is an $x \in X$ such that

$$\lim_{k\to\infty} \|x_k - x, y\| = 0,$$

for all $y \in X$. If $\{x_k\}$ converges to x, write $x_k \to x$ with $k \to \infty$ and call x the limit of $\{x_k\}$. In this case, we also write $\lim_{k\to\infty} x_k = x$.

Definition 1.3. A sequence $\{x_k\}$ in a 2-normed space X is said to be a *Cauchy sequence* with respect to the 2-norm if

$$\lim_{k,l\to\infty} \|x_k - x_l, y\| = 0,$$

for all $y \in X$. If every Cauchy sequence in X converges to some $x \in X$, then X is said to be *complete* with respect to the 2-norm. Any complete 2-normed space is said to be a 2*-Banach space*.

The following lemma is one of the tools whose we need in our main results.

Lemma 1.2 ([31])**.** *Let X be a* 2*-normed space. Then*

(a) $\Big|\|x, z\| - \|y, z\|\Big| \leq \|x - y, z\|$ *for all $x, y, z \in X$;*

(b) *if $\|x, z\| = 0$ for all $z \in X$, then $x = 0$;*

(c) *for a convergent sequence x_n in X*

$$\lim_{n\longrightarrow\infty} \|x_n, z\| = \left\|\lim_{n\longrightarrow\infty} x_n, z\right\|,$$

for all $z \in X$.

The problem of the stability of functional equations is caused by the question of S. M. Ulam [38] about the stability in group homomorphisms. The first affirmative partial answer to the Ulam's problem for Banach spaces was provided by D. H. Hyers [28]. The result of Hyers was generalizable. Namely, it was generalized by T. Aoki [3] for additive mappings and by Th. M. Rassias [34] for linear mappings by considering an unbounded Cauchy difference. In 1994, P. Găvruţa [25] introduced the generalization of the Th. M. Rassias theorem was obtained by replacing the unbounded Cauchy difference by a general control function in the spirit of Th. M. Rassias' approach.

Within that, a special kind of stability was introduced. This kind is the hyperstability which was given by the following definition.

Definition 1.4 ([13]). Let S be a nonempty set, (Y,d) be a metric space, $\mathcal{E} \subset \mathcal{C} \subset \mathbb{R}_+^{S^n}$ be nonempty, $\mathcal{T}$ be an operator mapping $\mathcal{C}$ into $\mathbb{R}_+^S$ and $\mathcal{F}_1, \mathcal{F}_2$ be operators mapping a nonempty set $\mathcal{D} \subset Y^S$ into Y^{S^n}. We say that the operator equation

$$\mathcal{F}_1\varphi(x_1,\dots,x_n) = \mathcal{F}_2\varphi(x_1,\dots,x_n), \quad x_1,\dots,x_n \in S, \tag{1.1}$$

is $(\mathcal{E},\mathcal{T})$-hyperstable provided for any $\varepsilon \in \mathcal{E}$ and $\varphi_0 \in \mathcal{D}$ with

$$d\Big(\mathcal{F}_1\varphi_0(x_1,\dots,x_n), \mathcal{F}_2\varphi_0(x_1,\dots,x_n)\Big) \le \varepsilon(x_1,\dots,x_n), \quad x_1,\dots,x_n \in S,$$

there is a solution $\varphi \in \mathcal{D}$ of equation (1.1) such that

$$d\Big(\varphi(x), \varphi_0(x)\Big) \le \mathcal{T}\varepsilon(x), \quad x \in S.$$

In [5] the first result of hyperstability has been published, however, the term *hyperstability* was first used in [30].

There are many papers concerning the hyperstability of functional equations, see for example [4, 7–9, 13, 16–20, 26, 27, 30, 33]. In 2013, Brzdęk [6] gave an important result that will be a basic tool to study the stability and hyperstability of functional equations.

Theorem 1.1 ([6]). *Let X be a nonempty set, (Y,d) a complete metric space $f_1,\dots,f_s\colon X \to X$ and $L_1,\dots,L_s\colon X \to \mathbb{R}_+$ be given mappings. Let $\Lambda\colon \mathbb{R}_+^X \to \mathbb{R}_+^X$ be a linear operator defined by*

$$\Lambda\delta(x) := \sum_{i=1}^{s} L_i(x)\delta(f_i(x)),$$

for $\delta \in \mathbb{R}_+^X$ and $x \in X$. If $\mathcal{T}\colon Y^X \to Y^X$ is an operator satisfying the inequality

$$d\Big(\mathcal{T}\xi(x), \mathcal{T}\mu(x)\Big) \le \sum_{i=1}^{s} L_i(x) d\Big(\xi(f_i(x)), \mu(f_i(x))\Big), \quad \xi,\mu \in Y^X, x \in X,$$

and a function $\varepsilon\colon X \to \mathbb{R}_+$ and a mapping $\varphi\colon X \to Y$ satisfies

$$d\left(\mathcal{T}\varphi(x), \varphi(x)\right) \le \varepsilon(x), \quad x \in X,$$

$$\varepsilon^*(x) := \sum_{k=0}^{\infty} \Lambda^k\varepsilon(x) < \infty, \quad x \in X,$$

then for every $x \in X$ the limit

$$\psi(x) := \lim_{n\to\infty} \mathcal{T}^n\varphi(x)$$

exists and the function $\psi \in Y^X$ is a unique fixed point of $\mathcal{T}$ with

$$d\left(\varphi(x), \psi(x)\right) \le \varepsilon^*(x), \quad x \in X.$$

In 2019, M. Almahalebi et al. [2] introduced and proved an analogue of Theorem 1.1 in 2-Banach spaces.

Theorem 1.2 ([2]). *Let X be a nonempty set, $\big(Y, \|\cdot,\cdot\|\big)$ be a 2-Banach space, $g : X \to Y$ be a surjective mapping and let $f_1, \ldots, f_r : X \to X$ and $L_1, \ldots, L_r : X \to \mathbb{R}_+$ be given mappings. Suppose that $\mathcal{T} : Y^X \to Y^X$ and $\Lambda : \mathbb{R}_+^{X\times X} \to \mathbb{R}_+^{X\times X}$ are two operators satisfying the conditions*

$$\Big\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x), g(z)\Big\| \leq \sum_{i=1}^{r} L_i(x)\Big\|\xi\Big(f_i(x)\Big) - \mu\Big(f_i(x)\Big), g(z)\Big\|,$$

for all $\xi, \mu \in Y^X$, $x, z \in X$ and

$$\Lambda\delta(x,z) := \sum_{i=1}^{r} L_i(x)\delta\Big(f_i(x), z\Big), \quad \delta \in \mathbb{R}_+^{X\times X},\ x, z \in X. \tag{1.2}$$

If there exist functions $\varepsilon : X \times X \to \mathbb{R}_+$ and $\varphi : X \to Y$ such that

$$\Big\|\mathcal{T}\varphi(x) - \varphi(x), g(z)\Big\| \leq \varepsilon(x,z)$$

and

$$\varepsilon^*(x,z) := \sum_{n=0}^{\infty} \Big(\Lambda^n\varepsilon\Big)(x,z) < \infty,$$

for all $x, z \in X$, then the limit

$$\lim_{n\to\infty} \Big((\mathcal{T}^n\varphi)\Big)(x)$$

exists for each $x \in X$. Moreover, the function $\psi : X \to Y$ defined by

$$\psi(x) := \lim_{n\to\infty} \Big((\mathcal{T}^n\varphi)\Big)(x)$$

is a fixed point of $\mathcal{T}$ with

$$\Big\|\varphi(x) - \psi(x), g(z)\Big\| \leq \varepsilon^*(x,z),$$

for all $x, z \in X$.

Another version of Theorem 1.2 in 2-Banach space can be found in [14]. Also, J. Brzdęk and K. Ciepliński extended their fixed point result to the n-normed spaces in [15].

In this paper, we consider and solve the following equation

$$f(x+ky) + f(x-ky) = 2f(x) + k^2 f(y) + k^2 f(-y), \tag{1.3}$$

where $k \in \mathbb{N}$. This equation can be reduced to the Drygas equation

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y). \tag{1.4}$$

In addition, we use Theorem 1.2 to investigate some stability and hyperstability results of equation (1.3) in 2-Banach spaces.

2. Solution of (1.3)

Throughout this section, X and Y will be real vector spaces. The functional equation (1.3) is connected with the functional equation (1.4) as it is shown below.

Theorem 2.1. *A function $f : X \to Y$ satisfies the functional equation* (1.3) *if and only if f satisfies the Drygas functional equation* (1.4) *for all $x, y \in X$.*

Proof. Suppose that $f : X \to Y$ satisfies (1.3) for all $x, y \in X$. Letting $x = y = 0$ in (1.3), we get $f(0) = 0$. Also, by setting $x = 0$ in (1.3), we obtain that

$$f(ky) + f(-ky) = k^2 f(y) + k^2 f(-y), \quad y \in X.$$

To prove that f satisfies the Drygas functional equation (1.4) for all $x, y \in X$, we assume that $x' = x$ and $y' = ky$ be two elements in X. Then we get

$$\begin{aligned} f(x' + y') + f(x' - y') &= f(x + ky) + f(x - ky) \\ &= 2f(x) + k^2 f(y) + k^2 f(-y) \\ &= 2f(x) + f(ky) + f(-ky) \\ &= 2f(x') + f(y') + f(-y'), \quad x, y \in X, \end{aligned}$$

which means that f satisfies the Drygas functional equation (1.4) for all $x, y \in X$. On the other hand, let f be a function satisfying the Drygas functional equation (1.4) for all $x, y \in X$ with $f(0) = 0$ and $f(x) = B(x, x) + A(x)$. Then

$$\begin{aligned} f(x + ky) + f(x - ky) &= 2f(x) + f(ky) + f(-ky) \\ &= 2f(x) + B(ky, ky) + A(ky) + B(-ky, -ky) + A(-ky) \\ &= 2f(x) + k^2 B(y, y) + k^2 B(-y, -y) + \underbrace{A(ky) + A(-ky)}_{=0} \\ &= 2f(x) + k^2 B(y, y) + k^2 B(-y, -y) + k^2 \underbrace{\Big(A(y) + A(-y)\Big)}_{=0} \\ &= 2f(x) + k^2 \Big(B(y, y) + A(y)\Big) + +k^2 \Big(B(-y, -y) + A(-y)\Big) \\ &= 2f(x) + k^2 f(y) + k^2 f(-y), \quad x, y \in X, \end{aligned}$$

which means that f satisfies (1.3) for all $x, y \in X$. $\square$

3. Stability Results

In this section, we give some investigations on the stability and hyperstability results of the equation (1.3) by using Theorem 1.2 in 2-Banach spaces.

Theorem 3.1. *Let X be a normed space, $\Big(Y, \|\cdot, \cdot\|\Big)$ be a 2-Banach space and $h_1, h_2 : X_0^2 \to \mathbb{R}_+$ be two functions such that*

$$\mathcal{U} := \Big\{n \in \mathbb{N} : \alpha_n < 1\Big\} \neq \emptyset,$$

where

$$\alpha_n = \frac{1}{2}\lambda_1(1+kn)\lambda_2(1+kn)+\frac{1}{2}\lambda_1(1-kn)\lambda_2(1-kn)+\frac{k^2}{2}\lambda_1(n)\lambda_2(n)+\frac{k^2}{2}\lambda_1(-n)\lambda_2(-n)$$

and

$$\lambda_i(n) := \inf\Big\{t \in \mathbb{R}_+ : h_i(nx,z) \le t h_i(x,z),\ x,z \in X_0\Big\},$$

for all $n \in \mathbb{N}$ with $i \in \{1,2\}$. Assume that $f : X \to Y$ satisfies the inequality

$$\Big\|f(x+ky)+f(x-ky)-2f(x)-k^2f(y)-k^2f(-y), g(z)\Big\| \le h_1(x,z)h_2(y,z), \tag{3.1}$$

for all $x,y,z \in X_0$ such that $x+ky \neq 0$ and $x-ky \neq 0$, where $g : X \to Y$ is a surjective mapping. Then there exists a unique function $D : X \to Y$ that satisfies the equation (1.3) *such that*

$$\Big\|f(x)-D(x), g(z)\Big\| \le \lambda_0 h_1(x,z)h_2(x,z),$$

for all $x,z \in X_0$, where

$$\lambda_0 = \frac{\lambda_2(n)}{2(1-\alpha_m)}.$$

Proof. Let us fix $m \in \mathbb{N}$. Replacing x by mx, where $x \in X_0$, in the inequality (3.1), we obtain

$$\begin{aligned}&\Bigg\|\frac{1}{2}f\Big((1+km)x\Big)+\frac{1}{2}f\Big((1-km)x\Big)-\frac{k^2}{2}f(mx)-\frac{k^2}{2}f(-mx)-f(x)\ ,\ g(z)\Bigg\|\\ &\le\frac{1}{2}h_1(x,z)h_2(mx,z),\end{aligned} \tag{3.2}$$

for all $x,z \in X_0$. Define the operator $\mathcal{T}_m\colon Y^{X_0} \to Y^{X_0}$ by

$$\mathcal{T}_m\xi(x) := \frac{1}{2}\xi\Big((1+km)x\Big)+\frac{1}{2}\xi\Big((1-km)x\Big)-\frac{k^2}{2}\xi(mx)-\frac{k^2}{2}\xi(-mx),$$

for all $x \in X_0$ and $\xi \in Y^{X_0}$. Further put

$$\varepsilon_m(x,z) := \frac{1}{2}h_1(x,z)h_2(mx,z), \quad x,z \in X_0, \tag{3.3}$$

and observe that

$$\varepsilon_m(x,z) = \frac{1}{2}h_1(x,z)h_2(mx,z) \le \frac{1}{2}\lambda_2(m)h_1(x,z)h_2(x,z), \quad x,z \in X_0,\ m \in \mathbb{N}. \tag{3.4}$$

Thus, the inequality (3.2) becomes

$$\|\mathcal{T}_m f(x) - f(x), g(z)\| \le \varepsilon_m(x,z), \quad x,z \in X_0.$$

Furthermore, for every $x,z \in X_0$ and $\xi,\mu \in Y^{X_0}$, we have

$$\begin{aligned}&\Big\|\mathcal{T}_m\xi(x)-\mathcal{T}_m\mu(x), g(z)\Big\|\\ =&\Bigg\|\frac{1}{2}\xi\Big((1+km)x\Big)+\frac{1}{2}\xi\Big((1-km)x\Big)-\frac{k^2}{2}\xi(mx)\end{aligned}$$

$$-\frac{k^2}{2}\xi(-mx)-\frac{1}{2}\mu\Big((1+km)x\Big)-\frac{1}{2}\mu\Big((1-km)x\Big)+\frac{k^2}{2}\mu(mx)+\frac{k^2}{2}\mu(-mx),g(z)\Big\|$$
$$\leq\frac{1}{2}\left\|(\xi-\mu)\Big((1+km)x\Big),g(z)\right\|+\frac{1}{2}\left\|(\xi-\mu)\Big((1-km)x\Big),g(z)\right\|$$
$$+\frac{k^2}{2}\left\|(\xi-\mu)(mx)\ ,\ g(z)\right\|+\frac{k^2}{2}\left\|(\xi-\mu)(-mx),g(z)\right\|,$$

for all $x,z\in X_0$ and $\xi,\mu\in Y^{X_0}$. It means that the condition (1.2) is satisfied and this brings us to define the operator $\Lambda_m\colon \mathbb{R}_+^{X_0\times X_0}\to\mathbb{R}_+^{X_0\times X_0}$ by

$$\Lambda_m\delta(x,z):=\frac{1}{2}\delta\Big((1+km)x,z\Big)+\frac{1}{2}\delta\Big((1-km)x,z\Big)+\frac{k^2}{2}\delta(mx,z)+\frac{k^2}{2}\delta(-mx,z),$$

for all $x,z\in X_0$ and $\delta\in\mathbb{R}_+^{X_0\times X_0}$. This operator has the form given by (1.2) with $f_1(x)=(1+km)x$, $f_2(x)=(1-km)x$, $f_3(x)=mx$, $f_4(x)=-mx$, $L_1(x)=L_2(x)=\frac{1}{2}$ and $L_3(x)=L_4(x)=\frac{k^2}{2}$ for all $x\in X_0$.

By induction on $n\in\mathbb{N}$, it is easy to show that

$$\Big(\Lambda_m^n\varepsilon_m\Big)(x,z)\leq\frac{1}{2}\lambda_2(m)\alpha_m^n h_1(x,z)h_2(x,z), \tag{3.5}$$

for all $x,z\in X_0$ and all $m\in\mathcal{U}$, where

$$\alpha_m=\frac{1}{2}\lambda_1(1+km)\lambda_2(1+km)+\frac{1}{2}\lambda_1(1-km)\lambda_2(1-km)+\frac{k^2}{2}\lambda_1(m)\lambda_2(m)$$
$$+\frac{k^2}{2}\lambda_1(-m)\lambda_2(-m).$$

Indeed, (3.3) and (3.4) imply that the inequality (3.5) holds for $n=0$. Next, we assume that (3.5) holds for $n=r$, where $r\in\mathbb{N}_1$. Then we obtain

$$\begin{aligned}\Big(\Lambda_m^{r+1}\varepsilon_m\Big)(x,z)=&\Lambda_m\left(\Big(\Lambda_m^r\varepsilon_m\Big)(x,z)\right)\\
=&\frac{1}{2}\Big(\Lambda_m^r\varepsilon_m\Big)\Big((1+km)x,z\Big)+\frac{1}{2}\Big(\Lambda_m^r\varepsilon_m\Big)\Big((1-km)x,z\Big)\\
&+\frac{k^2}{2}\Big(\Lambda_m^r\varepsilon_m\Big)\Big(mx,z\Big)+\frac{k^2}{2}\Big(\Lambda_m^r\varepsilon_m\Big)\Big(-mx,z\Big)\\
\leq&\frac{1}{4}\lambda_2(m)\alpha_m^r h_1\Big((1+km)x,z\Big)h_2\Big((1+km)x,z\Big)\\
&+\frac{1}{4}\lambda_2(m)\alpha_m^r h_1\Big((1-km)x,z\Big)h_2\Big((1-km)x,z\Big)\\
&+\frac{k^2}{4}\lambda_2(m)\alpha_m^r h_1\Big(mx,z\Big)h_2\Big(mx,z\Big)\\
&+\frac{k^2}{4}\lambda_2(m)\alpha_m^r h_1\Big(-mx,z\Big)h_2\Big(-mx,z\Big)\\
\leq&\frac{1}{2}\lambda_2(m)\Big(\frac{1}{2}\lambda_1(1+km)\lambda_2(1+km)+\frac{1}{2}\lambda_1(1-km)\lambda_2(1-km)\end{aligned}$$

$$
\begin{aligned}
&+\frac{k^2}{2}\lambda_1(m)\lambda_2(m)+\frac{k^2}{2}\lambda_1(-m)\lambda_2(-m)\Big)\alpha_m^r h_1(x,z)h_2(x,z)\\
=&\frac{1}{2}\lambda_2(m)\alpha_m^{r+1}h_1(x,z)h_2(x,z),
\end{aligned}
$$

for all $x, z \in X_0$ and all $m \in \mathcal{U}$. It means that (3.5) holds for $n = r+1$ which implies that (3.5) holds for all $n \in \mathbb{N}$. Hence, in view of (3.5), we obtain

$$
\begin{aligned}
\varepsilon^*(x,z) &:= \sum_{n=0}^{\infty} \Lambda_m^n \varepsilon_m(x,z)\\
&\le \sum_{n=0}^{\infty} \frac{1}{2}\lambda_2(m)\alpha_m^n h_1(x,z)h_2(x,z)\\
&= \frac{\lambda_2(m)h_1(x,z)h_2(x,z)}{2(1-\alpha_m)} < \infty,
\end{aligned}
$$

for all $x, z \in X_0$ and all $m \in \mathcal{U}$. Therefore, according to Theorem 1.2, with $\varphi = f$ and using the surjectivity of g, we get that the limit

$$
\lim_{n\to\infty} \Big(\mathcal{T}_m^n f\Big)(x)
$$

exists and defined a function $D_m : X \to Y$ such that

$$
\Big\|f(x) - D_m(x), g(z)\Big\| \le \frac{\lambda_2(m)h_1(x,z)h_2(x,z)}{2(1-\alpha_m)}, \quad x, z \in X_0,\ m \in \mathcal{U}. \tag{3.6}
$$

To prove that F_m satisfies the functional equation (1.3), just prove the following inequality by the induction on $n \in \mathbb{N}_0$

(3.7)

$$
\begin{aligned}
&\Big\|(\mathcal{T}_m^n f)\Big(x+ky) + (\mathcal{T}_m^n f)\Big(x-ky\Big) - 2(\mathcal{T}_m^n f)\Big(x\Big) - k^2(\mathcal{T}_m^n f)\Big(y\Big) - k^2(\mathcal{T}_m^n f)(-y), g(z)\Big\|\\
\le&\alpha_m^n h_1(x,z)h_2(y,z),
\end{aligned}
$$

for every $x, y, z \in X_0$ such that $x+ky \neq 0$, $x-ky \neq 0$ and every $m \in \mathcal{U}$.

First, for $n = 0$, we just find (3.1). Next, take $r \in \mathbb{N}$ and assume that (3.7) holds for $n = r$ and every $x, y, z \in X_0$ such that $x+ky \neq 0$ and $x-ky \neq 0$, $m \in \mathcal{U}$. Then, for each $x, y, z \in X_0$ and $m \in \mathcal{U}$, we have

$$
\begin{aligned}
&\Big\|(\mathcal{T}_m^{r+1} f)\Big(x+ky) + (\mathcal{T}_m^{r+1} f)\Big(x-ky\Big) - 2(\mathcal{T}_m^{r+1} f)\Big(x\Big)\\
&- k^2(\mathcal{T}_m^{r+1} f)\Big(y\Big) - k^2(\mathcal{T}_m^{r+1} f)(-y)\ ,\ g(z)\Big\|\\
=&\Big\|\frac{1}{2}(\mathcal{T}_m^r f)\Big((1+km)(x+ky)\Big) + \frac{1}{2}(\mathcal{T}_m^r f)\Big((1-km)(x+ky)\Big)\\
&- \frac{k^2}{2}(\mathcal{T}_m^r f)\Big(m(x+ky)\Big) - \frac{k^2}{2}(\mathcal{T}_m^r f)\Big(-m(x+ky)\Big)\\
&+ \frac{1}{2}(\mathcal{T}_m^r f)\Big((1+km)(x-ky)\Big) + \frac{1}{2}(\mathcal{T}_m^r f)\Big((1-km)(x-ky)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{k^2}{2}(\mathcal{T}_m^r f)\Big(m(x-ky)\Big) - \frac{k^2}{2}(\mathcal{T}_m^r f)\Big(-m(x-ky)\Big)\\
&- (\mathcal{T}_m^r f)\Big((1+km)(x)\Big) - (\mathcal{T}_m^r f)\Big((1-km)(x)\Big)\\
&+ k^2(\mathcal{T}_m^r f)\Big(mx\Big) + k^2(\mathcal{T}_m^r f)\Big(-mx\Big)\\
&- \frac{k^2}{2}(\mathcal{T}_m^r f)\Big((1+km)(y)\Big) - \frac{k^2}{2}(\mathcal{T}_m^r f)\Big((1-km)(y)\Big) + \frac{k^4}{2}(\mathcal{T}_m^r f)\Big(my\Big)\\
&+ \frac{k^4}{2}(\mathcal{T}_m^r f)\Big(-my\Big) - \frac{k^2}{2}(\mathcal{T}_m^r f)\Big((1+km)(-y)\Big) - \frac{k^2}{2}(\mathcal{T}_m^r f)\Big((1-km)(-y)\Big)\\
&+ \frac{k^4}{2}(\mathcal{T}_m^r f)\Big(-my\Big) + \frac{k^4}{2}(\mathcal{T}_m^r f)\Big(my\Big), g(z)\Big\|\\
\leq&\frac{1}{2}\Big\|(\mathcal{T}_m^r f)\Big((1+km)(x+ky)\Big) + (\mathcal{T}_m^r f)\Big((1+km)(x-ky)\Big)\\
&- 2(\mathcal{T}_m^r f)\Big((1+km)(x)\Big) - k^2(\mathcal{T}_m^r f)\Big((1+km)(y)\Big)\\
&- k^2(\mathcal{T}_m^r f)\Big((1+km)(-y)\Big), g(z)\Big\|\\
&+ \frac{1}{2}\Big\|(\mathcal{T}_m^r f)\Big((1-km)(x+ky)\Big) + (\mathcal{T}_m^r f)\Big((1-km)(x-ky)\Big)\\
&- 2(\mathcal{T}_m^r f)\Big((1-km)(x)\Big) - k^2(\mathcal{T}_m^r f)\Big((1-km)(y)\Big)\\
&- k^2(\mathcal{T}_m^r f)\Big((1-km)(-y)\Big), g(z)\Big\|\\
&+ \frac{k^2}{2}\Big\|(\mathcal{T}_m^r f)\Big(m(x+ky)\Big) + (\mathcal{T}_m^r f)\Big(m(x-ky)\Big) - 2(\mathcal{T}_m^r f)\Big(mx\Big)\\
&- k^2(\mathcal{T}_m^r f)\Big(my\Big) - k^2(\mathcal{T}_m^r f)\Big(-my\Big), g(z)\Big\|\\
&+ \frac{k^2}{2}\Big\|(\mathcal{T}_m^r f)\Big(-m(x+ky)\Big) + (\mathcal{T}_m^r f)\Big(-m(x-ky)\Big) - 2(\mathcal{T}_m^r f)\Big(-mx\Big)\\
&- k^2(\mathcal{T}_m^r f)\Big(-my\Big) - k^2(\mathcal{T}_m^r f)\Big(my\Big), g(z)\Big\|\\
\leq&\frac{1}{2}\alpha_m^r h_1\Big((1+km)x, z\Big)h_2\Big((1+km)y, z\Big) + \frac{1}{2}\alpha_m^r h_1\Big((1-km)x, z\Big)h_2\Big((1-km)y, z\Big)\\
&+ \frac{k^2}{2}\alpha_m^r h_1\Big(mx, z\Big)h_2\Big(my, z\Big) + \frac{k^2}{2}\alpha_m^r h_1\Big(-mx, z\Big)h_2\Big(-my, z\Big)\\
=&\alpha_m^{r+1} h_1(x, z)h_2(y, z).
\end{aligned}
$$

Thus, by induction, we have shown that (3.7) holds for every $x, y, z \in X_0$, $n \in \mathbb{N}_0$, and $m \in \mathcal{U}$ such that $x + ky \neq 0$ and $x - ky \neq 0$. Letting $n \to \infty$ in (3.7), we obtain the equality

$$D_m(x + ky) + D_m(x - ky) = 2D_m(x) + k^2 D_m(y) + k^2 D_m(-y),$$

for all $x, y \in X_0$ and $m \in \mathcal{U}$ such that $x + ky \neq 0$ and $x - ky \neq 0$. This implies that $D_m : X \to Y$, defined in this way, is a solution of the equation

$$D(x) = \frac{1}{2}D\Big((1+km)x\Big) + \frac{1}{2}D\Big((1-km)x\Big) - \frac{k^2}{2}D(mx) - \frac{k^2}{2}D\Big(-mx\Big), \tag{3.8}$$

for all $x \in X_0$ and all $m \in \mathcal{U}$. Next, we will prove that each cubic functional equation $D : X \to Y$ satisfying the inequality

$$\Big\|f(x) - D(x), g(z)\Big\| \leq L\ h_1(x,z)h_2(x,z), \quad x, z \in X_0, \tag{3.9}$$

with some $L > 0$, is equal to D_m for each $m \in \mathcal{U}$. To this end, we fix $m_0 \in \mathcal{U}$ and $D : X \to Y$ satisfying (3.9). From (3.6), for each $x \in X_0$, we get

$$\begin{aligned}\Big\|D(x) - D_{m_0}(x), g(z)\Big\| \leq& \Big\|D(x) - f(x), g(z)\Big\| + \Big\|f(x) - D_{m_0}(x), g(z)\Big\|\\ \leq& L\ h_1(x,z)h_2(x,z) + \varepsilon^*_{m_0}(x,z)\\ \leq& L_0\ h_1(x,z)h_2(x,z)\sum_{n=0}^{\infty}\alpha^n_{m_0},\end{aligned} \tag{3.10}$$

where $L_0 := 2(1-\alpha_{m_0})L + \lambda_2(m_0) > 0$ and we exclude the case that $h_1(x,z) \equiv 0$ or $h_2(x,z) \equiv 0$ which is trivial. Observe that D and D_{m_0} are solutions to equation (3.8) for all $m \in \mathcal{U}$. Next, we show that, for each $j \in \mathbb{N}_0$, we have

$$\Big\|D(x) - D_{m_0}(x), g(z)\Big\| \leq L_0\ h_1(x,z)h_2(x,z)\sum_{n=j}^{\infty}\alpha^n_{m_0}, \quad x, z \in X_0. \tag{3.11}$$

The case $j = 0$ is exactly (3.10). We fix $r \in \mathbb{N}$ and assume that (3.11) holds for $j = r$. Then, in view of (3.10), for each $x, z \in X_0$, we get

$$\begin{aligned}\Big\|D(x) - D_{m_0}(x), g(z)\Big\| =& \Big\|\frac{1}{2}D\Big((1+km_0)x\Big) + \frac{1}{2}D\Big((1-km_0)x\Big) - \frac{k^2}{2}D(m_0x)\\ &- \frac{k^2}{2}D\Big(-m_0x\Big) - \frac{1}{2}D_{m_0}\Big((1+km_0)x\Big)\\ &- \frac{1}{2}D_{m_0}\Big((1-km_0)x\Big) + \frac{k^2}{2}D_{m_0}(m_0x)\\ &+ \frac{k^2}{2}D_{m_0}\Big(-m_0x\Big), g(z)\Big\|\\ \leq& \frac{1}{2}\Big\|D\Big((1+km_0)x\Big) - D_{m_0}\Big((1+km_0)x\Big), g(z)\Big\|\\ &+ \frac{1}{2}\Big\|D\Big((1-km_0)x\Big) - D_{m_0}\Big((1-km_0)x\Big), g(z)\Big\|\\ &+ \frac{k^2}{2}\Big\|D\Big(m_0x\Big) - D_{m_0}\Big(m_0x\Big), g(z)\Big\|\\ &+ \frac{k^2}{2}\Big\|D\Big(-m_0x\Big) - D_{m_0}\Big(-m_0x\Big), g(z)\Big\|\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{2}L_0\, h_1\Big((1+km_0)x,z\Big)h_2\Big((1+km_0)x,z\Big)\sum_{n=r}^{\infty}\alpha_{m_0}^n\\
&\quad+\frac{1}{2}L_0\, h_1\Big((1-km_0)x,z\Big)h_2\Big((1-km_0)x,z\Big)\sum_{n=r}^{\infty}\alpha_{m_0}^n\\
&\quad+\frac{k^2}{2}L_0\, h_1\Big(m_0x,z\Big)h_2\Big(m_0x,z\Big)\sum_{n=r}^{\infty}\alpha_{m_0}^n\\
&\quad+\frac{k^2}{2}L_0\, h_1\Big(-m_0x,z\Big)h_2\Big(-m_0x,z\Big)\sum_{n=r}^{\infty}\alpha_{m_0}^n\\
&\le L_0\,\alpha_{m_0}\, h_1(x,z)h_2(x,z)\sum_{n=r}^{\infty}\alpha_{m_0}^n\\
&= L_0\, h_1(x,z)h_2(x,z)\sum_{n=r+1}^{\infty}\alpha_{m_0}^n.
\end{aligned}
$$

This shows that (3.11) holds for $j=k+1$. Now we can conclude that the inequality (3.11) holds for all $j\in\mathbb{N}_0$. Now, letting $j\to\infty$ in (3.11), we get

$$D = D_{m_0}. \tag{3.12}$$

Thus, we have also proved that $D_m = D_{m_0}$ for each $m\in\mathcal{U}$, which (in view of (3.6)) yields

$$\Big\|f(x)-D_{m_0}(x),g(z)\Big\| \le \frac{\lambda_2(m)h_1(x,z)h_2(x,z)}{2(1-\alpha_m)},\quad x,z\in X_0,\ m\in\mathcal{U}.$$

This implies (1.3) with $D=D_{m_0}$ and (3.12) confirms the uniqueness of D. □

4. Hyperstaility Results

The following theorems and corollaries concern the η-hyperstability of (1.3) in 2-Banach spaces. Namely, we consider functions $f : X\to Y$ fulfilling (1.3) approximately, i.e., satisfying the inequality

$$\Big\|f(x+ky)+f(x-ky)-2f(x)-k^2f(y)-k^2f(-y),g(z)\Big\| \le \eta(x,y,z), \tag{4.1}$$

for all $x,y,z\in X_0$ such that $x+ky\neq 0$ and $x-ky\neq 0$ with $\eta : X_0\times X_0\times X_0\to\mathbb{R}_+$ is a given mapping. Then we find a unique cubic function $F : X\to Y$ which is close to f. Then, under some additional assumptions on η, we prove that the conditional functional equation (1.3) is η-hyperstable in the class of functions $f : X\to Y$, i.e., each $f : X\to Y$ satisfying inequality (4.1), with such η, must fulfil equation (1.3).

Theorem 4.1. *Let X be a normed space, $(Y,\|\cdot,\cdot\|)$ be a real 2-Banach space, h_1,h_2 and $\mathcal{U}$ be as in Theorem 3.1. Assume that*

$$
\begin{cases}
\lim_{n\to\infty}\lambda_2(n)=\lim_{n\to\infty}\lambda_1(1+kn)\lambda_2(1+kn)=\lim_{n\to\infty}\lambda_1(-n)\lambda_2(-n)=0,\\
\lim_{n\to\infty}\lambda_1(1-kn)\lambda_2(1-kn)=\lim_{n\to\infty}\lambda_1(n)\lambda_2(n)=0.
\end{cases} \tag{4.2}
$$

Then every $f : X\to Y$ satisfying (3.1) is a solution of (1.3) on X_0.

Proof. Suppose that $f : X \to Y$ satisfies (3.1). Then, by Theorem 3.1, there exists a mapping $D : X \to Y$ satisfying (1.3) and

$$\|f(x) - D(x), g(z)\| \leq \lambda_0 h_1(x,z)h_2(x,z),$$

for all $x, z \in X_0$, where $g : X \to Y$ is a surjective mapping and

$$\lambda_0 = \frac{\lambda_2(n)}{2(1-\alpha_m)},$$

with

$$\alpha_n = \frac{1}{2}\lambda_1(1+kn)\lambda_2(1+kn)+\frac{1}{2}\lambda_1(1-kn)\lambda_2(1-kn)+\frac{k^2}{2}\lambda_1(n)\lambda_2(n)+\frac{k^2}{2}\lambda_1(-n)\lambda_2(-n).$$

Since, in view of (4.2), $\lambda_0 = 0$, this means that $f(x) = D(x)$ for all $x \in X_0$, whence

$$f(x+ky) + f(x-ky) = 2f(x) + k^2 f(y) + k^2 f(-y),$$

for all $x, y \in X_0$ such that $x + ky \neq 0$ and $x - ky \neq 0$, which implies that f satisfies the functional equation (1.3) on X_0. $\square$

Corollary 4.1. *Let $\big(X, \|\cdot\|\big)$ be a normed space, $\big(Y, \|\cdot,\cdot\|\big)$ be a real 2-Banach space and $\theta \geq 0$, $s \geq 0$, $p, q \in \mathbb{R}$ such that $p + q < 0$. Suppose that $f : X \to Y$ such that $f(0) = 0$ satisfies the inequality*

(4.3)
$$\Big\|f(x+ky) + f(x-ky) - 2f(x) - k^2 f(y) - k^2 f(-y), g(z)\Big\| \leq \theta\|x\|^p \; \|y\|^q \; \|z\|^s,$$

for all $x, y, z \in X_0$ such that $x + ky \neq 0$ and $x - ky \neq 0$, where $g : X \to Y$ is a surjective mapping. Then f satisfies (1.3) on X_0.

Proof. The proof follows from Theorem 3.1 by defining $h_1, h_2 : X_0 \times X_o \to \mathbb{R}_+$ by $h_1(x,z) = \theta_1\|x\|^p\|z\|^{s_1}$, $h_2(y,z) = \theta_2\|y\|^q\|z\|^{s_2}$ and $h_1(0,z) = h_2(0,z) = 0$ with $\theta_1, \theta_2 \in \mathbb{R}_+$, $s_1, s_2 \in \mathbb{R}_+$ and $p, q \in \mathbb{R}$ such that $\theta_1\theta_2 = \theta$, $s_1 + s_2 = s$ and $p + q < 0$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned}\lambda_1(n) &= \inf\{t \in \mathbb{R}_+ \colon h_1(nx,z) \leq t\, h_1(x,z),\ x,z \in X_0\}\\ &= \inf\{t \in \mathbb{R}_+ \colon \theta_1\|nx\|^p\|z\|^{s_1} \leq t\, \theta_1\|x\|^p\|z\|^{s_1},\ x,z \in X_0\}\\ &= n^p.\end{aligned}$$

Also, we have $\lambda_2(n) = n^q$ for all $n \in \mathbb{N}$. Clearly, we can find $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}&\frac{1}{2}\lambda_1(1+kn)\lambda_2(1+kn)+\frac{1}{2}\lambda_1(1-kn)\lambda_2(1-kn)+\frac{k^2}{2}\lambda_1(n)\lambda_2(n)+\frac{k^2}{2}\lambda_1(-n)\lambda_2(-n)\\ =&\frac{1}{2}(1+kn)^{p+q} + \frac{1}{2}(1-kn)^{p+q} + k^2 n^{p+q} < 1,\end{aligned}$$

for all $n \geq n_0$. According to Theorem 3.1, there exists a unique Drygas function $D : X \to Y$ such that

$$\Big\|f(x) - D(x), g(z)\Big\| \leq \theta\lambda_0 h_1(x,z)h_2(x,z),$$

for all $x, z \in X_0$, where

$$\lambda_0 = \frac{\lambda_2(n)}{2(1-\alpha_m)},$$

with

$$\alpha_n = \frac{1}{2}\lambda_1(1+kn)\lambda_2(1+kn)+\frac{1}{2}\lambda_1(1-kn)\lambda_2(1-kn)+\frac{k^2}{2}\lambda_1(n)\lambda_2(n)+\frac{k^2}{2}\lambda_1(-n)\lambda_2(-n).$$

Since $p+q<0$, one of p and q must be negative. Assume that $q<0$. Then

$$\begin{aligned}
&\lim_{n\to\infty}\lambda_2(n) = \lim_{n\to\infty} n^q = 0,\\
&\lim_{n\to\infty}\lambda_1(1+kn)\lambda_2(1+kn) = \lim_{n\to\infty}(1+kn)^{p+q} = 0,\\
&\lim_{n\to\infty}\lambda_1(1-kn)\lambda_2(1-kn) = \lim_{n\to\infty}(1+kn)^{p+q} = 0,\\
&\lim_{n\to\infty}\lambda_1(n)\lambda_2(n) = \lim_{n\to\infty} n^{p+q} = 0.
\end{aligned}$$

Thus by Theorem 4.1, we get the desired results. □

The next corollary prove the hyperstability results for the inhomogeneous Drygas functional equation

$$f(x+ky)+f(x-ky) = 2f(x)+k^2f(y)+k^2f(-y)+G(x,y).$$

Corollary 4.2. *Let $\big(X,\|\cdot\|\big)$ be a normed space, $\big(Y,\|\cdot,\cdot\|\big)$ be a real 2-Banach space and $\theta\ge 0$, $s\ge 0$, $p,q\in\mathbb{R}$ such that $p+q<0$. Assume that $G: X^2\to Y$ and $f: X\to Y$ such that $f(0)=0$ and satisfies the inequality*

$$\Big\|f(x+ky)+f(x-ky)-2f(x)-k^2f(y)-k^2f(-y)-G(x,y),g(z)\Big\| \le \theta\|x\|^p\,\|y\|^q\,\|z\|^s, \tag{4.4}$$

for all $x,y,z\in X_0$ such that $x+ky\neq 0$ and $x-ky\neq 0$, where $g: X\to Y$ is a surjective mapping. If the functional equation

$$f(x+ky)+f(x-ky) = 2f(x)+k^2f(y)+k^2f(-y)+G(x,y), \tag{4.5}$$

for all $x,y\in X_0$ such that $x+ky\neq 0$ and $x-ky\neq 0$ has a solution $f_0: X\to Y$ on X_0, then f is a solution to (4.5) *on X_0.*

Proof. From (4.4) we get that the function $K: X\to Y$ defined by $K := f - f_0$ satisfies (4.3). Consequently, Corollary 4.1 implies that K is a solution to Drygas functional equation (1.3) on X_0. Therefore,

$$\begin{aligned}
&f(x+ky)+f(x-ky)-2f(x)-k^2f(y)-k^2f(-y)-G(x,y)\\
=&K(x+ky)+f_0(x+ky)+K(x-ky)+f_0(x-ky)-2K(x)-2f_0(x)\\
&-k^2K(y)-k^2f_0(y)-k^2K(-y)-k^2f_0(-y)-G(x,y)\\
=&0,
\end{aligned}$$

for all $x,y\in X_0$ such that $x+ky\neq 0$ and $x-ky\neq 0$ which means f is a solution to (4.5) on X_0. □

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