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APPROXIMATING SOLUTIONS OF MONOTONE VARIATIONAL INCLUSION, EQUILIBRIUM AND FIXED POINT PROBLEMS OF CERTAIN NONLINEAR MAPPINGS IN BANACH SPACES

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Abstract. In this paper, motivated by the works of Timnak et al. [Filomat **31**(15) (2017), 4673–4693], Ogbuisi and Izuchukwu [Numer. Funct. Anal. **40**(13) (2019)] and some other related results in literature, we introduce an iterative algorithm and employ a Bregman distance approach for approximating a zero of the sum of two monotone operators, which is also a common solution of equilibrium problem involving pseudomonotone bifunction and a fixed point problem for an infinite family of Bregman quasi-nonexpansive mappings in the framework of a reflexive Banach space. Using our iterative algorithm, we state and prove a strong convergence result for approximating a common solution of the aforementioned problems. Furthermore, we give some applications of the consequences of our main result to convex minimization problem and variational inequality problem. Lastly, we display a numerical example to show the applicability of our main result. The result presented in this paper extends and complements many related results in the literature.

1. Introduction

Let E be a reflexive Banach space with E^* its dual and Q be a nonempty closed and convex subset of E. Let $f : E \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function, then the Fenchel conjugate of f denoted as $f^* : E^* \to (-\infty, +\infty]$ is defined as

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}, \quad x^* \in E^*.$$

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For more information on Legendre functions, see [37]. Let the domain of f be denoted as $\operatorname{dom} f = \{x \in E : f(x) < +\infty\}$, hence for any $x \in \operatorname{int}(\operatorname{dom} f)$ and $y \in E$, we define the right-hand derivative of f at x in the direction of y by

$$f^0(x,y) = \lim_{t \to 0^+} \frac{f(x+ty) - f(x)}{t}.$$

Let $f : E \to (-\infty, +\infty]$ be a function, then f is said to be:

(i) essentially smooth, if the subdifferential of f denoted as ∂f is both locally bounded and single-valued on its domain;

(ii) essentially strictly convex, if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every convex subset of $\operatorname{dom} \partial f$;

(iii) Legendre, if it is both essentially smooth and essentially strictly convex. See [10, 51] for more details on Legendre functions.

The function f is said to be:

(i) Gâteaux differentiable at x if $\lim_{t \to 0^+} \frac{f(x+ty)-f(x)}{t}$ exists for any y. In this case, $f^0(x,y)$ coincides with $\bigtriangledown f(x)$ (the value of the gradient $\bigtriangledown f$ of f at x);

(ii) Gâteaux differentiable, if it is Gâteaux differentiable for any $x \in \operatorname{int}(\operatorname{dom} f)$;

(iii) Frechet differentiable at x, if its limit is attained uniformly in $||y|| = 1$.

f is said to be uniformly Frechet differentiable on a subset Q of E, if the above limit is attained uniformly for $x \in Q$ and $||y|| = 1$. The function f is said to be Legendre if it satisfies the following conditions.

(i) The $\operatorname{int}(\operatorname{dom} f)$ is nonempty, f is Gâteaux differentiable on $\operatorname{int}(\operatorname{dom} f)$ and $\operatorname{dom} \bigtriangledown f = \operatorname{int}(\operatorname{dom} f)$.

(ii) The $\operatorname{int}(\operatorname{dom} f^*)$ is nonempty, f^* is Gâteaux differentiable on $\operatorname{int}(\operatorname{dom} f^*)$ and $\operatorname{dom} \bigtriangledown f^* = \operatorname{int}(\operatorname{dom} f)$.

Let E be a Banach space and $B_s := \{z \in E : ||z|| \leq s\}$ for $s > 0$. Then, a function $f : E \to \mathbb{R}$ is said to be uniformly convex on bounded subsets of E, [55, page 203 and 221] if $\rho_s(t) > 0$ for all $s, t > 0$, where $\rho_s : [0, +\infty) \to [0, +\infty]$ is defined by

$$\rho_s(t) = \inf_{x,y \in B_s, ||x-y||=t, \alpha \in (0,1)} \frac{\alpha f(x) + (1-\alpha) f(y) - f(\alpha(x) + (1-\alpha)y)}{\alpha(1-\alpha)},$$

for all $t \geq 0$, where ρ_s denote the gauge of uniform convexity of f. The function f is also said to be uniformly smooth on bounded subsets of E [55, page 221], if $\lim_{t \downarrow 0} \frac{\sigma_s}{t}$ for all $s > 0$, where $\sigma_s : [0, +\infty) \to [0, +\infty]$ is defined by

$$\sigma_s(t) = \sup_{x \in B, y \in S_E, \alpha \in (0,1)} \frac{\alpha f(x) + (1-\alpha) ty) + (1-\alpha) g(x - \alpha t y) - g(x)}{\alpha(1-\alpha)},$$

for all $t \geq 0$. The function f is said to be uniformly convex if the function $\delta f : [0, +\infty) \to [0, +\infty)$ defined by

$$\delta f(t) := \sup \left\{ \frac{1}{2} f(x) + \frac{1}{2} f(y) - f\left(\frac{x+y}{2}\right) : ||y - x|| = t \right\},$$

satisfies $\lim_{t \downarrow 0} \frac{\delta f(t)}{t} = 0$.

Recall that the function f is said to be totally convex at a point $x \in Domf$, if the function $v_f : \text{int}(\text{dom}\, f) \times [0, +\infty) \to [0, +\infty)$ defined by

$$v_f(x,t) := \inf\{D_f(y,x) : y \in \text{int}(\text{dom}\, f), ||y-x|| = t\},$$

is positive whenever $t > 0$. For details on uniformly convex and totally convex functions, see [12, 15, 18].

Definition 1.1 ([12]). Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function $D_f : E \times E \to [0, +\infty)$ defined by

$$D_f(x,y) := f(x) - f(y) - \langle \bigtriangledown f(y), x-y\rangle$$

is called the Bregman distance with respect of f.

It is well-known that Bregman distance D_f does not satisfy the properties of a metric because D_f fail to satisfy the symmetric and triangular inequality property. However, the Bregman distance satisfies the following so-called three point identity: for any $x \in \text{dom}\, f$ and $y, z \in \text{int}(\text{dom}\, f)$

$$D_f(x,y) + D_f(y,z) - D_f(x,z) = \langle \bigtriangledown f(z) - \bigtriangledown f(y), x-y\rangle. \tag{1.1}$$

For more information on Bregman functions and Bregman distances, see [38, 45]. Let $T : Q \to Q$ be a mapping, a point $x \in Q$ is called a fixed point of T, if $Tx = x$. We denote by $F(T)$ the set of all fixed points of T. Moreso, a point $p \in Q$ is called an asymptotic fixed point of T if Q contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to+\infty} ||Tx_n - x_n|| = 0$. The notion of asymptotic fixed point was introduced by [40]. We denote by $\hat{F}(T)$ the set of asymptotic fixed points of T.

Let Q be a nonempty closed and convex subset of E. An operator $T : Q \to Q$ is said to be:

(i) Bregman relatively nonexpansive, if $F(T) \neq \emptyset$ and

$$D_f(p, Tx) \leq D_f(p,x), \quad \text{for all } p \in F(T),\ x \in Q \text{ and } F(\hat{T}) = F(T);$$

(ii) Bregman quasi-nonexpansive mapping if $F(T) \neq \emptyset$ and

$$D_f(p, Tx) \leq D_f(p,x), \quad \text{for all } x \in Q \text{ and } p \in F(T);$$

(iii) Bregman Strongly Nonexpansive (BSNE) with $\hat{F}(T) \neq \emptyset$, if

$$D_f(p, Tx) \leq D_f(p,x), \quad \text{for all } x \in C,\ p \in F(\hat{T})$$

and for any bounded sequence $\{x_n\}_{n\geq 1} \subset Q$,

$$\lim_{n\to+\infty}(D_f(p,x_n) - D_f(p, Tx_n)) = 0$$

implies that $\lim_{n\to+\infty} D_f(Tx_n, x_n) = 0$. For more information on these classes of mappings, see [29, 30].

Let $B : E \to 2^{E^*}$ be a set-valued mapping, the domain and range of B are denoted by $\text{dom}\, B = \{x \in E : Bx \neq \emptyset\}$ and $\text{ran}\, B = \cup_{x\in B} Bx$, respectively. The graph of B is denoted as $G(B) = \{(x, x^*) \in E \times E^* : x^* \in Bx\}$. Recall that B is called a monotone mapping, if for any $x, y \in \operatorname{dom} B$, we have $\xi \in Bx$ and $\zeta \in By$ implies $\langle \xi - \zeta, x - y\rangle \geq 0$. B is said to be maximal monotone if it is monotone and its graph is not contained in the graph of any other monotone mapping. Let $f : E \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function and B be a maximal monotone mapping from E to E^*. For any $\lambda > 0$, the mapping $Res^f_{\lambda B} : E \to \operatorname{dom} B$ defined by

$$Res^f_{\lambda B} = (\bigtriangledown f + \lambda B)^{-1} \circ \bigtriangledown f,$$

is called the f-resolvent of B. It is well known that $B^{-1}(0) = F(Res^f_{\lambda B})$ for each $\lambda > 0$.

Let Q be a nonempty closed and convex subset of a reflexive Banach space E, the mapping $A : E \to 2^{E^*}$ is called Bregman Inverse Strongly Monotone (BISM) on the set Q if

$$Q \cap (\operatorname{dom} f) \cap (\operatorname{int}(\operatorname{dom} f)) \neq \emptyset,$$

and for any $x, y \in Q \cap (\operatorname{int}(\operatorname{dom} f))$, $\xi \in Ax$ and $\zeta \in Ay$, we have that

$$\langle \xi - \zeta, \bigtriangledown f^*(\bigtriangledown f(x) - \xi) - \bigtriangledown f^*(\bigtriangledown f(y) - \zeta)\rangle \geq 0.$$

Let $A : E \to E^*$ be a single-valued monotone mapping and $B : E \to 2^{E^*}$ be a multivalued monotone mapping. Then, the Monotone Variational Inclusion Problem (MVIP) (also known as the problem of finding a zero of sum of two monotone mappings) is to find $x \in E$ such that

$$0^* \in A(x) + B(x). \tag{1.2}$$

We denote by Ω, the solution set of problem (1.2).

It is well known that many interesting problems arising from mechanics, economics, applied sciences, optimization such as equilibrium and variational inequality problems can be solved using MVIP.

Suppose $A = 0$ in (1.2), we obtain the following Monotone Inclusion Problem (MIP), which is to find $x \in E$ such that

$$0^* \in B(x). \tag{1.3}$$

Many algorithms have been introduced by several authors for solving the MVIP and related optimization problems in Hilbert, Banach, Hadamard and p-uniformly convex metric spaces, see [1,7–9,17,28,31,34,35,44,53]. For instance, Reich and Sabach [27,42] introduced some iterative algorithms and proved two strong convergence results for approximating a common solution of a finite family of MIP (1.3) in a reflexive Banach space. Recently, Timnak et al. [51] introduced a new Halpern-type iterative scheme for finding a common zero of finitely many maximal monotone mappings in a reflexive Banach space and prove the following strong convergence theorem.

Theorem 1.1. *Let E be a reflexive Banach space and $f : E \to \mathbb{R}$ be a strongly coercive Bregman function which is bounded on bounded subsets and uniformly convex and uniformly smooth on bounded subset of E. Let $A_i : E \to 2^{E^*}$, $i = 1, 2, \ldots,$ be*

N maximal monotone operators such that $Z := \cap_{i=1}^{N} A_i^{-1}(0^*) \neq \emptyset$. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ be two sequences in $(0,1)$ satisfying the following control conditions:

(i) $\lim_{n\to+\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{+\infty} \alpha_n = \infty$*;*

(ii) $0 < \liminf_{n\to+\infty} \beta_n \leq \limsup_{n\to+\infty} \beta_n < 1$*.*

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by

$$\begin{cases} u \in E,\ x_1 \in E \text{ chosen arbitrarily}, \\ y_n = \nabla f^*[\beta_n \nabla f(x_n) + (1-\beta_n) \nabla f(Res^f_{r_N A_N}) \cdots (Res^f_{r_1 A_1}(x_n))], \\ x_{n+1} = \nabla f^*[\alpha_n \nabla f(u) + (1-\alpha_n) \nabla f(y_n)], \end{cases} \tag{1.4}$$

for $n \in \mathbb{N}$, where ∇f is the gradient of f. If $r_i > 0$, for each $i = 1, 2, \ldots, N$, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined in (1.4) converges strongly to $proj^f_Z u$ as $n \to +\infty$.

Very recently, Ogbuisi and Izuchukwu [33] introduced the following iterative algorithm to obtain a strong convergence result for approximating a zero of sum of two maximal monotone operators which is also a fixed point of a Bregman strongly nonexpansive mapping in the framework of a reflexive Banach space. Let $u, x_0 \in Q$ be arbitrary and the sequence $\{x_n\}$ be generated by

$$\begin{cases} Q_0 = Q, \\ y_n = \nabla f^*(\alpha_n \nabla f(u) + \beta_n \nabla f(x_n) + \gamma_n \nabla f(T(x_n))), \\ u_n = (Res^f_{\lambda B} \circ A^f_\lambda) y_n, \\ Q_{n+1} = \{z \in Q_n : D_f(z, u_n) \leq \alpha_n D_f(z, u) + (1-\alpha_n) D_f(z, x_n)\}, \\ x_{n+1} = P^f_{Q_{n+1}}(x_0), \quad n \geq 0, \end{cases} \tag{1.5}$$

with conditions $\lim_{n\to\infty} \alpha_n = 0,\ \alpha_n + \beta_n + \gamma_n = 1$ and $0 < a < \beta_n,\ \gamma_n < b < 1$. Then, $\{x_n\}$ converges strongly to $P^f_{F(T)\cap\Gamma} x_0$, where $\Gamma := (A+B)^{-1}(0)$.

Equilibrium Problem (EP) involving monotone bifunctions and related optimization problems have been studied extensively by many authors, (see [2, 3, 11, 19, 22, 23, 36, 39, 46, 47, 49, 50] and other references contained in). Very recently, Eskandani et al. [18] introduced an EP involving a pseudomonotone bifunction in the framework of a reflexive Banach space.

Let C be a nonempty closed and convex subset of a reflexive Banach space E, the EP for a bifunction $g : C \times C \to \mathbb{R}$ satisfying condition $g(x,x) = 0$ for every $x \in C$ is defined as follows: find $x^* \in C$ such that

$$g(x^*, y) \geq 0, \quad \text{for all } y \in C. \tag{1.6}$$

We denote by Δ, the set of solutions of (1.6).

Recall that a bifunction g is called monotone on C, if for all $x, y \in C$, $g(x,y) + g(y,x) \leq 0$ and the mapping $A : C \to E^*$ is pseudomonotone if and only if the bifunction $g(x,y) = \langle A(x), y - x\rangle$ is pseudomonotone on C (see [18]). To solve an EP involving a pseudomonotone bifunction, we need the following assumptions:

L1. g is pseudomonotone, i.e., for all $x, y \in C$:

$$g(x,y) \geq 0 \quad \text{implies} \quad g(y,x) \leq 0;$$

L2. g is Bregman-Lipschitz type continuous, i.e., there exist two positive constants c_1, c_2 such that

$$g(x,y)+g(y,z)\geq g(x,z)-c_1D_f(y,x)-c_2D_f(z,y),\quad \text{for all } x,y,z\in C;$$

L3. g is weakly continuous on $C\times C$, i.e., if $x,y\in C$ and $\{x_n\}$ and $\{y_n\}$ are two sequences in C converging weakly to x and y respectively, then $g(x_n,y_n)\to g(x,y)$;

L4. $g(x,\cdot)$ is convex, lower semicontinuous and subdifferential on C for every fixed $x\in C$;

L5. for each $x,y,z\in C$, $\limsup_{t\downarrow 0} g(tx+(1-t)y,z)\leq g(y,z)$.

Using assumptions L1-L5, Eskandani et al. [18] introduced an hybrid iterative algorithm to approximate a common element of the set of solutions of finite family of EPs involving pseudomonotone bifunctions and the set of common fixed points for a finite family of Bregman relatively nonexpansive mappings in the framework of reflexive Banach spaces. They proved the following strong convergence theorem.

Theorem 1.2. *Let C be a nonempty closed convex subset of a reflexive Banach space E and $f: E\to\mathbb{R}$ be a super coercive Legendre function which is bounded, uniformly Frechet differentiable and totally convex on bounded subset of E. Let for $i=1,2,\ldots,N, g_i: C\times C\to\mathbb{R}$ be a bifunction satisfying* L1-L5. *Assume that for each $1\leq r\leq M$, $T_r: C\to CB(C)$ be a multivalued Bregman relatively nonexpansive mapping, such that $\Gamma=(\cap_{r=1}^M F(T_r))\cap(\cap_{i=1}^N EP(g_i))\neq\emptyset$. Suppose that $\{x_n\}$ is a sequence generated by $x_1\in C$ and*

$$\begin{cases} w_n^i=\operatorname{argmin}\{\lambda_n g_i(x_n,w)+D_f(w,x_n): w\in C\}, \quad i=1,\ldots,N,\\ z_n^i=\operatorname{argmin}\{\lambda_n g_i(w_n^i,z)+D_f(z,x_n): z\in C\}, \quad i=1,\ldots,N,\\ i_n\in \operatorname{Argmax}\{D_f(z_n^i,x_n), i=1,2,\ldots,N\}, \quad \overline{z_n}:=z_n^{i_n},\\ y_n=\bigtriangledown f^*(\beta_{n,0}\bigtriangledown f(\overline{z_n})+\sum_{r=1}^M\beta_{n,r}\bigtriangledown f(z_{n,r})), \quad z_{n,r}\in T_r\overline{z_n},\\ x_{n+1}=\overleftarrow{P_C^f}(\bigtriangledown f^*(\alpha_n\bigtriangledown f(u_n)+(1-\alpha_n)\bigtriangledown f(z_{n,r})), \end{cases}$$

where $CB(C)$ denotes the family of a nonempty, closed and convex subsets of C, $\{\alpha_n\},\{\beta_{n,r}\},\{\lambda_n\}$ and $\{u_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\}\subset(0,1)$, $\lim_{n\to+\infty}\alpha_n=0$, $\sum_{n=1}^{+\infty}\alpha_n=\infty$;

(ii) $\{\beta_{n,r}\}\subset(0,1)$, $\sum_{r=0}^M\beta_{n,r}=1$, $\liminf_{n\to+\infty}\beta_{n,0}\beta_{n,r}>0$ for all $1\leq r\leq M$ and $n\in\mathbb{N}$;

(iii) $\{\lambda_n\}\subset[a,b]\subset(0,p)$, where $p=\min\{\frac{1}{c_1},\frac{1}{c_2}\}$, $c_1=\max_{1\leq i\leq N}c_{i,1}$;

$c_2=\max_{1\leq i\leq N}c_{i,2}$ and $c_{i,1},c_{i,2}$ are the Bregman-Lipschitz coefficients of g_i for all $1\leq i\leq N$;

(iv) $\{u_n\}\subset E, \lim_{n\to+\infty}u_n=u$ for some $u\in E$.

Then, the sequence $\{x_n\}$ converges strongly to $\overleftarrow{P_\Gamma^f}u$.

Remark 1.1. We will like to emphasize that approximating a common solution of MVIP and EP have some possible applications to mathematical models whose constraints can be expressed as MVIP and EP. In fact, this happens in practical problems like signal processing, network resource allocation, image recovery, to mention a few, (see [20]).

Inspired by the works of Eskandani et al. [18], Timnak et al. [51], Ogbuisi and Izuchukwu [33] and other related results in literature, we introduce a Halpern type iteration process to approximate a zero of sum of two monotone operators, which is also a common solution of equilibrium problem involving pseudomonotone bifunction and fixed point problem for an infinite family of Bregman quasi-nonexpansive mappings in the framework of a reflexive Banach space. We state and prove a strong convergence result for finding a common solution of the aforementioned problems and give applications to the consequences of our main results. Finally, we display a numerical example to show the applicability of our main result. The result presented in this paper improve and generalize some known results in the literature.

2. Preliminaries

We give some known and useful results which will be needed in the proof of our main theorem. In the sequel, we denote strong and weak convergence by $\to$ and $\rightharpoonup$, respectively.

Definition 2.1. A function $f : E \to \mathbb{R}$ is said to be super coercive if

$$\lim_{x\to+\infty} \frac{f(x)}{||x||} = +\infty$$

and strongly coercive if

$$\lim_{||x_n||\to+\infty} \frac{f(x_n)}{||x_n||} = +\infty.$$

Definition 2.2. Let Q be a nonempty subset of a real Banach space E and $\{T_n\}_{n=1}^{+\infty}$ a sequence of mappings from Q into E such that $\cap_{n=1}^{+\infty} F(T_n) \neq \emptyset$. Then $\{T_n\}_{n=1}^{+\infty}$ is said to satisfy the AKTT-condition if for each bounded subset K of Q,

$$\sum_{n=1}^{+\infty} \sup\{||T_{n+1}z - T_n z|| : z \in K\} < +\infty.$$

Lemma 2.1 ([6]). *Let C be a nonempty subset of a real Banach space E and $\{T_n\}_{n=1}^{\infty}$ be a sequence of mappings from C into E which satisfies the AKTT condition. Then, for each $x \in C$, $\{T_n x\}_{n=1}^{+\infty}$ is convergent. Furthermore, if we define a mapping $T : C \to E$ by*

$$Tx := \liminf_{n\to+\infty} T_n x, \quad \text{for all } x \in C,$$

then, for each bounded subset K of C,

$$\limsup_{n\to+\infty} \{||T_n z - Tz|| : z \in K\} = 0.$$

In this sequel, we write that $(\{T_n\}_{n=1}^{+\infty}, T)$ satisfies the AKTT-condition if $\{T_n\}_{n=1}^{+\infty}$ satisfies the AKTT-condition and $F(T) = \cap_{n=1}^{+\infty} F(T_n)$.

Lemma 2.2 ([51])**.** *Let E be a Banach space, $s > 0$ a constant, ρ_s the gauge of uniform convexity of g and $g : E \to \mathbb{R}$ a convex function which is uniformly convex on bounded subset of E. Then*

(i) for any $x, y \in B_s$ and $\alpha \in (0,1)$, we have

$$g(\alpha x + (1-\alpha)y) \leq \alpha g(x) + (1-\alpha)g(y) - \alpha(1-\alpha)\rho_s(||x-y||),$$

(ii) for any $x, y \in B_s$

$$\rho_s(||x-y||) \leq D_g(x,y).$$

Here, $B_s := \{z \in E : ||z|| \leq s\}$.

Lemma 2.3 ([14])**.** *Let E be a reflexive Banach space, $f : E \to \mathbb{R}$ a strongly coercive Bregman function and V a function defined by*

$$V(x, x^*) = f(x) - \langle x, x^* \rangle + f^*(x^*), \quad x \in E,\, x^* \in E^*.$$

The following assertions also hold:

$$\begin{aligned} D_f(x, \bigtriangledown f^*(x^*)) =& V(x,x^*), \quad \text{for all } x \in E \text{ and } x^* \in E^*, \\ V(x,x^*) + \langle \bigtriangledown g^*(x^*) - x, y^* \rangle \leq& V(x, x^* + y^*), \quad \text{for all } x \in E \text{ and } x^*, y^* \in E^*. \end{aligned}$$

Lemma 2.4 ([14])**.** *Let E be a Banach space and $f : E \to \mathbb{R}$ a Gâteaux differentiable function which is uniformly convex on bounded subsets of E. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be bounded sequences in E. Then*

$$\lim_{n\to+\infty} D_f(y_n, x_n) = 0 \Rightarrow \lim_{n\to+\infty} ||y_n - x_n|| = 0.$$

Lemma 2.5 ([18])**.** *Let C be a nonempty closed convex subset of a reflexive Banach space E and $f : E \to \mathbb{R}$ a Legendre and super coercive function. Suppose that $g : C \times C \to \mathbb{R}$ is a bifunction satisfying* L1-L4*. For arbitrary sequence $\{x_n\} \subset C$ and $\{\lambda_n\} \subset (0, +\infty)$, let $\{w_n\}$ and $\{z_n\}$ be sequences generated by*

$$\begin{cases} w_n = \operatorname{argmin}_{y\in C}\{\lambda_n g(x_n, y) + D_f(y, x_n)\}, \\ z_n = \operatorname{argmin}_{y\in C}\{\lambda_n g(w_n, y) + D_f(y, x_n)\}. \end{cases}$$

Then, for all $x^* \in \Delta$, we have that

$$D_f(x^*, z_n) \leq D_f(x^*, x_n) - (1 - \lambda_n c_1)D_f(w_n, x_n) - (1 - \lambda_n c_2)D_f(z_n, w_n),$$

where c_1 and c_2 are the Bregman-Lipschitz coefficients of g.

Lemma 2.6 ([42])**.** *Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

Lemma 2.7 ([55])**.** *Let $f : E \to \mathbb{R}$ be a continuous convex function which is bounded on bounded subsets of E. Then, the following are equivalent:*

(i) f is super coercive and uniformly convex on bounded subset of E;

(ii) $\operatorname{dom} f^* = E^*$, f^* is bounded and uniformly smooth on bounded subsets of E^*;

(iii) $\operatorname{dom} f^* = E^*, f^*$ *is Fréchet differentiable and* $\bigtriangledown f^*$ *is uniformly norm-to-norm continuous on bounded subset of* E^*.

Lemma 2.8 ([55])**.** *Let* $f : E \to \mathbb{R}$ *be a continuous convex function which is super coercive. Then, the following are equivalent:*

(i) f *is bounded and uniformly smooth on bounded subsets of* E*;*

(ii) f *is Fréchet differentiable and* $\bigtriangledown f$ *is uniformly norm-to-norm continuous on bounded subset of* E*;*

(iii) $\operatorname{dom} f^* = E^*$, f^* *is super coercive and uniformly convex on bounded subsets of* E^*.

Lemma 2.9 ([33])**.** *Let* $B : E \to 2^{E^*}$ *be a maximal monotone operator and* $A : E \to E^*$ *be a BISM mapping such that* $(A+B)^{-1}(0^*) \neq \emptyset$. *Let* $f : E \to \mathbb{R}$ *be a Legendre function, which is uniformly Fréchet differentiable and bounded on bounded subset of* E. *Then*

$$D_f(u, Res^f_{\lambda B} \circ A^f(x)) + D_f(Res^f_{\lambda B}(x), x) \leq D_f(u, x),$$

for any $u \in (A+B)^{-1}(0^*)$, $x \in E$ *and* $\lambda > 0$.

Lemma 2.10 ([33])**.** *Let* $B : E \to 2^{E^*}$ *be a maximal monotone operator and* $A : E \to E^*$ *be a BISM mapping such that* $(A+B)^{-1}(0^*) \neq \emptyset$. *Let* $f : E \to \mathbb{R}$ *be a Legendre function, which is uniformly Fréchet differentiable and bounded on bounded subset of* E. *Then*

(i) $(A+B)^{-1}(0^*) = F(Res^f_{\lambda B} \circ A^f_\lambda)$;

(ii) $Res^f_{\lambda B} \circ A^f_\lambda$ *is a BSNE operator with* $F(Res^f_{\lambda B} \circ A^f_\lambda) = \hat{F}(Res^f_{\lambda B} \circ A^f_\lambda)$.

Definition 2.3. Let E be a reflexive Banach space and C a nonempty closed and convex subset of E. A Bregman projection of $x \in \operatorname{int}(\operatorname{dom} f)$ onto $C \subset \operatorname{int}(\operatorname{dom} f)$ is the unique vector $P^f_C(x) \in C$ satisfying

$$D_f(P^f_C(x), x) = \inf\{D_f(y, x) : y \in C\}.$$

Lemma 2.11 ([41])**.** *Let* C *be a nonempty closed and convex subset of a reflexive Banach space* E *and* $x \in E$. *Let* $f : E \to \mathbb{R}$ *be a Gâteaux differentiable and totally convex function. Then*

(i) $z = P^f_C(x)$ *if and only if* $\langle \bigtriangledown f(x) - \bigtriangledown f(z), y - z\rangle \leq 0$ *for all* $y \in C$*;*

(ii) $D_f(y, P^f_C(x)) + D_f(P^f_C(x), x) \leq D_f(y, x)$ *for all* $y \in C$.

Lemma 2.12 ([52])**.** *Let* C *be a nonempty convex subset of a reflexive Banach space* E *and* $f : C \to \mathbb{R}$ *be a convex and subdifferential function on* C. *Then* f *attains its minimum at* $x \in C$ *if and only if* $0 \in \partial f(x) + N_C(x)$, *where* $N_C(x)$ *is the normal cone of* C *at* x, *that is*

$$N_C(x) := \{x^* \in E^* : \langle x - z, x^*\rangle \geq 0 \textit{ for all } z \in C\}.$$

Lemma 2.13 ([16]). *If f and g are two convex functions on E such that there is a point $x_0 \in \operatorname{dom} f \cap \operatorname{dom} g$ where f is continuous, then*

$$\partial(f+g)(x) = \partial f(x) + \partial g(x), \quad \text{for all } x \in E. \tag{2.1}$$

Lemma 2.14 ([48]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\sigma_n)a_n + \sigma_n\delta_n, \quad n > 0,$$

where $\{\sigma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a real sequence such that
(i) $\sum_{n=1}^{+\infty} \sigma_n = +\infty$;
(ii) $\limsup_{n\to+\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{+\infty} |\sigma_n\delta_n| < +\infty$.
Then $\lim_{n\to+\infty} a_n = 0$.

3. Main Results

In what follows, Ω and Δ denote the solution set of MVIP (1.2) and EP (1.6) respectively.

Algorithm 3.1. Choose $u, x_1 \in E$. Assume that the control parameters $\{\mu_n\}$, $\{\beta_n\}$ and $\{\alpha_n\}$ satisfy the following conditions:
(i) $\alpha_n \in (0,1)$, $\lim_{n\to+\infty} \alpha_n = 0$ and $\sum_{n=1}^{+\infty} \alpha_n = +\infty$;
(ii) $\beta_n \in (0,1)$ and $0 < \liminf_{n\to+\infty} \beta_n \leq \limsup_{n\to+\infty} \beta_n < 1$;
(iii) $0 < \underline{\mu} \leq \mu_n \leq \overline{\mu} < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$, where c_1, c_2 are positive constants.
Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} u_n = \nabla f^*(\beta_n \nabla f(x_n) + (1-\beta_n) \nabla f(T_n x_n)), \\ y_n = (Res^f_{\lambda B} \circ A^f_\lambda) u_n, \\ z_n = \operatorname{argmin}_{a\in Q}\{\mu_n g(y_n, a) + D_f(a, y_n)\}, \\ w_n = \operatorname{argmin}_{a\in Q}\{\mu_n g(z_n, a) + D_f(a, y_n)\}, \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n) \nabla f(w_n)), \quad n \geq 1. \end{cases} \tag{3.1}$$

Theorem 3.2. *Let E be a reflexive Banach space with E^* its dual and $Q \subseteq E$ a nonempty closed convex set. For $n \in \mathbb{N}$, let $T_n : E \to E$ be an infinite family of Bregman quasi-nonexpansive mapping such that $(\{T_n\}_{n=1}^{+\infty}, T)$ satisfy the AKTT-condition and $F(T) = \hat{F}(T)$. Let $A : E \to E^*$ be a BISM mapping, $B : E \to 2^{E^*}$ a maximal monotone operator and $g : Q \times Q \to \mathbb{R}$ a bifunction satisfying* L1-L5. *Assume that $f : E \to \mathbb{R}$ is a strongly coercive Legendre function, which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E such that $Q \subset \operatorname{int}(\operatorname{dom} f)$ with $\Gamma := \cap_{n=1}^{+\infty} F(T_n) \cap \Omega \cap \Delta \neq \emptyset$. Then, the sequence $\{x_n\}$ generated by Algorithm* 3.1 *converges strongly to $v = P^f_\Gamma u$, where P^f_Γ is the Bregman projection from E to Γ.*

Proof. Let $p \in \Gamma$, then we have from (3.1), Lemma 2.5 and Lemma 2.10 *(ii)* that

$$D_f(p, w_n) \leq D_f(p, y_n) - (1-\mu_n c_1)D_f(z_n, y_n) - (1-\mu_n c_2)D_f(w_n, z_n)$$

$$
\begin{aligned}
&= D_f(p, Res^f_{\lambda B} \circ A^f_\lambda(u_n)) - (1-\mu_n c_1) D_f(z_n, y_n) - (1-\mu_n c_2) D_f(w_n, z_n) \\
&\leq D_f(p, u_n) - (1-\mu_n c_1) D_f(z_n, y_n) - (1-\mu_n c_2) D_f(w_n, z_n) \\
&= D_f(p, \bigtriangledown f^*(\beta_n \bigtriangledown f(x_n) + (1-\beta_n) \bigtriangledown f(T_n x_n))) \\
&\quad - (1-\mu_n c_1) D_f(z_n, y_n) - (1-\mu_n c_2) D_f(w_n, z_n) \\
&\leq \beta_n D_f(p, x_n) + (1-\beta_n) D_f(p, T_n x_n) \\
&\quad - (1-\mu_n c_1) D_f(z_n, y_n) - (1-\mu_n c_2) D_f(w_n, z_n) \\
&\leq \beta_n D_f(p, x_n) + (1-\beta_n) D_f(p, x_n) \\
&\quad - (1-\mu_n c_1) D_f(z_n, y_n) - (1-\mu_n c_2) D_f(w_n, z_n) \\
&\leq D_f(p, x_n). \qquad (3.2)
\end{aligned}
$$

Now, we conclude from (3.1) and (3.2) that

$$
\begin{aligned}
D_f(p, x_{n+1}) &= D_f(p, \bigtriangledown f^*(\alpha_n \bigtriangledown f(u) + (1-\alpha_n) \bigtriangledown f(w_n))) \\
&\leq \alpha_n D_f(p, u) + (1-\alpha_n) D_f(p, w_n) \\
&\leq \alpha_n D_f(p, u) + (1-\alpha_n) D_f(p, u_n) \\
&\leq \alpha_n D_f(p, u) + (1-\alpha_n) D_f(p, x_n) \\
&\leq \max\Big\{ D_f(p, u), D_f(p, x_n) \Big\} \\
&\ \ \vdots \\
&\leq \max\Big\{ D_f(p, u), D_f(p, x_1) \Big\}. \qquad (3.3)
\end{aligned}
$$

From Lemma 2.8, we have that f^* is bounded on bounded subset of E^*. Hence, $\bigtriangledown f^*$ is also bounded on bounded subset of E^*. From Lemma 2.6, the following sequences $\{x_n\}_{n=1}^{+\infty}$, $\{(T_n x_n)\}_{n=1}^{+\infty}$, $\{(\bigtriangledown f^* u_n)\}_{n=1}^{+\infty}$, $\{(\bigtriangledown f^* w_n)\}_{n=1}^{+\infty}$, $\{(\bigtriangledown f^* z_n)\}_{n=1}^{+\infty}$ and $\{(\bigtriangledown f^* y_n)\}_{n=1}^{+\infty}$ are all bounded. In view of Lemma 2.7 and Lemma 2.8, dom $f^* = E^*$ and f^* is super coercive and uniformly convex on bounded subset of E^*. Let $s \geq \sup\{||x_n||, || \bigtriangledown (T_n x_n)|| : n \in \mathbb{N}\}$ be large enough and let $\rho_s^* : [0, +\infty) \to [0, +\infty)$ be the gauge of uniform convexity of f^*. Now, we have from Lemma 2.2 and (3.1) that

$$
\begin{aligned}
D_f(p, u_n) &= D_f(p, \bigtriangledown f^*(\beta_n \bigtriangledown f(x_n) + (1-\beta_n) \bigtriangledown f(T_n x_n))) \\
&= f(p) + f(\beta_n \bigtriangledown f(x_n) + (1-\beta_n) \bigtriangledown f(T_n x_n)) \\
&\quad - \langle p, \beta_n \bigtriangledown f(x_n) + (1-\beta_n) \bigtriangledown f(T_n x_n) \rangle \\
&\leq \beta_n f(p) + (1-\beta_n) f(p) + \beta_n f^*(\bigtriangledown f(x_n)) + (1-\beta_n) f^*(\bigtriangledown f(T_n x_n) \\
&\quad - \beta_n (1-\beta_n) \rho_s^*(|| \bigtriangledown f(x_n) - \bigtriangledown f(T_n x_n)||) \\
&\quad - \langle p, \beta_n \bigtriangledown f(x_n) + (1-\beta_n) \bigtriangledown f(T_n x_n) \rangle \\
&\leq \beta_n D_f(p, x_n) + (1-\beta_n) D_f(p, T_n x_n) \\
&\quad - \beta_n (1-\beta_n) \rho_s^*(|| \bigtriangledown f(x_n) - \bigtriangledown f(T_n x_n)||) \\
&\leq D_f(p, x_n) - \beta_n (1-\beta_n) \rho_s^*(|| \bigtriangledown f(x_n) - \bigtriangledown f(T_n x_n)||). \qquad (3.4)
\end{aligned}
$$

From (3.1), (3.2) and (3.4) and Lemma 2.3, we obtain that

$$\begin{aligned}
D_f(p, x_{n+1}) =& D_f(p, \triangledown f^*(\alpha_n \triangledown f(u) + (1-\alpha_n) \triangledown f(w_n))) \\
=& V_p(p, \alpha_n \triangledown f(u) + (1-\alpha_n) \triangledown f(w_n)) \\
\leq& V_p(p, \alpha_n \triangledown f(u) + (1-\alpha_n) \triangledown f(w_n)) - \alpha_n(\triangledown f(u) - \triangledown f(p)) \\
&- \langle \triangledown f^*(\alpha_n \triangledown f(u) + (1-\alpha_n) \triangledown f(w_n)) - p, -\alpha_n(\triangledown f(u) - \triangledown f(p))\rangle \\
=& V(p, \alpha_n \triangledown f(u) + (1-\alpha_n) \triangledown f(w_n)) + \alpha_n \langle x_{n+1} - p, \triangledown f(u) - \triangledown f(p)\rangle \\
=& D_f(p, \triangledown f^*(\alpha_n \triangledown f(p) + (1-\alpha_n) \triangledown f(w_n))) \\
&+ \alpha_n \langle x_{n+1} - p, \triangledown f(u) - \triangledown f(p)\rangle \\
\leq& \alpha_n D_f(p,p) + (1-\alpha_n) D_f(p, w_n) + \alpha_n \langle x_{n+1} - p, \triangledown f(u) - \triangledown f(p)\rangle \\
\leq& (1-\alpha_n) D_f(p, x_n) - (1-\alpha_n)(1-\mu_n c_1) D_f(z_n, y_n) \\
&- (1-\alpha_n)(1-\mu_n c_2) D_f(w_n, z_n) \\
&- \beta_n(1-\beta_n)\rho_s^*(|| \triangledown f(x_n) - \triangledown f(T_n x_n))|| \\
&+ \alpha_n \langle x_{n+1} - p, \triangledown f(u) - \triangledown f(p)\rangle.
\end{aligned} \tag{3.5}$$

We now consider two cases to prove a strong convergence result.

CASE 1. Assume that the sequence $\{D_f(p, x_n)\}$ is a monotone decreasing sequence, then $\{D_f(p, x_n)\}$ is convergent. Clearly, we have that $D_f(p, x_n) - D_f(p, x_{n+1}) \to 0$, as $n \to +\infty$.

Now, we have from (3.5), Lemma 2.4, conditions (i)-(iii) of (3.1) that

$$\lim_{n \to +\infty} ||z_n - y_n|| = 0. \tag{3.6}$$

Also,

$$\lim_{n \to +\infty} ||w_n - z_n|| = 0 \tag{3.7}$$

and

$$\lim_{n \to +\infty} \rho_s^*(|| \triangledown f(x_n) - \triangledown f(T_n x_n)||) = 0. \tag{3.8}$$

Applying the property of ρ_s^* on (3.8), we obtain that

$$\lim_{n \to +\infty} || \triangledown f(x_n) - \triangledown f(T_n x_n)|| = 0.$$

Since $\triangledown f^*$ is uniformly norm-to-norm continuous on bounded subset of E^*, we obtain that

$$\lim_{n \to +\infty} ||x_n - T_n x_n|| = 0. \tag{3.9}$$

Since $(\{T_n\}_{n=1}^{+\infty}, T)$ satisfies the AKTT condition, we then conclude that

$$\begin{aligned}
||x_n - Tx_n|| &\leq ||x_n - T_n x_n|| + ||T_n x_n - Tx_n|| \\
&\leq ||x_n - T_n x_n|| + \sup\{||T_n x - Tx|| : x \in K\},
\end{aligned} \tag{3.10}$$

where $K = rB = \{x \in E : ||x|| \leq r\}$. By applying Lemma 2.1, (3.9) and (3.10), we have that

$$\lim_{n\to+\infty} ||x_n - Tx_n|| = 0. \tag{3.11}$$

From (3.9), the boundedness of $\bigtriangledown f$ and the uniform continuity of f on bounded subsets of E, we have that

$$D_f(T_nx_n, x_n) = f(T_nx_n) - f(x_n) - \langle T_nx_n - x_n, \bigtriangledown fx_n\rangle \to 0, \quad n \to +\infty. \tag{3.12}$$

From Lemma 2.9, (3.1), (3.2) and (3.3), we have that

$$\begin{aligned} D_f(y_n, u_n) &\leq D_f(p, u_n) - D_f(p, y_n) \\ &\leq \beta_n D_f(p, x_n) + (1-\beta_n)D_f(p, T_nx_n) - D_f(p, u_n) \\ &\leq \beta_n D_f(p, x_n) + (1-\beta_n)D_f(p, x_n) - D_f(p, w_n) \\ &= D_f(p, x_n) + \alpha_n D_f(p, u) - D_f(p, x_{n+1}). \end{aligned}$$

Using condition (i) and Lemma 2.4, we have that

$$\lim_{n\to+\infty} ||y_n - u_n|| = 0. \tag{3.13}$$

From (3.1) and (3.12), we have that

$$\begin{aligned} D_f(x_n, u_n) &= D_f(x_n, \bigtriangledown f^*(\beta_n f(x_n) + (1-\beta_n) \bigtriangledown f(T_nx_n))) \\ &\leq \beta_n D_f(x_n, x_n) + (1-\beta_n)D_f(x_n, T_nx_n) \to 0, \quad n \to +\infty. \end{aligned}$$

Hence, we have from Lemma 2.4 that

$$\lim_{n\to+\infty} ||u_n - x_n|| = 0. \tag{3.14}$$

From (3.13) and (3.14), we have that

$$\lim_{n\to+\infty} ||y_n - x_n|| = 0. \tag{3.15}$$

From (3.6) and (3.15), we obtain that

$$\lim_{n\to+\infty} ||z_n - x_n|| = 0. \tag{3.16}$$

From (3.7) and (3.16), we have that

$$\lim_{n\to+\infty} ||w_n - x_n|| = 0. \tag{3.17}$$

Using (3.1), we have that

$$\begin{aligned} D_f(w_n, x_{n+1}) &= D_f(w_n, \bigtriangledown f^*(\alpha_n \bigtriangledown f(u) + (1-\alpha_n) \bigtriangledown f(w_n))) \\ &\leq \alpha_n D_f(w_n, u) + (1-\alpha_n)D_f(w_n, w_n) \to 0, \quad n \to +\infty. \end{aligned}$$

We have from Lemma 2.4 that

$$\lim_{n\to+\infty} ||x_{n+1} - w_n|| = 0. \tag{3.18}$$

We conclude from (3.17) and (3.18) that

$$\lim_{n\to+\infty} ||x_{n+1} - x_n|| = 0. \tag{3.19}$$

Since $\{x_n\}$ is bounded in E, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$. From (3.14), (3.15), (3.16) and (3.17), we have that $\{u_{n_k}\}$, $\{z_{n_k}\}$, $\{y_{n_k}\}$ and $\{w_{n_k}\}$ converges weakly to x^*. Also, from (3.11), we obtain that $x^* \in F\hat{(T)} = F(T)$. Next, we show that $x^* \in \Omega$. Since

$$z_n = \operatorname{argmin}_{a\in Q}\{\mu_n g(y_n, a) + D_f(a, y_n)\},$$

then by Lemma 2.12 and 2.13 and condition L4, we obtain that

$$0 \in \partial \mu_n g(y_n, z_n) + \bigtriangledown D_f(z_n, y_n) + N_C(z_n).$$

Therefore, there exist $\bar{\theta_n} \in \partial g(y_n, z_n)$ and $\theta_n \in N_C(z_n)$ such that

$$\mu_n \bar{\theta_n} + \bigtriangledown f(z_n) - \bigtriangledown f(y_n) + \theta_n = 0. \tag{3.20}$$

Observe that $\theta_n \in N_C(z_n)$ and $\langle q - z_n, \theta_n\rangle \leq 0$ for all $q \in Q$. Since $\bar{\theta_n} \in \partial g(y_n, z_n)$, we have

$$g(y_n, q) - g(y_n, z_n) \geq \langle q - z_n, \theta_n\rangle, \tag{3.21}$$

for all $q \in Q$. Using (3.20) and (3.21), we obtain that

$$\mu_n[g(y_n, q) - g(y_n, z_n)] \geq \langle z_n - q, \nabla f(z_n) - \bigtriangledown f(y_n)\rangle, \quad \text{for all } q \in C,$$

this implies that

$$[g(y_n, q) - g(y_n, z_n)] \geq \frac{1}{\mu_n}\langle z_n - q, \nabla f(z_n) - \bigtriangledown f(y_n)\rangle, \quad \text{for all } q \in C. \tag{3.22}$$

Using (3.6), (3.15), (3.16), condition L3 and letting $n \to \infty$ in (3.22), we conclude that $g(x^*, q) \geq 0$, for all $q \in Q$. Hence $x^* \in \Delta$. We will also show that $0^* \in A(x^*) + B(x^*)$. From (3.13) and Lemma 2.10, we have that $x^* \in \hat{F}(Res^f_{\lambda B} \circ A^f_\lambda) = F(Res^f_{\lambda B} \circ A^f_\lambda) = (A + B)^{-1}(0^*)$. That is $0^* \in A(x^*) + B(x^*)$. Hence, $x^* \in \Omega$. We conclude that $x^* \in \Gamma$. We prove that $\{x_n\}$ converges strongly to $v = P^f_\Gamma u$.

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$ and

$$\limsup_{n\to+\infty}\langle x_{n+1} - v, \bigtriangledown f(u) - \bigtriangledown f(v)\rangle = \lim_{k\to\infty}\langle x_{n_k+1} - v, \bigtriangledown f(u) - \bigtriangledown f(v)\rangle. \tag{3.23}$$

Since $x^* \in \Gamma$, we obtain from Lemma 2.11 and (3.23) that

$$\limsup_{n\to+\infty}\langle x_{n+1} - v, \bigtriangledown f(u) - \bigtriangledown f(v)\rangle = \langle x^* - v, \bigtriangledown f(u) - \bigtriangledown f(v)\rangle \leq 0. \tag{3.24}$$

From (3.5), we have that

$$D_f(v, x_{n+1}) \leq (1 - \alpha_n)D_f(v, x_n) + \alpha_n\langle x_{n+1} - v, \bigtriangledown f(u) - \bigtriangledown f(v)\rangle. \tag{3.25}$$

On applying Lemma 2.14 in (3.25), we conclude that $D_f(x_n, v) \to 0,\ n \to \infty$. Therefore, $\{x_n\}$ converges strongly to $v = P^f_\Gamma u$.

CASE 2. Suppose that $\{D_f(p, x_n)\}$ is not a monotone decreasing sequence. Let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined for $n \geq n_0$ for some sufficiently large n_0 by

$$\tau(n) = \max\{j \in \mathbb{N} : j \leq n,\, D_f(p, x_{n_k}) \leq D_f(p, x_{n_j+1})\}.$$

Then τ is a non-decreasing sequence such that $\tau(n) \to +\infty$ and $D_f(p, x_{\tau(n)}) \leq D_f(p, x_{\tau(n)+1})$ for $n \geq n_0$.

We have from (3.5), conditions (i), (ii), (iii) that

$$\lim_{\tau(n)\to+\infty} ||z_{\tau(n)} - y_{\tau(n)}|| = 0 = \lim_{\tau(n)\to+\infty} ||w_{\tau(n)} - z_{\tau(n)}|| \tag{3.26}$$

and

$$\lim_{\tau(n)\to+\infty} || \bigtriangledown f(x_{\tau(n)}) - \bigtriangledown(T_{\tau(n)} x_{\tau(n)})|| = 0. \tag{3.27}$$

Following the same argument as in CASE 1, we have

$$\begin{aligned} \lim_{\tau(n)\to+\infty} ||u_{\tau(n)} - x_{\tau(n)}|| &= 0, \\ \lim_{\tau(n)\to+\infty} ||y_{\tau(n)} - x_{\tau(n)}|| &= 0, \\ \lim_{\tau(n)\to+\infty} ||x_{\tau(n+1)} - x_{\tau(n)}|| &= 0 \end{aligned}$$

and

$$\limsup_{\tau(n)\to+\infty} \langle x_{\tau(n)+1} - v, \bigtriangledown f(u) - \bigtriangledown f(v)\rangle = \langle x^* - v, \bigtriangledown f(u) - \bigtriangledown f(v)\rangle \leq 0. \tag{3.28}$$

It follows from (3.5) that

$$D_f(v, x_{\tau(n)+1}) \leq (1 - \alpha_{\tau(n)}) D_f(v, x_{\tau(n)}) + \alpha_{\tau(n)} \langle x_{\tau(n)+1} - v, \bigtriangledown f(u) - \bigtriangledown f(v)\rangle. \tag{3.29}$$

Since $\alpha_{\tau(n)} > 0$, we obtain that

$$D_f(v, x_{\tau(n)}) \leq \langle x_{\tau(n)+1} - v, \bigtriangledown f(u) - \bigtriangledown f(v)\rangle.$$

Hence, we deduce from (3.28) that $D_f(v, x_{\tau(n)}) = 0$. This implies that $\{x_{\tau(n)}\}$ converges strongly to v. Thus, $\{x_n\}$ converges strongly to $v \in \Gamma$. $\square$

We give the following consequences of our main result. In the next result, we consider a fixed point problem of relatively nonexpansive mapping and an EP involving a pseudomonotone bifunction.

Corollary 3.1. *Let E be a reflexive Banach space with E^* its dual and $Q \subseteq E$ be a nonempty closed convex set. Let $T : E \to E$ be a Bregman relatively nonexpansive mapping with $F(T) = \hat{F}(T)$ and $g : Q \times Q \to \mathbb{R}$ be a bifunction satisfying* L1-L5. *Assume that $f : E \to \mathbb{R}$ is a strongly coercive Legendre function, which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E such that*

$Q \subset \operatorname{int}(\operatorname{dom} f)$ with $\Gamma := F(T) \cap \Delta \neq \emptyset$. Let the sequence $\{x_n\}$ be generated iteratively by

$$\begin{cases} u_n = \bigtriangledown f^*(\beta_n \bigtriangledown f(x_n) + (1-\beta_n) \bigtriangledown f(Tx_n)), \\ z_n = \operatorname{argmin}_{a \in Q}\{\mu_n g(u_n, a) + D_f(a, u_n)\}, \\ w_n = \operatorname{argmin}_{a \in Q}\{\mu_n g(z_n, a) + D_f(a, u_n)\}, \\ x_{n+1} = \bigtriangledown f^*(\alpha_n \bigtriangledown f(u) + (1-\alpha_n) \bigtriangledown f(w_n)), \quad n \geq 1. \end{cases} \tag{3.30}$$

If conditions (i)-(iii) in (3.1) still hold, then the sequence $\{x_n\}$ converges strongly to $x^* = P_\Gamma^f u$.

Here, we consider a common solution of fixed point problem for an infinite family of Bregman quasi-nonexpansive mappings which is a zero of sum of monotone operators.

Corollary 3.2. *Let E be a reflexive Banach space and E^* be its dual space. For $n \in \mathbb{N}$, let $T_n : E \to E$ be an infinite family of Bregman quasi-nonexpansive mapping such that $(\{T_n\}_{n=1}^{+\infty}, T)$ satisfy the AKTT-condition and $F(T) = \hat{F}(T)$. Let $A : E \to E^*$ be a BISM mapping and $B : E \to 2^{E^*}$ be a maximal monotone operator. Assume that $f : E \to \mathbb{R}$ is a strongly coercive Legendre function, which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E such that $\Gamma := \cap_{n=1}^{+\infty} F(T_n) \cap \Omega \neq \emptyset$. Let the sequence $\{x_n\}$ be generated iteratively by*

$$\begin{cases} u_n = \bigtriangledown f^*(\beta_n \bigtriangledown f(x_n) + (1-\beta_n) \bigtriangledown f(T_n x_n)), \\ y_n = (Res_{\lambda B}^f \circ A_\lambda^f) u_n, \\ x_{n+1} = \bigtriangledown f^*(\alpha_n \bigtriangledown f(u) + (1-\alpha_n) \bigtriangledown f(y_n)), \quad n \geq 1. \end{cases} \tag{3.31}$$

If conditions (i)-(ii) in (3.1) still hold, then the sequence $\{x_n\}$ converges strongly to $v = P_\Gamma u$.

Remark 3.1. In our result, we employed a Halpern type iterative algorithm due to its flexibility in defining the algorithm parameters, which is important from the numerical implementation perspective. The iteration process employed in this result has an advantage over the ones used in [18, 33] and some known results in the literature in the sense that we do not use any projection of a point on the intersection of closed and convex sets which creates some difficulties in practical computation of the iterative sequence. In fact, the results presented in Corollary 3.1 and 3.2 coincide with the results of [18] and [33], and in one way or the other extend their result based on their choice of iterative algorithm.

4. Applications and Numerical Example

4.1. **Convex Minimization Problem (CMP).** Let Q be a nonempty closed and convex subset of a reflexive Banach space E and $g : E \to (-\infty, +\infty]$ be a proper, convex and lower semi-continuous function which attains its minimum over E. Let $T_n : Q \to E$ be an infinite family of Bregman quasi-nonexpansive mapping such that $(\{T_n\}_{n=1}^{+\infty}, T)$ satisfy the AKTT-condition with $F(T) = \hat{F(T)}$ and $f : E \to \mathbb{R}$ be a strongly coercive Legendre function, which is bounded, uniformly Frechet differentiable and totally convex on bounded subsets of E. Then, the CMP is to find $x \in F(T)$ such that

$$g(x) = \min_{y \in E} g(y). \tag{4.1}$$

It is generally known that (4.1) can be formulated as follows: find $x \in F(T)$ such that

$$0^* \in \partial g(x), \tag{4.2}$$

where $\partial g = \{\xi \in E^* : \langle \xi, y - x \rangle \leq g(y) - g(x) \text{ for all } x \in E\}$. It is known that ∂g is a maximal monotone operator whenever g is a proper, convex and lower semi-continuous function. Hence, by taking $\partial g = B$ and $A = 0$ in Theorem 3.1, we obtain a strong convergence result for approximation solutions of EP involving pseudomonotone bifunction and CMP (4.1).

4.2. **Variational Inequality Problem.** Let C be a nonempty closed and convex subset of a reflexive Banach space E with E^* its dual. Let $A : C \to E^*$ be a mapping and the function g defined as $g(x, y) = \langle y - x, Ax \rangle$. Then, the classical Variational Inequality Problem (VIP) is to find $z \in C$ such that

$$\langle y - z, Az \rangle \geq 0, \quad \text{for all } y \in C. \tag{4.3}$$

VIP is one of the most important problems in optimization as it is used in studying differential equations, minimax problems, and has certain applications to mechanics and economic theory, see [4, 5, 21, 24, 25]. We denote by $VI(C, A)$, the set of solutions of VIP (4.3).

Lemma 4.1 ([18]). *Let C be a nonempty closed convex subset of a reflexive Banach space E, $A : C \to E^*$ be a mapping and $f : E \to \mathbb{R}$ be a Legendre function. Then*

$$\overleftarrow{P}_C^f(\nabla f^*[\nabla f(x) - \mu A(y)]) = \operatorname{argmin}_{w \in C}\{\mu\langle w - y, A(y)\rangle\} + D_f(w, x)\},$$

for all $x \in E$, $y \in C$ and $\mu \in (0, +\infty)$.

Theorem 4.1. *Let E be a reflexive Banach space with E^* its dual and $Q \subseteq E$ be a nonempty closed convex set. Let $T : E \to E$ be a Bregman relatively nonexpansive mapping with $F(T) = \hat{F}(T)$. Let A is a pseudomonotone and L-Lipschitz continuous mapping from Q to E^*. Assume that $f : E \to \mathbb{R}$ is a strongly coercive Legendre function, which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E such that $Q \subset \operatorname{int}(\operatorname{dom} f)$ with $\Gamma := \{F(T) \cap VI(Q, A)\} \neq \emptyset$. Let the sequence $\{x_n\}$ be generated iteratively by*

$$\begin{cases} u_n = \nabla f^*(\beta_n \nabla f(x_n) + (1 - \beta_n) \nabla f(Tx_n)), \\ z_n = \overleftarrow{P}_Q^f(\nabla f^*(\nabla f(x_n) - \mu_n A(u_n))), \\ w_n = \overleftarrow{P}_Q^f(\nabla f^*(\nabla f(x_n) - \mu_n A(z_n))), \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(w_n)), \quad n \geq 1. \end{cases} \tag{4.4}$$

Suppose that conditions (i)*-*(ii) *in* (3.1) *hold and* $\{\mu_n\} \subset [a,b] \subset (0,p)$*, where* $p = \min \frac{2\tau}{L}$ *and* τ *is given by* (1.2) *holds, then the sequence* $\{x_n\}$ *converges strongly to* $v = P_\Gamma^f u$*.*

4.3. **Numerical Example.** We now display a numerical example of our algorithm to show its applicability.

Let $X = \mathbb{R}$ and $C = [0,1]$. Now, define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{2x^4}{27}$, then $\bigtriangledown f(x) = \frac{8x^3}{27}$. Thus, by the definition of Fenchel conjugate of f, we obtain that $f^*(x^*) = \frac{9}{8}{x^*}^{\frac{4}{3}}$ and $\bigtriangledown f^*(x^*) = \frac{36}{24}{x^*}^{\frac{1}{3}}$. Note that f satisfies the assumptions in Theorem 3.1 (see [13]), and that $\bigtriangledown f = (\bigtriangledown f^*)^{-1}$. Let $B : \mathbb{R} \to \mathbb{R}$ be defined by $B(x) = 7x - 2$ and $A : \mathbb{R} \to \mathbb{R}$ be defined by$A(x) = 5x$, then A and B are BISM and maximal monotone mappings respectively. Therefore, we compute their resolvents as follows:

$$\begin{aligned} Res_\lambda^f(A_\lambda^f x) &= (\bigtriangledown f + \lambda B)^{-1} \bigtriangledown f(\bigtriangledown f^*(\bigtriangledown f - \lambda A)(x)) \\ &= (\bigtriangledown f + \lambda B)^{-1}((\bigtriangledown f - \lambda A)(x)) \\ &= (\bigtriangledown f + \lambda B)^{-1}(x^3 - 5\lambda x). \end{aligned}$$

Now, define $g : C \times C \to \mathbb{R}$ by $g(x,y) = M(x)(y-x)$, where

$$M(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{100}, \\ \sin(x - \frac{1}{100}), & \frac{1}{100} \le x \le 1, \end{cases}$$

then g satisfies assumptions L1-L5 with $c_1 = 1 = c_2$ (see [18]). Also, define $T_n : \mathbb{R} \to \mathbb{R}$ by $T_n = \frac{1}{n}x$ for all $x \in \mathbb{R}$. Take $\alpha_n = \frac{1}{100n+1}$ and $\beta_n = \frac{n+1}{2n+7}$. Then, all assumptions of Theorem 3.1 are satisfied. Hence, Algorithm 3.1 becomes

$$\begin{cases} u_n = \bigtriangledown f^*(\beta_n f(x_n) + (1-\beta_n) \bigtriangledown f(\frac{x_n}{n})), \\ y_n = (\bigtriangledown f + \lambda B)^{-1}(u_n^3 - 5\lambda u_n), \\ z_n = y_n - \mu_n M(y_n), \\ w_n = y_n - \mu_n M(z_n), \\ x_{n+1} = \bigtriangledown f^*(\alpha_n \bigtriangledown f(u) + (1-\alpha_n) \bigtriangledown f(w_n)). \end{cases}$$

Case 1: $x_1 = 2,\ u = 0.5,\ \lambda = 10$ and $\frac{n+1}{4n+5}$.
Case 2: $x_1 = 0.5,\ u = 2,\ \lambda = 0.1$ and $\frac{n+1}{4n+5}$.
Case 3: $x_1 = 0.5,\ u = 2,\ \lambda = 0.1$ and $\frac{2n+1}{6n+7}$.
Case 4: $x_1 = 3,\ u = -7,\ \lambda = 2$ and $\frac{2n+1}{6n+7}$.

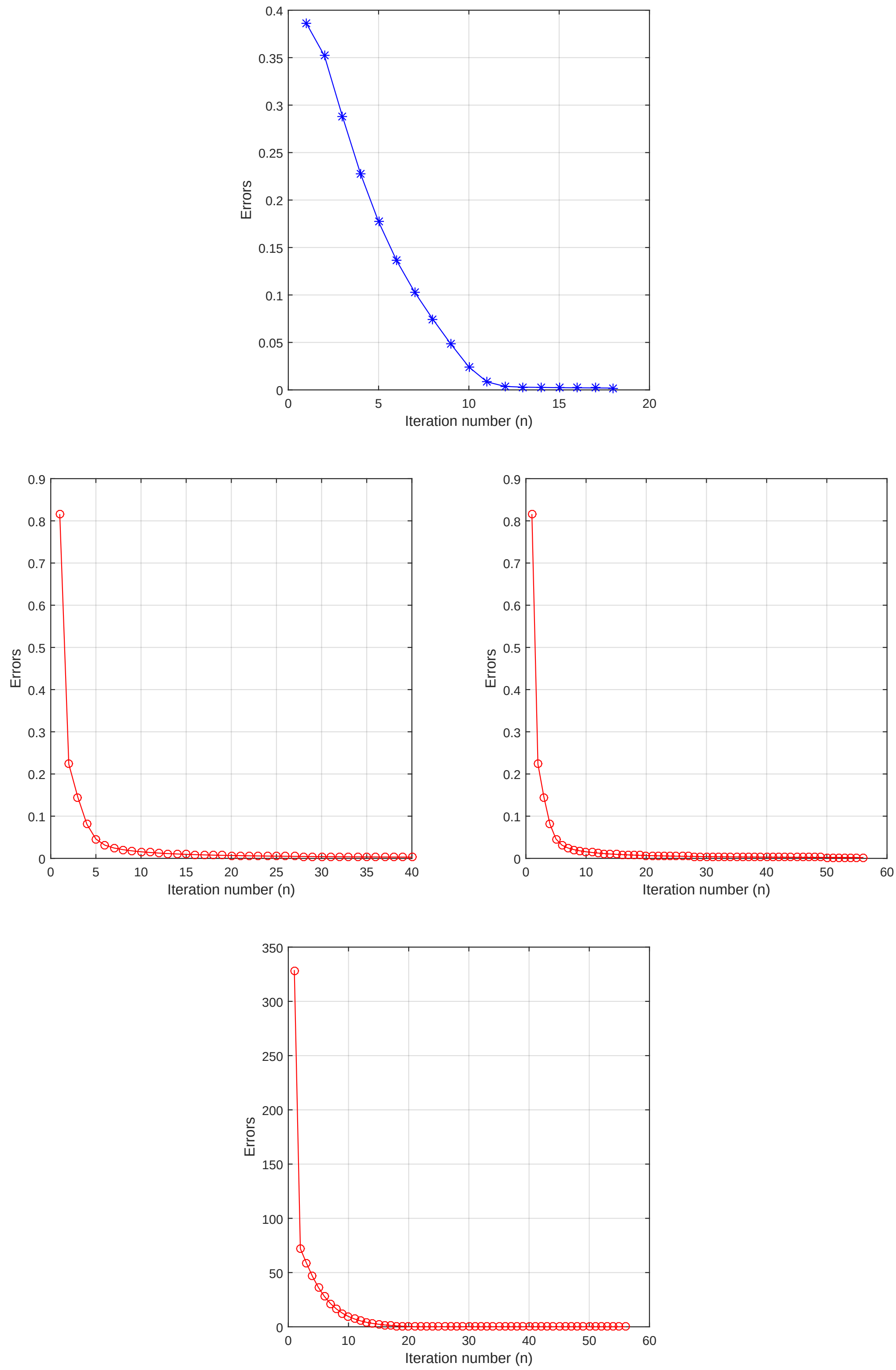

FIGURE 1. Errors vs Iteration numbers(n): Case 1 (top), Case 2 (middle left), Case 3 (middle right), Case 4 (bottom).

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