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DIFFERENCE ANALOGUES OF SECOND MAIN THEOREM AND PICARD TYPE THEOREM FOR SLOWLY MOVING PERIODIC TARGETS

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Abstract. In this paper, we show some Second main theorems for linearly non-degenerate meromorphic mappings over the field $\mathcal{P}_c^1$ of c-periodic meromorphic functions having their hyper-orders strictly less than one in $\mathbb{C}^m$ intersecting slowly moving targets in $\mathbb{P}^n(\mathbb{C})$. As an application, we give some Picard type theorems for meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ under the growth condition hyper-order less than one.

1. Introduction

In 2006, R. Halburd and R. Korhonen [5] considered the Second main theorem for complex difference operator with finite order in complex plane. Later, difference analogues of the Second main theorem for holomorphic curves or for meromorphic mappings into $\mathbb{P}^n(\mathbb{C})$ were obtained independently by the authors such as P. M. Wong, H. F. Law, P. P. W. Wong, R. Halburd, R. Korhonen, K. Tohge, T. B. Cao (see [2, 6–8]). Recently, T. B. Cao and R. Korhonen [3] obtained a new natural difference analogue of the Cartan's theorem [1], in which the counting function $N(r, \nu^0_{W(f)})$ of Wronskian determinant of f is replaced by the counting function $N(r, \nu^0_{C(f)})$ of Casorati determinant of f (it was called the finite difference Wronskian determinant in [6]).

Key words and phrases. Second main theorem, meromorphic mappings, Nevanlinna theory, Casorati determinant, moving targets.

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In particular, under the growth condition hyper-order < 1, R. Korhonen, N. Li-K. Tohge [9] obtained the Second main theorem of holomorphic curves of $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$ for slowly moving periodic targets which is similar to the Cartan's theorem.

To state some results in this direction, we recall some notations in [3, 9].

Let $c \in \mathbb{C}^m$, we denote by $\mathcal{M}_m$ the set of all meromorphic functions on $\mathbb{C}^m$, by $\mathcal{P}_c$ the set of all meromorphic functions of $\mathcal{M}_m$ periodic with period c, and by $\mathcal{P}_c^\lambda$ the set of all meromorphic functions of $\mathcal{M}_m$ periodic with period c and having their hyper-orders strictly less than λ. Obviously, $\mathcal{P}_c^\lambda \subset \mathcal{P}_c \subset \mathcal{M}_m$.

Definition 1.1. Let f be a meromorphic mapping from $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ with a reduced representation $f = (f_0 : \cdots : f_n)$. Then the map f is said to be linearly nondegenerate over a field $\mathcal{K}$ if the entire functions $f_0, \ldots, f_n$ are linearly independent over the field $\mathcal{K}$.

For $c = (c_1, \ldots, c_m)$ and $z = (z_1, \ldots, z_m)$, we write $c + z = (c_1 + z_1, \ldots, c_m + z_m)$. Let f be a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ with a reduced representation $f = (f_0 : \cdots : f_n)$. Denote

$$f(z) \equiv f := \bar{f}^{[0]}, f(z+c) \equiv \bar{f} := \bar{f}^{[1]}, f(z+2c) \equiv \bar{\bar{f}} := \bar{f}^{[2]}, \ldots, f(z+kc) \equiv \bar{f}^{[k]}.$$

Let

$$D^{(j)} = \left(\frac{\partial}{\partial z_1}\right)^{\alpha_1(j)} \cdots \left(\frac{\partial}{\partial z_m}\right)^{\alpha_m(j)}$$

be a partial differentiation operator of order at most $j = \sum_{k=1}^m \alpha_k(j)$. Similarly as the Wronskian determinant

$$W(f) = W(f_0, \ldots, f_n) = \begin{vmatrix} f_0 & f_1 & \cdots & f_n \\ D^{(1)} f_0 & D^{(1)} f_1 & \cdots & D^{(1)} f_n \\ \vdots & \vdots & \ddots & \vdots \\ D^{(n)} f_0 & D^{(n)} f_1 & \cdots & D^{(n)} f_n \end{vmatrix},$$

the Casorati determinant is defined by

$$C(f) = C(f_0, \ldots, f_n) = \begin{vmatrix} f_0 & f_1 & \cdots & f_n \\ \bar{f}_0 & \bar{f}_1 & \cdots & \bar{f}_n \\ \vdots & \vdots & \ddots & \vdots \\ \bar{f}_0^{[n]} & \bar{f}_1^{[n]} & \cdots & \bar{f}_n^{[n]} \end{vmatrix}.$$

Let $H_1, H_2, \ldots, H_q$ be (fixed) hyperplanes in $\mathbb{P}^n(\mathbb{C})$ given by

$$H_j = \{[\omega_0 : \cdots : \omega_n] \in \mathbb{P}^n(\mathbb{C}) : a_{j0}\omega_0 + \cdots + a_{jn}\omega_n = 0\} \quad (1 \le j \le q),$$

where the constants $a_{j0}, \ldots, a_{jn} \in \mathbb{C}$ are not simultaneously zero.

Definition 1.2. Let $N \geqslant n$ and $q \geqslant N + 1$. The family $\{H_j\}_{j=1}^q$ is said to be in N-subgeneral position in $\mathbb{P}^n(\mathbb{C})$ if any $N + 1$ of the vectors $(a_{j0}, \ldots, a_{jn})$ $(1 \le j \le q)$ are linearly independent over $\mathbb{C}$.

If they are in n-subgeneral position, we simply say that they are in general position.

Let $a_1, \ldots, a_q$ $(q \geq n+1)$ be q meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ with reduced representations $a_j = (a_{j0} : \cdots : a_{jn})$ $(1 \leq j \leq q)$. The moving hyperplane H_j associated with a_j is defined by

$$H_j(z) = \{[\omega_0 : \cdots : \omega_n] \in \mathbb{P}^n(\mathbb{C}) : L_{H_j}(z, a_j(z)) := a_{j0}(z)\omega_0 + \cdots + a_{jn}(z)\omega_n = 0\},$$

with $z \in \mathbb{C}^m \setminus I(a_j)$, where $I(a_j)$ is the locus of indeterminacy of a_j.

Similarly to the above definition, we have the following.

Definition 1.3. Let $k \geq n$ and $q \geq k+1$ and let $\mathcal{K}$ be a field such that $\mathbb{C} \subset \mathcal{K}$. We say that the moving targets $a_1, \ldots, a_q$ (also say that the moving hyperplanes $H_1(z), \ldots, H_q(z)$) are in k-subgeneral position over $\mathcal{K}$ if any $k+1$ of vectors $(a_{j0}(z), \ldots, a_{jn}(z))$ $(1 \leq j \leq q)$ are linearly independent over $\mathcal{K}$.

If they are in n-subgeneral position over $\mathcal{K}$, we also simply say that they are in general position over $\mathcal{K}$.

Let f, a be two meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ with reduced representations $f = (f_0, \ldots, f_n),\ a = (a_0, \ldots, a_n)$, respectively. We define $(f, a) := \sum_{i=0}^n a_i f_i$.

Definition 1.4. We say that a is a small moving target or a slowly moving target with respect to f if $T(r, a) = o(T(r, f))$ as $r \to \infty$, where the notations $T(r, a)$ and $T(r, f)$ are characteristic functions of a and f, respectively.

When the entire functions a_j $(0 \leq j \leq n)$ are periodic with period c then we say that a_j are moving periodic targets with period c.

Definition 1.5. Let $n \in \mathbb{N}, c \in \mathbb{C}^m \setminus \{0\}$ and $a \in \mathbb{C}$. An a-point z_0 of a meromorphic function $f(z)$ is said to be n-successive with separation c, if the n mappings $f(z+kc)$ $(k = 1, \ldots, n)$ take the value a at $z = z_0$ with multiplicity not less than that of $f(z)$ there. All the other a-points of $f(z)$ are called n-aperiodic of pace c.

By $\tilde{N}^{[n,c]}(r, (f, a))$, we denote the counting function of all n-aperiodic zeros of the function (f, a) of pace c.

Note that $\tilde{N}_{[n,c]}(r, (f, a)) \equiv 0$ when all zeros of (f, a) with taking their multiplicities into account are located periodically with period c. This is also the case when the moving target a is forward invariant by f with respect to the translation $\tau_c = z + c$, i.e., $\tau_c(f^{-1}(a)) \subset f^{-1}(a)$ and $f^{-1}(a)$ are considered to be multi-sets in which each point is repeated according to its multiplicity. In fact, it follows by the definition that any zero with a forward invariant preimage of the function (f, a) must be n-successive with separation c, since

$$f^{-1}(a) \subset \tau_{-c}(f^{-1}(a)) \subset \cdots \subset \tau_{-(n-1)c}(f^{-1}(a)).$$

With these definitions, the Second main theorem of holomorphic curves for slowly moving periodic hyperplanes is stated as follows.

Theorem A. ([9]) (Difference analogue of the Cartan's Second main theorem) *Let $n \geq 1$ and let $g = (g_0 : \cdots : g_n)$ be a holomorphic curve of $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$ with* *hyper-order* $\zeta = \zeta_2(g) < 1$, *where* $g_0, \ldots, g_n$ *are linearly independent over* $\mathcal{P}_c^1$. *Let*

$$a_j(z) = (a_{j0} : \cdots : a_{jn}) \quad (j \in \{0, \ldots, q\}),$$

where $a_{jk}(z)$ *are c-periodic entire functions satisfying* $T(r, a_{jk}) = o(T_g(r))$ *for all* $j, k \in \{0, \ldots, q\}$. *If the moving hyperplanes*

$$H_j(z) = \{(\omega_0, \ldots, \omega_n) : L_{H_j}(z, a_j(z)) = 0\} \quad (j \in \{0, \ldots, q\})$$

are located in general position over $\mathcal{P}_c^1$, *then*

$$\Big|\Big| \quad (q-n)T_g(r) \le \sum_{j=0}^{q} \tilde{N}_g^{[n,c]}(r, L_{H_j}) + o\left(T_g(r)\right).$$

Here, by the notation "$||\ P$" we mean the assertion P holds for all $r \in [0, \infty)$ outside of an exceptional set with finite logarithmic measure.

Firstly, by using the idea proposed by D. D. Thai, S. D. Quang [11], we will extend Theorem A to meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. Namely, we have the following.

Theorem 1.1. *Let* $c \in \mathbb{C}^m$ *and* $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ *be a linearly nondegenerate meromorphic mapping over* $\mathcal{P}_c^1$ *with hyper-order* $\zeta_2(f) < 1$. *Let* a_j $(1 \le j \le q)$ *be* q *slowly moving periodic targets with respect to* f *with period* c, *located in general position over* $\mathcal{P}_c^1$. *Then we have*

$$\Big|\Big| \quad \frac{q}{n+2} T(r, f) \le \sum_{j=1}^{q} \tilde{N}_{(f,a_j)}^{[n,c]}(r) + o\left(T(r, f)\right),$$

where $\tilde{N}_{(f,a_j)}^{[n,c]}(r)$ *is the counting function of all* n*-aperiodic zeros of the function* (f, a_j).

We now consider $\mathcal{A} = \{a_1, \ldots, a_q\}$ be the set of meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$.

Definition 1.6. We say that $\mathcal{A}$ is nondegenerate over a field $\mathcal{K}$ if $\dim(\mathcal{A})_{\mathcal{K}} = n + 1$ and for each nonempty proper subset $\mathcal{A}_1$ of $\mathcal{A}$

$$(\mathcal{A}_1)_{\mathcal{K}} \cap (\mathcal{A} \setminus \mathcal{A}_1)_{\mathcal{K}} \cap \mathcal{A} \ne \emptyset,$$

where $(\mathcal{A})_{\mathcal{K}}$ is the linear span of $\mathcal{A}$ over the field $\mathcal{K}$.

With the above definitions, we have the following theorem.

Theorem 1.2. *Let* $c \in \mathbb{C}^m$ *and let* $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ *be a linearly nondegenerate meromorphic mapping over* $\mathcal{P}_c^1$ *with hyper-order* $\zeta_2(f) < 1$. *Let* a_j $(1 \le j \le q)$ *be* q *slowly moving periodic targets with respect to* f *with period* c *such that* $(f, a_j) \not\equiv 0$ $(1 \le j \le q)$. *Assume that* $\mathcal{A} = \{a_1, \ldots, a_q\}$ *is nondegenerate over* $\mathcal{M}_m$. *Then we have*

$$\Big|\Big| \quad T(r, f) \le \sum_{j=1}^{q} \tilde{N}_{(f,a_j)}^{[n,c]}(r) + o\left(T(r, f)\right),$$

where $\tilde{N}_{(f,a_j)}^{[n,c]}(r)$ *is the counting function of all* n*-aperiodic zeros of the function* (f, a_j).

We would like note that if the mapping f is forward invariant over a_j for all $j \in \{1, \dots, q\}$ with respect to the translation $\tau_c(z) = z+c$, i.e., $\tau((f, a_j)^{-1}) \subset (f, a_j)^{-1}$ (counting multiplicity) holds for all j, then $\tilde{N}^{[n,c]}_{(f,a_j)}(r) = 0$ for all j. It follows from Theorem 1.2 that there exists no linearly nondegenerate meromorphic mapping over $\mathcal{P}_c$ which is periodic with period c.

By considering the uniqueness problem for $f(z)$ and $f(z+c)$ intersecting hyperplanes, the authors of [3, 7] obtained an unicity theorem for linearly degenerate meromorphic mappings over $\mathcal{P}_c$. That result is an extension of Picard's theorem under the growth condition hyper-order less than 1.

Theorem B ([3]). *Let f be a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ with hyper-order $\zeta_2(f) < 1$, and let $\tau(z) = z + c$, where $c \in \mathbb{C}^m$. Assume that $\tau((f, H_j)^{-1}) \subset (f, H_j)^{-1}$ (counting multiplicity) holds for q distinct hyperplanes $\{H_j\}_{j=1}^q$ in N-subgeneral position in $\mathbb{P}^n(\mathbb{C})$. If $q > 2N$, then $f(z) = f(z + c)$.*

Finally, we would like to extend the above result to the case of slowly moving periodic targets.

Theorem 1.3. *Let f be a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ with hyper-order $\zeta_2(f) < 1$. Let $\mathcal{A} = \{a_1, \dots, a_q\}$ be the set of slowly moving periodic targets with respect to f with period c which is nondegenerate over $\mathcal{M}_m$ and satisfying $(f, a_j) \not\equiv 0$ $(1 \le j \le q)$. Assume that f is forward invariant over a_j for all $j \in \{1, \dots, q\}$ with respect to the translation $\tau_c(z) = z + c$. Then the image of f is contained in a projective linear subspace over $\mathcal{P}_c^1$ of dimension $\le \left[\frac{n}{q-n}\right]$. Particularly, if $q > 2n$, then f is periodic with period c, i.e., $f(z) = f(z + c)$.*

Theorem 1.4. *Let f be a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ with hyper-order $\zeta_2(f) < 1$. Let $c \in \mathbb{C}^m$ and $k \in \mathbb{N}$, $k \ge n$. Let $\mathcal{A} = \{a_1, \dots, a_q\}$ be the set of slowly moving periodic targets with respect to f with period c such that $(f, a_j) \not\equiv 0$ $(1 \le j \le q)$ and satisfies condition: $\dim(\mathcal{A})_{\mathcal{M}_m} = n + 1$ and for each a proper subset $\mathcal{A}_1$ of $\mathcal{A}$ with $|\mathcal{A}_1| \ge k + 1$ then $(\mathcal{A}_1)_{\mathcal{M}_m} \cap (\mathcal{A} \setminus \mathcal{A}_1) \ne \emptyset$. Assume that f is forward invariant over a_j for all $j \in \{1, \dots, q\}$ with respect to the translation $\tau_c(z) = z+c$. Then the image of f is contained in a projective linear subspace over $\mathcal{P}_c^1$ of dimension $\le \left[\frac{k}{q-k}\right]$. Particularly, if $q > 2k$ then f is periodic with period c, i.e., $f(z) = f(z + c)$.*

Denote by $\mathcal{R} = \mathcal{R}(\{a_i\}_{i=1}^q) \subset \mathcal{M}_m$ the smallest subfield which contains $\mathbb{C}$ and all $\frac{a_{ik}}{a_{il}}$ with $a_{il} \not\equiv 0$. Obviously, $\mathcal{R} \subset \mathcal{P}_c^1 \subset \mathcal{M}_m$. Since the proof of Theorem 1.4, we can see that this theorem also holds when the field M_m is replaced by the field $\mathcal{R}$ or the field $\mathcal{P}_c^1$. Therefore, when the moving targets are in k-subgeneral position over $\mathcal{P}_c^1$, it is easy to see that they satisfy the hypothesis of Theorem 1.4. Immediately, we have the following corollary which is an extension of Theorem B.

Corollary 1.1. *Let $c \in \mathbb{C}^m$ and $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic mapping with hyper-order $\zeta_2(f) < 1$. Let a_j $(1 \le j \le q)$ be q slowly moving periodic targets with period c, located in k-subgeneral position over $\mathcal{P}_c^1$ such that $(f, a_j) \not\equiv 0$ $(1 \le j \le q)$.*

Assume that f is forward invariant over a_j for all $j \in \{1, \dots, q\}$ with respect to the translation $\tau_c(z) = z + c$. Then the image of f is contained in a projective linear subspace over $\mathcal{P}_c^1$ of dimension $\leq \left[\frac{n}{q-k}\right]$. Particularly, if $q \geq 2k + 1$, then f is periodic with period c, i.e., $f(z) = f(z + c)$.

2. Preliminaries

2.1. **Divisor.** We set $||z|| = \Big(|z_1|^2 + \cdots + |z_n|^2\Big)^{1/2}$ for $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and define

$$B_m(r) := \{z \in \ \mathbb{C}^m : ||z|| < r\}, \quad S_m(r) := \{z \in \mathbb{C}^m : ||z|| = r\} \quad (0 < r < \infty).$$

Define

$$\sigma_m(z) := \Big(dd^c||z||^2\Big)^{m-1} \qquad \text{and}$$

$$\eta_m(z) := d^c\text{log}||z||^2 \wedge \Big(dd^c\text{log}||z||^2\Big)^{m-1} \quad \text{on} \quad \mathbb{C}^m \setminus \{0\}.$$

Let F be a nonzero holomorphic function on a domain Ω in $\mathbb{C}^m$. For a set $\alpha = (\alpha_1, \dots, \alpha_m)$ of nonnegative integers, we set $|\alpha| = \alpha_1 + \cdots + \alpha_m$ and $\mathcal{D}^\alpha F = \frac{\partial^{|\alpha|} F}{\partial^{\alpha_1} z_1 \cdots \partial^{\alpha_m} z_m}$. We define the map $\nu_F : \Omega \to \mathbb{Z}$ by

$$\nu_F(z) := \max\ \{n : \mathcal{D}^\alpha F(z) = 0 \text{ for all } \alpha \text{ with } |\alpha| < n\} \quad (z \in \Omega).$$

We mean by a divisor on a domain Ω in $\mathbb{C}^m$ a map $\nu : \Omega \to \mathbb{Z}$ such that for each $a \in \Omega$, there are nonzero holomorphic functions F and G on a connected neighbourhood $U \subset \Omega$ of a such that $\nu(z) = \nu_F(z) - \nu_G(z)$ for each $z \in U$ outside an analytic set of dimension $\leq m - 2$. Two divisors are regarded as the same if they are identical outside an analytic set of dimension $\leq m - 2$. For a divisor ν on Ω, we set $|\nu| := \overline{\{z : \nu(z) \neq 0\}}$, which is a purely $(m - 1)$-dimensional analytic subset of Ω or empty.

Take a nonzero meromorphic function φ on a domain Ω in $\mathbb{C}^n$. For each $a \in \Omega$, we choose nonzero holomorphic functions F and G on a neighbourhood $U \subset \Omega$ such that $\varphi = \frac{F}{G}$ on U and $\dim(F^{-1}(0) \cap G^{-1}(0)) \leq m - 2$, and we define the divisors $\nu^0_\varphi,\ \nu^\infty_\varphi$ by $\nu^0_\varphi := \nu_F,\ \nu^\infty_\varphi := \nu_G$, which are independent of choices of F and G and so globally well-defined on Ω.

2.2. **Counting function.** For a divisor ν on $\mathbb{C}^m$, we define the counting function of ν by

$$n(t) = \begin{cases} \int\limits_{|\nu| \cap B(t)} \nu(z)\sigma_{m-1}, & \text{if } m \geq 2, \\ \sum\limits_{|z| \leq t} \nu(z), & \text{if } m = 1. \end{cases}$$

Define

$$N(r, \nu) = \int\limits_1^r \frac{n(t)}{t^{2m-1}} dt \quad (1 < r < \infty).$$

Let $\varphi : \mathbb{C}^m \to \mathbb{C}$ be a meromorphic function. Define

$$N_\varphi(r) = N(r, \nu_\varphi), \quad N_\varphi^{(M)}(r) = N^{(M)}(r, \nu_\varphi).$$

2.3. **Characteristic function.** Let $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic mapping. For arbitrarily fixed homogeneous coordinates $(w_0 : \cdots : w_n)$ on $\mathbb{P}^n(\mathbb{C})$, we take a reduced representation $f = (f_0 : \cdots : f_n)$, which means that each f_i is a holomorphic function on $\mathbb{C}^m$ and $f(z) = \big(f_0(z) : \cdots : f_n(z)\big)$ outside the analytic set $\{f_0 = \cdots = f_n = 0\}$ of codimension greater or equal to 2. Set $\|f\| = \Big(|f_0|^2 + \cdots + |f_n|^2\Big)^{1/2}$.

The characteristic function of f is defined by

$$T(r, f) = \int\limits_{S_m(r)} \log\|f\|\eta_m - \int\limits_{S_m(1)} \log\|f\|\eta_m.$$

Note that $T(r, f)$ is independent of the choice of the representation of f. The order and hyper-order of f are respectively defined by

$$\zeta(f) := \limsup_{r\to\infty} \frac{\log^+ T(r, f)}{\log r} \quad \text{and} \quad \zeta_2(f) := \limsup_{r\to\infty} \frac{\log^+ \log^+ T(r, f)}{\log r},$$

where $\log^+ x := \max\{\log x, 0\}$ for any $x > 0$.

2.4. **Some lemmas.** It is known that the holomorphic functions $f_0, \ldots, f_n$ on $\mathbb{C}^m$ are linearly dependent over $\mathbb{C}$ if and only if their Wronskian determinant $W(f_0, \ldots, f_n)$ vanishes identically [10]. The similar result was proved by T. B. Cao, R. Korhonen [3] as follows.

Lemma 2.1 ([3])**.** (i) *Let $c \in \mathbb{C}^m$. A meromorphic mapping $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ with a reduced representation $f = (f_0 : \cdots : f_n)$ satisfies $C(f) \not\equiv 0$ if and only if f is linearly nondegenerate over the field $\mathcal{P}_c$.*

(ii) *Let $c \in \mathbb{C}^m$. If a meromorphic mapping $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ with a reduced representation $f = (f_0 : \cdots : f_n)$ satisfies $\zeta_2(f) < \lambda < +\infty$, then $C(f) \not\equiv 0$ if and only if f is linearly nondegenerate over the field $\mathcal{P}_c^\lambda \subset \mathcal{P}_c$.*

The lemma on the Logarithmic derivative [1, 4] plays an important role in the Nevanlinna theory. Here, it is replaced by the following lemma due to T. B. Cao, R. Korhonen [3].

Lemma 2.2 ([3])**.** *Let f be a nonconstant meromorphic function on $\mathbb{C}^m$ such that $f(0) \neq 0, \infty$, and let $\epsilon > 0$. If $\zeta_2(f) := \zeta < 1$, then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = \int_{S_m(r)} \log^+ \left|\frac{f(z+c)}{f(z)}\right| \eta_m(z) = o\left(\frac{T(r, f)}{r^{1-\zeta-\epsilon}}\right),$$

where ϵ is some positive constant.

Lemma 2.3. ([7, Lemma 8.3]). *Let $T : [0; +\infty) \to [0; +\infty)$ be a non-decreasing continuous function and let $s \in (0; +\infty)$. If the hyper-order $\zeta = \limsup_{r\to\infty} \frac{\log\log T(r)}{\log r} < 1$ and $\delta \in (0; 1-\zeta)$, then*

$$|| \quad T(r+s) = T(r) + o\left(\frac{T(r,f)}{r^{\delta}}\right).$$

3. Proof of Theorem 1.1

We recall the Second main theorem of meromorphic mappings with hyper-order $\zeta_2(f) < 1$ intersecting hyperplanes in N-subgeneral position in $\mathbb{P}^n(\mathbb{C})$.

Lemma 3.1 ([3]). *Let $c \in \mathbb{C}^m$ and $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a linearly nondegenerate meromorphic mapping over $\mathcal{P}_c$ with hyper-order $\zeta = \zeta_2(f) < 1$, and let H_j $(1 \le j \le q)$ be q $(q > 2N - n + 1)$ hyperplanes in N-subgeneral position in $\mathbb{P}^n(\mathbb{C})$. Then we have*

$$\Big\|(q - 2N + n - 1)T(r,f) \le \sum_{j=1}^{q} N(r, \nu^0_{(f,H_j)}) - \frac{N}{n} N(r, \nu^0_{(C(f))}) + o\left(\frac{T(r,f)}{r^{1-\zeta-\epsilon}}\right),$$

where $\nu^0_{(f,H_j)}$ is the zero divisor of function $H_j(f)$ and ϵ is some positive constant.

Proof of Theorem 1.1. Consider $n+2$ meromorphic mappings $a_{j_0}, \dots, a_{j_{n+1}}$ with reduced representations $a_{j_k} = (a_{j_k0} : \cdots : a_{j_kn})$ $(1 \le j_0 < \cdots < j_{n+1} \le q)$. We may assume that $a_{j_k0} \not\equiv 0$ for all $0 \le k \le n+1$. We put $\tilde{a}_{j_ki} = \frac{a_{j_ki}}{a_{j_k0}}$, $\tilde{a}_{jk} = (\tilde{a}_{j_k0} : \cdots : \tilde{a}_{j_kn})$. Take a reduced representation $f = (f_0 : \cdots : f_n)$ of f and define $(f, \tilde{a}_{j_k}) = \sum_{i=0}^n f_i \tilde{a}_{j_ki}$. Since $\{a_j\}_{j=1}^q$ is in general position over $\mathcal{P}_c^1$, we have $\tilde{a}_{n+1} = \sum_{k=0}^n c_k \tilde{a}_{j_k}$, where

$$c_k \in \mathcal{R}(\{a_j\}_{j=1}^q) \setminus \{0\} \quad \text{and} \quad T(r, c_k) = O\left(\max_{1\le j\le q} T(r, a_j)\right) = o(T(r,f)).$$

Moreover, we can see that $c_k \in \mathcal{P}_c^1$ for all $0 \le k \le n$.

Define $\tilde{f} = (c_0(f, \tilde{a}_{j_0}) : \cdots : c_n(f, \tilde{a}_{j_n}))$. Then $\tilde{f}$ is a linearly nondegenerate meromorphic mapping of $\mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ over $\mathcal{P}_c^1$. Indeed, assume that $\sum_{k=0}^n \lambda_k c_k (f, \tilde{a}_{j_k}) \equiv 0$ with $\lambda_k \in \mathcal{P}_c^1$ $(0 \le k \le n)$. This implies that

$$\left(\sum_{k=0}^n \lambda_k c_k \tilde{a}_{j_k0}\right) f_0 + \cdots + \left(\sum_{k=0}^n \lambda_k c_k \tilde{a}_{j_kn}\right) f_n \equiv 0.$$

Since f is linearly nondegenerate over $\mathcal{P}_c^1$, we have

$$\sum_{k=0}^n \lambda_k c_k \tilde{a}_{j_ki} = 0 \quad (i = 0, \dots, n).$$

By $\det(\tilde{a}_{j_ki})_{0\le k\le n, 0\le i\le n} \not\equiv 0$, the above linearly equation system has solutions $\lambda_k c_k \equiv 0$ $(0 \le k \le n)$. Hence, $\lambda_k \equiv 0$ $(0 \le k \le n)$. This implies that $\tilde{f}$ is linearly nondegenerate over $\mathcal{P}_c^1$.

Let z_0 is a common zero of $c_k(f, \tilde{a}_{j_k})$ $(0 \le k \le n)$. There are two possibilities.

Case 1. If $(f, \tilde{a}_{j_k})(z_0) = 0$ for all $0 \le k \le n$, then z_0 is either in $I(f)$ which is an analytic subset of codim > 2 or z_0 is a zero of $\det(\tilde{a}_{j_ki})_{0\le k\le n, 0\le i\le n}$, where $I(f)$ is the locus of indeterminacy of f. Moreover, by the First main theorem and by the assumption of the theorem, we get

$$\begin{aligned} N_{\det(\tilde{a}_{j_k i})_{0\le k\le n, 0\le i\le n}}(r) &\le T(r, \det(\tilde{a}_{j_k i})_{0\le k\le n, 0\le i\le n}) + O(1) \\ &= O\left(\max_{1\le j\le q} T(r, a_j)\right) \\ &= o(T(r, f)). \end{aligned}$$

Case 2. If there exists $0 \le k \le n$ such that $c_k(z_0) = 0$, we also have

$$N_{c_k}(r) \le T(r, c_k) = O\left(\max_{1\le j\le q} T(r, a_j)\right) = o(T(r, f)).$$

We now take $\tilde{f} = (hc_0(f, \tilde{a}_{j_0}) : \cdots : hc_n(f, \tilde{a}_{j_n}))$ as a reduced representation of $\tilde{f}$, where h is a meromorphic function on $\mathbb{C}^m$. It is easy to see that

$$N_h(r) \le \sum_{k=0}^{n} \left(N_{1/c_k}(r) + N_{a_{j_k 0}}(r)\right) = O\left(\max_{1\le j\le q} T(r, a_j)\right) = o(T(r, f))$$

and

$$N_{1/h}(r) \le \sum_{k=0}^{n} N_{c_k}(r) + N_{\det(\tilde{a}_{j_k i})_{0\le k\le n, 0\le i\le n}}(r) = O\left(\max_{1\le j\le q} T(r, a_j)\right) = o(T(r, f)).$$

Define $F_k = (f, \tilde{a}_{j_k})$ $(0 \le k \le n)$. Then since the linearly equation system

$$\sum_{t=0}^{n} \tilde{a}_{j_k t} f_t = F_k \quad (0 \le k \le n),$$

we can see that $f_t = \sum_{i=0}^{n} b_{ti} F_i$ $(0 \le t \le n)$, where $b_{ti} \in \mathcal{R}(\{a_j\}_{j=1}^{q}) \cap \mathcal{P}_c^1$. Put

$$A = \left(\sum_{0\le k,t\le n} |\tilde{a}_{j_k t}|^2\right)^{1/2} \quad \text{and} \quad B = \left(\sum_{0\le k,t\le n} |b_{ti}|^2\right)^{1/2}.$$

Then

$$||f|| \le B\left(\sum_{t=0}^{n} |F_t|\right)^{1/2} \quad \text{and} \quad \left(\sum_{t=0}^{n} |F_t|\right)^{1/2} \le A||f||.$$

Therefore, we have

$$\begin{aligned} T(r, f) &= \int_{S(r)} \log\left(\sum_{t=0}^{n} |F_t|\right)^{1/2} \eta_n + O\left(\max_{1\le j\le q} T(r, a_j)\right) \\ &= \int_{S(r)} \log\left(\sum_{t=0}^{n} |F_t|\right)^{1/2} \eta_n + o(T(r, f)). \end{aligned}$$

On the other hand, we have

$$\sum_{k=0}^{n} |(f, \tilde{a}_{j_k})|^2 \le \left(\sum_{k=0}^{n} |hc_k(f, \tilde{a}_{j_k})|^2\right) \left(\sum_{k=0}^{n} \left|\frac{1}{c_k}\right|^2\right) \left|\frac{1}{h}\right|^2$$

and

$$\sum_{k=0}^{n}|hc_k(f,\tilde{a}_{j_k})|^2 \leq |h|^2\left(\sum_{k=0}^{n}|c_k|^2\right)\left(\sum_{k=0}^{n}|(f,\tilde{a}_{j_k})|^2\right).$$

Hence, we have

$$\begin{aligned}T(r,f) &= T(r,\tilde{f}) + O\left(\max_{1\leq j\leq q} T(r,a_j)\right) - N_h(r) + N_{1/h}(r)\\ &= T(r,\tilde{f}) + o(T(r,f)).\end{aligned}\tag{3.1}$$

It follows that

$$\begin{aligned}\zeta_2(\tilde{f}) &= \limsup_{r\to\infty}\frac{\log^+\log^+ T(r,\tilde{f})}{\log r} = \limsup_{r\to\infty}\frac{\log^+\log^+(T(r,f)+o(T(r,f)))}{\log r}\\ &\leq \limsup_{r\to\infty}\frac{\log^+\log^+(2T(r,f))}{\log r} \leq \limsup_{r\to\infty}\frac{\log^+(2+\log^+(T(r,f)))}{\log r}\\ &\leq \limsup_{r\to\infty}\frac{\log^+ 2+\log^+\log^+(T(r,f)))}{\log r} = \zeta_2(f) < 1.\end{aligned}$$

By applying Lemma 3.1 to the hyperplanes

$$H_0 = \{\omega_0 = 0\},\dots,H_n = \{\omega_n = 0\}, H_{n+1} = \{\omega_0+\cdots+\omega_n = 0\},$$

for $\tilde{f}$, we have

$$T(r,\tilde{f}) \leq \sum_{k=0}^{n} N_{hc_k(f,\tilde{a}_{j_k})}(r) + N_{h(f,\tilde{a}_{j_{n+1}})}(r) - N(r,\nu^0_{(C(f))}) + o\left(\frac{T(r,\tilde{f})}{r^{1-\zeta-\epsilon}}\right).$$

This, by going through all points $z_0 \in \mathbb{C}^m$ and by the definitions of $\tilde{N}^{[n,c]}(r,H(f))$, we obtain

$$\begin{aligned}T(r,\tilde{f}) &\leq \sum_{k=0}^{n}\tilde{N}^{[n,c]}_{hc_k(f,\tilde{a}_{j_k})}(r) + \tilde{N}^{[n,c]}_{h(f,\tilde{a}_{j_{n+1}})}(r) + o\left(\frac{T(r,\tilde{f})}{r^{1-\zeta-\epsilon}}\right)\\ &\leq \sum_{k=0}^{n+1}\tilde{N}^{[n,c]}_{(f,a_{j_k})}(r) + O\left(\max_{1\leq j\leq q} T(r,a_j)\right) + o\left(\frac{T(r,\tilde{f})}{r^{1-\zeta-\epsilon}}\right)\\ &\leq \sum_{k=0}^{n+1}\tilde{N}^{[n,c]}_{(f,a_{j_k})}(r) + o(T(r,f)).\end{aligned}\tag{3.2}$$

Combining inequality (3.1) with inequality (3.2), we get

$$T(r,f) \leq \sum_{k=0}^{n+1}\tilde{N}^{[n,c]}_{(f,a_{j_k})}(r) + o(T(r,f)).\tag{3.3}$$

We now take the sum of both of sides of (3.3) over all combinations $(j_0,\dots,j_{n+1})$ with $1 \leq j_0 < \cdots < j_{n+1} \leq q$, we get

$$\Big\|\quad \frac{q}{n+2}T(r,f) \leq \sum_{j=1}^{q}\tilde{N}^{[n,c]}_{(f,a_j)}(r) + o(T(r,f)).$$

The proof of Theorem 1.1 is completed. □

4. Proof of Theorem 1.2

In order to prove Theorem 1.2, we need the following.

Lemma 4.1 ([11]). *Assume that $\mathcal{A} = \{a_1, \dots, a_q\}$ is nondegenerate over $\mathcal{M}_m$. There exist subsets $I_1, \dots, I_k$ of $(f, \tilde{a}_j)_{j=1}^q$ such that the following are satisfied:*

(i) *I_1 is minimal over $\mathcal{R}$, i.e., I_1 is linearly dependent over $\mathcal{R}$ and each proper subset of I_1 is linearly independent over $\mathcal{R}$;*

(ii) *I_i is linearly independent over $\mathcal{R}$ for all $2 \le i \le k$;*

(iii) *$\left(\bigcup_{j=1}^k I_j\right)_{\mathcal{R}} = (\{(f, \tilde{a}_i)\}_{i=1}^q)_{\mathcal{R}}$;*

(iv) *for each $2 \le i \le k$, there exist meromorphic functions $c_\alpha \in \mathcal{R} \setminus \{0\}$ such that*

$$\sum_{(f,\tilde{a}_\alpha)\in I_i} c_\alpha (f, \tilde{a}_\alpha) \in \left(\bigcup_{j=1}^{i-1} I_j\right)_{\mathcal{R}},$$

where $\mathcal{R} = \mathcal{R}(\{a_j\}_{j=1}^q)$.

Proof of Theorem 1.2. Take subsets $I_1 = \{(f, \tilde{a}_1), (f, \tilde{a}_2), \dots, (f, \tilde{a}_{t_1})\}$ and $I_i = \{(f, \tilde{a}_{t_{i-1}+1}), \dots, (f, \tilde{a}_{t_i})\}$ $(2 \le i \le k)$ as in Lemma 4.1. Since I_1 is minimal, there exist $c_{1j} \in \mathcal{R} \setminus \{0\}$ such that

$$\sum_{j=1}^{t_1} c_{1j}(f, \tilde{a}_j) = 0.$$

Put $c_{1j} = 0$ for all $j > t_1$. We have $\sum_{j=1}^{t_k} c_{1j}(f, \tilde{a}_j) = 0$. Lemma 4.1 yields that $\{c_{1j}(f, \tilde{a}_j)\}_{j=2}^{t_1}$ is linearly independent over $\mathcal{R}$. It is easy to see that $\{\tilde{a}_j\}_{j=2}^{t_1}$ is linearly independent over $\mathbb{C}$. Since the assumption that f is linearly nondegenerate over $\mathcal{P}_c^1$, and using the arguments as in Theorem 1.1, we can see that $\{c_{1j}(f, \tilde{a}_j)\}_{j=2}^{t_1}$ are linearly independent over $\mathcal{P}_c^1$. Therefore, by Lemma 2.1, the Casorati determinant

$$\begin{aligned} C_1 &= C(c_{12}(f, \tilde{a}_1), \dots, c_{1t_1}(f, \tilde{a}_{t_1})) \\ &= \prod_{j=0}^{t_1-2} \bar{f}_0^{[j]} C\left(\frac{c_{12}(f, \tilde{a}_2)}{f_0}, \dots, \frac{c_{1t_1}(f, \tilde{a}_{t_1})}{f_0}\right) \\ &= \prod_{j=0}^{t_1-t_0-1} \bar{f}_0^{[j]} \tilde{C}_1 \not\equiv 0, \end{aligned}$$

where $t_0 = 1$.

We now consider $i \ge 2$. By the property of subset I_i, there exist meromorphic functions $c_{ij} \not\equiv 0$, $t_{i-1} + 1 \le j \le t_i$ in $\mathcal{R}$ such that $\sum_{j=t_{i-1}+1}^{t_i} c_{ij}(f, \tilde{a}_j) \in \left(\bigcup_{j=1}^{i-1} I_j\right)_{\mathcal{R}}$. Therefore, there exist meromorphic functions $c_{ij} \in \mathcal{R}$ $(1 \le j \le t_i)$ such that $c_{ij} \not\equiv 0$, $t_{i-1} + 1 \le j \le t_i$ and $\sum_{j=1}^{t_i} c_{ij}(f, \tilde{a}_j) = 0$. Put $c_{ij} = 0$ for all $j > t_i$, then $\sum_{j=1}^{t_k} c_{ij}(f, \tilde{a}_j) = 0$. Since $c_{ij}(f, \tilde{a}_j)_{j=t_{i-1}+1}^{t_i}$ are linearly independent over $\mathcal{R}$, they are linearly independent over $\mathcal{P}_c^1$, we have

$$\begin{aligned} C_i &= C(c_{it_{i-1}+1}(f,\tilde{a}_{t_{i-1}+1}),\dots,c_{it_i}(f,\tilde{a}_{t_i})) \\ &= \prod_{j=0}^{t_i-t_{i-1}-1} \bar{f}_0^{[j]} C\left(\frac{c_{it_{i-1}+1}(f,\tilde{a}_{t_{i-1}+1})}{f_0},\dots,\frac{c_{it_i}(f,\tilde{a}_{t_i})}{f_0}\right) \\ &= \prod_{j=0}^{t_i-t_{i-1}-1} \bar{f}_0^{[j]} \tilde{C}_i \not\equiv 0. \end{aligned}$$

Consider the $t_k \times t_k + 1$ minor matrices $\mathcal{T}$ and $\tilde{\mathcal{T}}$ given by

$$\mathcal{T} = \begin{pmatrix} c_{11}(f,\tilde{a}_1) & \cdots & c_{1t_k}(f,\tilde{a}_{t_k}) \\ c_{11}(\bar{f},\tilde{a}_1) & \cdots & c_{1t_k}(\bar{f},\tilde{a}_{t_k}) \\ \vdots & \vdots & \vdots \\ c_{11}(\bar{f}^{[t_1-t_0-1]},\tilde{a}_1) & \cdots & c_{1t_k}(\bar{f}^{[t_1-t_0-1]},\tilde{a}_{t_k}) \\ c_{21}(f,\tilde{a}_1) & \cdots & c_{2t_k}(f,\tilde{a}_{t_k}) \\ c_{21}(\bar{f},\tilde{a}_1) & \cdots & c_{2t_k}(\bar{f},\tilde{a}_{t_k}) \\ \vdots & \vdots & \vdots \\ c_{21}(\bar{f}^{[t_2-t_1-1]},\tilde{a}_1) & \cdots & c_{1t_k}(\bar{f}^{[t_2-t_1-1]},\tilde{a}_{t_k}) \\ c_{31}(f,\tilde{a}_1) & \cdots & c_{3t_k}(f,\tilde{a}_{t_k}) \\ c_{31}(\bar{f},\tilde{a}_1) & \cdots & c_{3t_k}(\bar{f},\tilde{a}_{t_k}) \\ \vdots & \vdots & \vdots \\ c_{k1}(\bar{f}^{[t_k-t_{k-1}-1]},\tilde{a}_1) & \cdots & c_{kt_k}(\bar{f}^{[t_k-t_{k-1}-1]},\tilde{a}_{t_k}) \end{pmatrix}$$

and

$$\tilde{\mathcal{T}} = \begin{pmatrix} \frac{c_{11}(f,\tilde{a}_1)}{f_0} & \cdots & \frac{c_{1t_k}(f,\tilde{a}_{t_k})}{f_0} \\ \frac{c_{11}(\bar{f},\tilde{a}_1)}{\bar{f}_0} & \cdots & \frac{c_{1t_k}(\bar{f},\tilde{a}_{t_k})}{\bar{f}_0} \\ \vdots & \vdots & \vdots \\ \frac{c_{11}(\bar{f}^{[t_1-t_0-1]},\tilde{a}_1)}{\bar{f}_0^{[t_1]}} & \cdots & \frac{c_{1t_k}(\bar{f}^{[t_1]},\tilde{a}_{t_k})}{\bar{f}_0^{[t_1-t_0-1]}} \\ \frac{c_{21}(f,\tilde{a}_1)}{f_0} & \cdots & \frac{c_{2t_k}(f,\tilde{a}_{t_k})}{f_0} \\ \frac{c_{21}(\bar{f},\tilde{a}_1)}{\bar{f}_0} & \cdots & \frac{c_{2t_k}(\bar{f},\tilde{a}_{t_k})}{\bar{f}_0} \\ \vdots & \vdots & \vdots \\ \frac{c_{21}(\bar{f}^{[t_2-t_1-1]},\tilde{a}_1)}{\bar{f}_0^{[t_2-t_1-1]}} & \cdots & \frac{c_{1t_k}(\bar{f}^{[t_2-t_1-1]},\tilde{a}_{t_k})}{\bar{f}_0^{[t_2-t_1-1]}} \\ \frac{c_{31}(f,\tilde{a}_1)}{f_0} & \cdots & \frac{c_{3t_k}(f,\tilde{a}_{t_k})}{f_0} \\ \frac{c_{31}(\bar{f},\tilde{a}_1)}{\bar{f}_0} & \cdots & \frac{c_{3t_k}(\bar{f},\tilde{a}_{t_k})}{\bar{f}_0} \\ \vdots & \vdots & \vdots \\ \frac{c_{k1}(\bar{f}^{[t_k-t_{k-1}-1]},\tilde{a}_1)}{\bar{f}_0^{[t_k-t_{k-1}-1]}} & \cdots & \frac{c_{kt_k}(\bar{f}^{[t_k-t_{k-1}-1]},\tilde{a}_{t_k})}{\bar{f}_0^{[t_k-t_{k-1}-1]}} \end{pmatrix}.$$

Denote by $\mathcal{D}_i$ (resp. $\tilde{\mathcal{D}}_i$) the determinant of the matrix obtained by deleting the $(i+1)$-th column of the minor matrix $\mathcal{T}$ (resp. $\tilde{\mathcal{T}}$). Since the sum of each row of $\mathcal{T}$ (resp. $\tilde{\mathcal{T}}$) is zero, we get

$$\mathcal{D}_i = (-1)^i \mathcal{D}_0 = (-1)^i \prod_{v=1}^{k} C_v = (-1)^i \prod_{v=1}^{k} \prod_{l=0}^{t_v - t_{v-1}-1} \bar{f}_0^{[l]} \tilde{C}_v = (-1)^i \prod_{v=1}^{k} \prod_{l=0}^{t_v - t_{v-1}-1} \bar{f}_0^{[l]} \tilde{\mathcal{D}}_0$$
$$= \prod_{v=1}^{k} \prod_{l=0}^{t_v - t_{v-1}-1} \bar{f}_0^{[l]} \tilde{\mathcal{D}}_i.$$

Since $\dim(\mathcal{A})_{\mathcal{R}} = n+1$, we can assume that $\tilde{a}_{i_1}, \dots, \tilde{a}_{i_{n+1}}$ is a basis of $(\mathcal{A})$ over $\mathcal{R}$. Then $\det(\tilde{a}_{i_j s})_{1 \le j \le n+1, 0 \le s \le n} \not\equiv 0$. By solving the linearly equation system

$$(f, \tilde{a}_{i_j}) = \tilde{a}_{i_j 0} f_0 + \cdots + \tilde{a}_{i_j n} f_n \quad (1 \le j \le n+1),$$

we get $f_v = \sum_{j=1}^{n+1} A_{vt}(f, \tilde{a}_{i_j})$ $(0 \le v \le n)$, with $A_{vt} \in \mathcal{R}$.

Take a basic $\{(f, \tilde{a}_{j_1})\}, \dots, (f, \tilde{a}_{j_d})\}$ of the space $\left(\bigcup_{i=1}^{k} I_i\right)_{\mathcal{R}}$. By

$$((f, \tilde{a}_{j_1})\}, \dots, (f, \tilde{a}_{j_d}))_{\mathcal{R}} = \left(\bigcup_{i=1}^{k} I_i\right)_{\mathcal{R}} = (\{(f, \tilde{a}_j)\}_{j=1}^{q})_{\mathcal{R}},$$

we have $f_v = \sum_{t=1}^{d} B_{vt}(f, \tilde{a}_{j_t})$ $(0 \le v \le n)$, with $B_{tv} \in \mathcal{R}$. Hence,

$$|f_v(z)| \le \sum_{t=1}^{d} |B_{vt}(z)| \max_{1 \le i \le t_k} \{|(f, \tilde{a}_i)(z)|\} \quad (z \in \mathbb{C}^m).$$

Define $A(z) = \sum_{t=1}^{d} \sum_{v=0}^{n} |B_{vt}(z)|$, then we have

$$||f(z)|| \le A(z) \max_{1 \le i \le t_k} \{|(f, \tilde{a}_i)(z)|\}$$

and

$$\begin{aligned} \int_{S(r)} \log^+ A(z) \eta_n &\le \sum_{t=1}^{d} \sum_{v=0}^{n} \log^+ |B_{tv}(z)| \eta_n + O(1) \\ &\le \sum_{t=1}^{d} \sum_{v=0}^{n} T(r, B_{vt}) + O(1) \\ &= O(\max_{1 \le j \le q} T(r, a_j)) + O(1) \\ &= o(T(r, f)). \end{aligned} \tag{4.1}$$

We now fix $z_0 \in \mathbb{C}^m$ and take i $(1 \le i \le t_k)$ such that

$$|(f, \tilde{a}_i)(z_0)| = \max_{1 \le i \le t_k} \{|(f, \tilde{a}_i)(z_0)|\}.$$

Then

$$\frac{|\mathcal{D}_0(z_0)|\cdot||f(z_0)||}{\prod_{j=1}^{t_k}|(f,\tilde{a_j})(z_0)|}=\frac{|\mathcal{D}_i(z_0)|}{\prod_{j=1,j\neq i}^{t_k}|(f,\tilde{a_j})(z_0)|}\cdot\frac{||f(z_0)||}{|(f,\tilde{a_i})(z_0)|}$$
$$\leq A(z_0)\frac{|\mathcal{D}_i(z_0)|}{\prod_{j=1,j\neq i}^{t_k}|(f,\tilde{a_j})(z_0)|}.$$

Therefore, we have

$$\log\frac{|\mathcal{D}_0(z_0)|\cdot||f(z_0)||}{\prod_{j=1}^{t_k}|(f,\tilde{a}_j)(z_0)|}\leq\log^+\left(A(z_0)\frac{|\mathcal{D}_i(z_0)|}{\prod_{j=1,j\neq i}^{t_k}|(f,\tilde{a}_j)(z_0)|}\right)$$
$$\leq\log^+\left(\frac{|\mathcal{D}_i(z_0)|}{\prod_{j=1,j\neq i}^{t_k}|(f,\tilde{a}_j)(z_0)|}\right)+\log^+A(z_0).$$

It implies that for each $z\in\mathbb{C}^m$, we have

$$\log\frac{|\mathcal{D}_0(z)|\cdot||f(z)||}{\prod_{j=1}^{t_k}|(f,\tilde{a}_j)(z)|}\leq\sum_{i=1}^{t_k}\log^+\left(\frac{|\mathcal{D}_i(z)|}{\prod_{j=1,j\neq i}^{t_k}|(f,\tilde{a}_j)(z)|}\right)+\log^+A(z)$$
$$=\sum_{i=1}^{t_k}\log^+\left(\frac{|\tilde{\mathcal{D}}_i(z)|}{\frac{\prod_{j=1,j\neq i}^{t_k}|(f,\tilde{a}_j)(z)|}{\prod_{v=1}^{k}\prod_{l=0}^{t_v-t_{v-1}-1}|\bar{f}_0^{[l]}|}}\right)+\log^+A(z).$$

Hence, we get
(4.2)

$$\log||f(z)||\leq\log\frac{\prod_{j=1}^{t_k}|(f,\tilde{a}_j)(z)|}{|\mathcal{D}_0(z)|}+\sum_{i=1}^{t_k}\log^+\left(\frac{|\tilde{\mathcal{D}}_i(z)|}{\frac{\prod_{j=1,j\neq i}^{t_k}|(f,\tilde{a}_j)(z)|}{\prod_{v=1}^{k}\prod_{l=0}^{t_v-t_{v-1}-1}|\bar{f}_0^{[l]}(z)|}}\right)+\log^+A(z).$$

Note that each element of the matrix of the determinant

$$\frac{\tilde{\mathcal{D}}_i(z)}{\frac{\prod_{j=1,j\neq i}^{t_k}(f,\tilde{a}_j)(z)}{\prod_{v=1}^{k}\prod_{l=0}^{t_v-t_{v-1}-1}\bar{f}_0^{[l]}(z)}}$$

has a form

$$c_{ij}\frac{\frac{(\bar{f}^{[l]},\tilde{a}_j)}{\bar{f}_0^{[l]}}}{\frac{(f,\tilde{a}_j)}{f_0}}\cdot\frac{\bar{f}_0^{[l]}}{f_0}\quad(1\leq i\leq k,1\leq j\leq t_k).$$

On the other hand, by the definition of the counting functions and the Jensen's formula, we have

$$\begin{aligned} T\left(r, \frac{(f,\tilde{a}_j)}{f_0}\right) &= \int_{S(r)} \log|1+\sum_{i=1}^{n} \tilde{f}_i \tilde{a}_{ji}|\eta_m + O(1) \\ &\le \int_{S(r)} \log((1+\sum_{i=1}^{n} |\tilde{f}_i|)^{1/2}(1+\sum_{i=1}^{n}|\tilde{a}_{ji}|)^{1/2})\eta_m \\ &\quad + N_{f_0}(r) + \sum_{i=1}^{n} N_{a_{i0}}(r) + O(1) \\ &= \int_{S(r)} \log(1+\sum_{i=1}^{n}|\tilde{f}_i|)^{1/2}\eta_m + N_{f_0}(r) + \int_{S(r)} \log(1+\sum_{i=1}^{n}|\tilde{a}_{ji}|)^{1/2}\eta_m \\ &\quad + \sum_{i=1}^{n} N_{a_{i0}}(r) + O(1) \\ &\le T(r,f) + T(r,a_j) + O(1) \\ &= T(r,f) + o(T(r,f)). \end{aligned}$$

Therefore, the hyper-order $\zeta_2\left(\frac{(f,\tilde{a}_j)}{f_0}\right) < 1$ $(1 \le j \le q)$. By Lemma 2.2, we have

$$\begin{aligned} \Big\| m\left(r, c_{ij}\frac{\frac{(\bar{f}^{[l]},\tilde{a}_j)}{\bar{f}_0^{[l]}}}{\frac{(f,\tilde{a}_j)}{f_0}} \cdot \frac{\bar{f}_0^{[l]}}{f_0}\right) &\le m\left(r, c_{ij}\right) + m\left(r, \frac{\frac{(\bar{f}^{[l]},\tilde{a}_j)}{\bar{f}_0^{[l]}}}{\frac{(f,\tilde{a}_j)}{f_0}}\right) + m\left(r, \frac{\bar{f}_0^{[l]}}{f_0}\right) \\ &\le O(\max_{1\le j\le q}\{T(r,a_j)\}) + o(T(r,f)) \\ &= o(T(r,f)). \end{aligned}$$

Hence,

$$\Big\| \quad m\left(r, \frac{\tilde{\mathcal{D}}_i(z)}{\frac{\prod_{j=1, j\ne i}^{t_k}(f,\tilde{a}_j)(z)}{\prod_{v=1}^{k}\prod_{l=0}^{t_v-t_{v-1}-1}\bar{f}_0^{[l]}(z)}}\right) \le o(T(r,f)). \tag{4.3}$$

By taking integrating both sides of inequality (4.2) and together this with (4.1) and (4.3), we get

(4.4)

$$\Big\| \quad T(r,f) \le \int_{S(r)} \log\frac{\prod_{j=1}^{t_k}|(f,\tilde{a}_j)|}{|\mathcal{D}_0|}\eta_m + o(T(r,f)) \le N\left(r, \nu^0_{\frac{\prod_{j=1}^{t_k}(f,\tilde{a}_j)}{\mathcal{D}_0}}\right) + o(T(r,f)).$$

Take z_0 as a zero of $\frac{\prod_{j=1}^{t_k}(f,\tilde{a}_j)}{\mathcal{D}_0}$. Then z_0 is a zero or a pole of some $(f,\tilde{a}_j)$ or a pole of some c_{sj}.

Case 1. Assume that z_0 is an n-successive with separation c of $(f,\tilde{a}_j)$ with multiplicity $v_j > 0$ for all j $(1 \le j \le t_k)$. Then $\nu^0_{(\bar{f}^{[v]},\tilde{a}_j)}(z_0) \ge \nu^0_{(f,\tilde{a}_j)}(z_0)$ for all $1 \le v \le n$

and $1 \le j \le t_k$. Without loss of generality, we may assume that $v_1 \le v_2 \le \cdots \le v_k$. For each $1 \le j \le t_k$, we have

$$C_i = (f, \tilde{a}_{t_{i-1}+1}) \cdots (f, \tilde{a}_{t_i}) c_{it_{i-1}+1} \cdots c_{it_i} \begin{vmatrix} 1 & \cdots & 1 \\ \frac{(\bar{f}, \tilde{a}_{t_{i-1}+1})}{(f, \tilde{a}_{t_{i-1}+1})} & \cdots & \frac{(\bar{f}, \tilde{a}_{t_i})}{(f, \tilde{a}_{t_i})} \\ \vdots & \ddots & \vdots \\ \frac{(\bar{f}^{[t_i - t_{i-1}-1]}, \tilde{a}_{t_{i-1}+1})}{(f, \tilde{a}_{t_{i-1}+1})} & \cdots & \frac{(\bar{f}^{[t_i - t_{i-1}-1]}, \tilde{a}_{t_i})}{(f, \tilde{a}_{t_i})} \end{vmatrix}$$
$$=: (f, \tilde{a}_{t_{i-1}+1}) \cdots (f, \tilde{a}_{t_i}) c_{it_{i-1}+1} \cdots c_{it_i} \hat{C}_i.$$

Put $J = \{(i,j) : c_{ij} \not\equiv 0, 1 \le i \le k, 1 \le j \le t_k\}$, then

$$\mathcal{D}_0 = \prod_{i=1}^{k} C_i = \prod_{j=2}^{t_k} (f, \tilde{a}_j) \prod_{(i,j) \in J} c_{ij} \prod_{i=1}^{k} \hat{C}_i.$$

Hence,

$$\nu_{\mathcal{D}_0}(z_0) = \sum_{j=2}^{t_k} \nu_{(f,\tilde{a}_j)}(z_0) + \sum_{(i,j)\in J} \nu_{c_{ij}}(z_0) + \sum_{i=1}^{k} \nu_{\hat{C}_i}(z_0) \ge \sum_{j=2}^{t_k} v_j + \sum_{(i,j)\in J} \nu_{c_{ij}}(z_0). \tag{4.5}$$

Take $\{a_{d_0}, \ldots, a_{d_n}\}$ as a basis of $(\mathcal{A})_{\mathcal{M}_m}$. Since $(\{(f, \tilde{a}_i)\}_{i=1}^{t_k})_{\mathcal{R}} = (\{(f, \tilde{a}_i)\}_{a \in \mathcal{A}})_{\mathcal{R}}$,

$$(f, \tilde{a}_{d_j}) = \sum_{i=1}^{t_k} \alpha_{ij} (f, \tilde{a}_{ij}) \quad (0 \le j \le n).$$

Note that $\alpha_{ij} \in \mathcal{R}$. Put $I = \{\alpha_{ij}, 1 \le i \le t_k, 0 \le j \le n \text{ such that } \alpha_{ij} \ne 0\}$ and $m = \max_{\alpha_{ij} \in I} \nu^{\infty}_{\alpha_{ij}}(z_0)$.

• If $m \ge v_1$, then $v_1 \le \sum_{\alpha_{ij} \in I} \nu^{\infty}_{\alpha_{ij}}(z_0)$.

• Otherwise, we have z_0 is a zero of $(f, \tilde{a}_{d_j})$ with multiplicity at least $v_1 - m$ for $0 \le j \le n$. Then z_0 is a zero of (f, a_{d_j}) with multiplicity at least $v_1 - m$ for $0 \le j \le n$. If $z_0 \notin I(f)$, then z_0 is a zero of $\det(a_{d_js})$ with multiplicity at least $v_1 - m$. It implies that $v_1 \le \sum_{\alpha_{ij} \in I} \nu^{\infty}_{\alpha_{ij}}(z_0) + \nu^0_{\det(a_{d_js})}(z_0)$ if $z_0 \notin I(f)$. Therefore, together these with (4.5), we have

$$\begin{aligned} \nu^0_{\frac{\prod_{j=1}^{t_k}(f\tilde{a}_j)}{\mathcal{D}_0}}(z_0) &\le \sum_{j=1}^{t_k} v_j - \left(\sum_{j=2}^{t_k} v_j + \sum_{(i,j)\in J} \nu_{c_{ij}}(z_0) \right) \\ &\le \sum_{\alpha_{ij} \in I} \nu^{\infty}_{\alpha_{ij}}(z_0) + \nu^0_{\det(a_{d_js})}(z_0) - \sum_{(i,j)\in J} \nu_{c_{ij}}(z_0). \end{aligned}$$

Case 2. Assume that there exists an index j_0 such that z_0 is not an n-successive with separation c of $(f, \tilde{a}_{j_0})$. Without loss of generality, we may assume that $j_0 = 1$. We can assume that z_0 is an n-successive with separation c of $(f, \tilde{a}_j)$ with all $2 \le j \le l$, and z_0 is an n-aperiodic with separation c of $(f, \tilde{a}_j)$ with all $l + 1 \le j \le t_{k_0}$, and z_0 is a pole of $(f,\tilde{a}_j)$ with all $t_{k_0}<j\le t_k$, where $k_0\le k$. Take i_0 satisfying $t_{i_0-1}\le l<t_{i_0}$, we have

$$C_i=(f,\tilde{a}_{t_{i-1}+1})\cdots(f,\tilde{a}_{t_i})c_{it_{i-1}+1}\cdots c_{it_i}\hat{C}_i\quad(1\le i\le t_0-1)$$

and

$$\begin{aligned}C_{i_0}=&(f,\tilde{a}_{t_{i_0-1}+1})\cdots(f,\tilde{a}_l)c_{i_0t_{i_0-1}+1}\cdots c_{i_0t_{i_0}}\\&\times\begin{pmatrix}1&\cdots&1&\cdots&(f,\tilde{a}_{l+1})&\cdots&(f,\tilde{a}_{t_0})\\ \frac{(\bar{f},\tilde{a}_{t_{i_0-1}+1})}{(f,\tilde{a}_{t_{i_0-1}+1})}&\cdots&\frac{(\bar{f},\tilde{a}_l)}{(f,\tilde{a}_l)}&\cdots&(\bar{f},\tilde{a}_{l+1})&\cdots&(\bar{f},\tilde{a}_{t_0})\\ \vdots&\ddots&\vdots&\ddots&\vdots&\ddots&\vdots\\ \frac{(\bar{f}^{[v]},\tilde{a}_{t_{i_0-1}+1})}{(f,\tilde{a}_{t_{i_0-1}+1})}&\cdots&\frac{(\bar{f}^{[v]},\tilde{a}_l)}{(f,\tilde{a}_l)}&\cdots&(\bar{f}^{[v]},\tilde{a}_{l+1})&\cdots&(\bar{f}^{[v]},\tilde{a}_{t_0})\end{pmatrix}\\=&(f,\tilde{a}_{t_{i_0-1}+1})\cdots(f,\tilde{a}_l)c_{i_0t_{i_0-1}+1}\cdots c_{i_0t_{i_0}}\hat{C}_{i_0},\end{aligned}$$

where $v=t_0-t_{i_0-1}-1$ and

$$C_i=c_{it_{i-1}+1}\cdots c_{it_i}C((f,\tilde{a}_{t_{i-1}+1}),\dots,(f,\tilde{a}_{t_i}))=c_{it_{i-1}+1}\cdots c_{it_i}\hat{C}_i,$$

for all $i_0+1\le i\le k$. Then we have

$$\frac{\prod_{j=1}^{t_k}(f,\tilde{a}_j)}{\mathcal{D}_0}=\frac{(f,\tilde{a}_1)\prod_{i=l+1}^{t_k}(f,\tilde{a}_i)}{\prod_{(i,j)\in J}c_{ij}\cdot\prod_{i=1}^{k}\hat{C}_i}$$

and therefore

$$\begin{aligned}\nu^0_{\frac{\prod_{j=1}^{t_k}(f,\tilde{a}_j)}{\mathcal{D}_0}}(z_0)=&\nu^0_{(f,\tilde{a}_1)(z_0)}+\sum_{i=l+1}^{t_k}\nu^0_{(f,\tilde{a}_i)}(z_0)-\sum_{(i,j)\in J}\nu_{c_{ij}}(z_0)+\sum_{i_0+1\le i\le k,0\le j\le n}\nu^\infty_{a_{t_ij}}(z_0)\\&-\sum_{i=1}^{k}\nu^0_{\hat{C}_i}(z_0)\\ \le&\nu^0_{(f,\tilde{a}_1)(z_0)}+\sum_{i=l+1}^{t_k}\nu^0_{(f,\tilde{a}_i)}(z_0)-\sum_{(i,j)\in J}\nu_{c_{ij}}(z_0)+\sum_{i_0+1\le i\le k,0\le j\le n}\nu^\infty_{a_{t_ij}}(z_0).\end{aligned}$$

By going through all points $z_0\in\mathbb{C}^m$ and by the definition of $\tilde{N}^{[n,c]}_{(f,\tilde{a}_i)}(r)$, two cases above imply that

$$\begin{aligned}N\left(r,\nu^0_{\frac{\prod_{j=1}^{t_k}(f,\tilde{a}_j)}{\mathcal{D}_0}}\right)\le&\sum_{j=1}^{t_k}\tilde{N}^{[n,c]}(r,\nu^0_{(f,\tilde{a}_j)})-\sum_{(i,j)\in J}N(r,\nu_{c_{ij}})+\sum_{1\le i\le t_k,0\le j\le n}N(r,\nu^\infty_{a_{ij}})\\&+\sum_{\alpha_{ij}\in I}N(r,\nu^\infty_{\alpha_{ij}})+N(r,\nu^0_{\det(a_{d_js})})\\ \le&\sum_{j=1}^{q}\tilde{N}^{[n,c]}_{(f,\tilde{a}_j)}(r)+o(T(r,f)).\end{aligned}$$

Together this with (4.4), we have

$$\Big\| \quad T(r,f) \le \sum_{j=1}^{q} \tilde{N}^{[n,c]}_{(f,\tilde{a}_j)}(r) + o(T(r,f)) \quad \le \sum_{j=1}^{q} \tilde{N}^{[n,c]}_{(f,a_j)}(r) + o(T(r,f)).$$

The proof of Theorem 1.2 is completed. □

5. Proof of Theorem 1.3

We recall the lemma due to T. B. Cao and R. Korhonen [3] as follows.

Lemma 5.1 ([3]). *Let $c \in \mathbb{C}^m$ and $f = (f_0 : \cdots : f_n)$ be a meromorphic mapping from $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ such that hyper-order $\zeta_2(f) < \lambda \le 1$ and all zeros of $f_0, \ldots, f_n$ forward invariant with respect to the translation $\tau(z) = z + c$. Let $S_1 \cup \cdots \cup S_l$ be the partition of $\{0, 1, \ldots, n\}$ formed in such a way that i and j are in the same class S_k if and only if $\frac{f_i}{f_j} \in \mathcal{P}^\lambda_c$. If $f_0 + \cdots + f_n = 0$, then*

$$\sum_{j \in S_k} f_j = 0,$$

for all $k \in \{1, \ldots l\}$.

Lemma 5.2. *If $\mathcal{A} = \{a_1, \ldots, a_q\}$ is linearly nondegenerate over $\mathcal{M}_m$, then $\mathcal{A}$ is linearly nondegenerate over $\mathcal{R}$, i.e., $\dim(\mathcal{A})_{\mathcal{R}} = n+1$ and for each nonempty proper subset $\mathcal{A}_1$ of $\mathcal{A}$, we have*

$$(\mathcal{A}_1)_{\mathcal{R}} \cap (\mathcal{A} \setminus \mathcal{A}_1)_{\mathcal{R}} \cap \mathcal{A} \ne \emptyset.$$

Proof. By the assumption, $\dim(\mathcal{A})_{\mathcal{M}_m} = n+1$, so $\dim(\mathcal{A})_{\mathcal{R}} = n+1$. Take any nonempty proper subset $\mathcal{A}_1$ of $\mathcal{A}$, we have

$$(\mathcal{A}_1)_{\mathcal{M}_m} \cap (\mathcal{A} \setminus \mathcal{A}_1)_{\mathcal{M}_m} \cap \mathcal{A} \ne \emptyset.$$

We consider two possibilities.

Case 1. There exists $a_t \in (\mathcal{A}_1)_{\mathcal{M}_m} \cap (\mathcal{A} \setminus \mathcal{A}_1)$. Then there exist $b_1, \ldots, b_k \in \mathcal{A}_1$ which are linearly independent over $\mathcal{M}_m$ and there exist $c_1, \ldots, c_k \in \mathcal{M}_m \setminus \{0\}$ such that $a_t = \sum_{i=1}^k c_i b_i$. Take reduced representations $a_t = (a_{t0} : \cdots : a_{tn})$ and $b_i = (b_{i0} : \cdots : b_{in})$ $(1 \le i \le k)$. We have a linear equation system $\sum_{i=1}^k c_i b_{ij} = a_{tj}$ $(0 \le j \le n)$. Since $\{\tilde{b}_1, \ldots, \tilde{b}_k\}$ is linearly independent over $\mathcal{M}_m$, $\mathrm{rank}(b_{ij})_{1 \le i \le k, 0 \le j \le n} = k$. By solving the above linear equation system, we have $c_i \in \mathcal{R}$ $(1 \le i \le k)$. It follows that $a_t \in (\mathcal{A}_1)_{\mathcal{R}}$ and hence $a_t \in (\mathcal{A}_1)_{\mathcal{R}} \cap (\mathcal{A} \setminus \mathcal{A}_1)$, i.e.,

$$(\mathcal{A}_1)_{\mathcal{R}} \cap (\mathcal{A} \setminus \mathcal{A}_1)_{\mathcal{R}} \cap \mathcal{A} \ne \emptyset.$$

Case 2. There exists $a_t \in \mathcal{A}_1 \cap (\mathcal{A} \setminus \mathcal{A}_1)_{\mathcal{M}_m}$. By the same arguments as in Case 1, we have $a_t \in (\mathcal{A} \setminus \mathcal{A}_1)_{\mathcal{R}}$ and therefore we also have

$$(\mathcal{A}_1)_{\mathcal{R}} \cap (\mathcal{A} \setminus \mathcal{A}_1)_{\mathcal{R}} \cap \mathcal{A} \ne \emptyset.$$

Lemma 5.2 is proved. □

Proof of Theorem 1.3. By the assumption of the theorem, the holomorphic functions $g_j = (f, a_j) = \sum_{i=0}^{n} f_i a_i$ satisfy $\{\tau(g_j^{-1}(0))\} \subset \{g_j^{-1}(0)\}, j \in \{1, \ldots, q\}$, where $\{\cdot\}$ denotes a multiset with counting multiplicities of its elements. We say that $i \sim j$ if $g_i = \alpha g_j$ for some $\alpha \in \mathcal{P}_c^1 \setminus \{0\}$. Therefore, the set of indexes $\{1, \ldots, q\}$ may be split into disjoint equivalent classes S_j, $\{1, \ldots, q\} = \cup_{j=1}^{l} S_j$ for some $1 \leq l \leq q$.

We assume that the complement of S_j has at least $n+1$ elements for some $j \in 1, \ldots, l$. Let $\mathcal{A}_1 = \{1, \ldots, q\} \setminus S_j$. Then $\mathcal{A}_1$ contains at least $n+1$ elements. By Lemma 5.2, there exist $\{s_0\}$ and $\{s_1, \ldots, s_u\}$ belonging to disjoint equivalent classes and $\alpha_1, \ldots, \alpha_u \in \mathcal{R} \setminus \{0\}$ such that $a_{s_0} + \sum_{j=1}^{u} \alpha_j a_{s_j} \equiv 0$. So, we have

$$(f, a_{s_0}) + \sum_{j=1}^{u} \alpha_j (f, a_{s_j}) = g_{s_0} + \sum_{j=1}^{u} \alpha_j g_{s_j} \equiv 0.$$

By the assumption of the theorem again, we can see that all zeros of $\alpha_j g_{s_j}$ are forward invariant with respect to the translation $\tau(z) = z + c$. This implies that

$$g := (g_{s_0} : \alpha_1 g_{s_1} : \cdots : \alpha_u g_{s_u})$$

is a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^u(\mathbb{C})$. We have

$$\begin{aligned} T(r, g) &= \int_{S_m(r)} \log ||g|| \eta_m + O(1) \\ &\leq \int_{S_m(r)} \log ||f|| \eta_m + \sum_{j=0}^{u} \int_{S_m(r)} \log ||a_{s_j}|| \eta_m + \sum_{j=1}^{u} \int_{S_m(r)} \log ||\alpha_{s_j}|| \eta_m + O(1) \\ &= T(r, f) + \sum_{j=0}^{u} T(r, a_{s_j}) + \sum_{j=1}^{u} T(r, \alpha_{s_j}) + O(1) \\ &= T(r, f) + o(T(r, f)). \end{aligned}$$

Therefore, $\zeta_2(g) \leq \zeta_2(f) < 1$. By Lemma 5.1, we get $g_{s_0} \equiv 0$ and $\sum_{j=1}^{u} \alpha_j g_{s_j} \equiv 0$. This is a contradiction. It implies that the complement of S_j has at most n elements for all $j \in 1, \ldots, l$, and hence S_j has at least $q - n$ elements for all $j \in 1, \ldots, l$. From this, we have

$$l \leq \frac{q}{q-n}.$$

Since $\dim(\mathcal{A})_{\mathcal{M}_m} = n+1$, we have $\dim(\mathcal{A})_{\mathcal{P}_c^1} = n+1$. Therefore, we can take a subset $V \subset \{1, \ldots, q\}$ with $|V| = n+1$ such that $\{a_j\}_{j \in V}$ is linearly independent. Put $V_j = V \cap S_j$ for each $1 \leq j \leq l$. Then we have $V = \cup_{j=1}^{l} V_j$. Since each V_j gives raise to $|V_j| - 1$ equations over the field $\mathcal{P}_c^1$, it is easy to see that there are at least

$$\sum_{j=1}^{l} (|V_j| - 1) = n + 1 - l \geq n + 1 - \frac{q}{q-n} = n - \frac{n}{q-n}$$

linear independent relations over the field $\mathcal{P}_c^1$. Hence, the image of f is contained in a projective linear subspace over $\mathcal{P}_c^1$ of dimension $\leq \left[\frac{n}{q-n}\right]$. If $q > 2n$, obviously $\left[\frac{n}{q-n}\right] = 0$. It follows that $f(z) = f(z+c)$. Theorem 1.3 is proved. $\square$

6. Proof of Theorem 1.4

Repeat the arguments as in Theorem 1.3. Assume that the complement $\mathcal{A}_1 = \{1 \ldots, q\} \setminus S_j$ of S_j has at least $k+1$ elements for some $j \in 1, \ldots, l$. By the assumption, we have $(\mathcal{A}_1)_{\mathcal{M}_m} \cap (\mathcal{A} \setminus \mathcal{A}_1) \neq \emptyset$. From Case 1 in the proof of Lemma 5.2, it is easy to see that $(\mathcal{A}_1)_{\mathcal{R}} \cap (\mathcal{A} \setminus \mathcal{A}_1) \neq \emptyset$. Therefore, there exist $s_0 \in S_j = \mathcal{A} \setminus \mathcal{A}_1, v_1, \ldots, v_t \in \mathcal{A}_1$ and $\beta_1, \ldots, \beta_t \in \mathcal{R} \setminus \{0\}$ such that $a_{s_0} + \sum_{j=1}^{t} \beta_j a_{v_j} = 0$. Similarly to the proof of Theorem 1.3, we can deduce that $(f, a_{s_0}) \equiv 0$. This is a contradiction. Therefore, $|\mathcal{A}_1| \leq k$, so $|S_j| \geq q - k$. Again using the discussion as in Theorem 1.3, we can show that the image of f is contained in a projective linear subspace over $\mathcal{P}_c^1$ of dimension $\leq \left[\frac{k}{q-k}\right]$ and if $q > 2k$, then $\left[\frac{k}{q-k}\right] = 0$. Therefore, $f(z) = f(z+c)$. Theorem 1.4 is proved.

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