Kragujevac Journal of Mathematics
Volume 47(5) (2023), Pages 739–754.

EXISTENCE AND STABILITY OF SOLUTIONS FOR NABLA FRACTIONAL DIFFERENCE SYSTEMS WITH ANTI-PERIODIC BOUNDARY CONDITIONS

JAGAN MOHAN JONNALAGADDA[1]

Abstract. In this paper, we propose sufficient conditions on existence, uniqueness and Ulam-Hyers stability of solutions for coupled systems of fractional nabla difference equations with anti-periodic boundary conditions, by using fixed point theorems. We also support these results through a couple of examples.

1. Introduction

The study of anti-periodic boundary value problems garnered significant interest due to their occurrence in the mathematical modelling of a variety of real-world problems in engineering and science. For example, we refer [19, 31, 32, 40] and the references therein.

The boundary value problems (BVPs) connected with nabla fractional difference equations can be tackled with almost similar methods as their continuous counterparts. Peterson et al. [15, 24] have initiated the study of BVPs for linear and nonlinear nabla fractional difference equations with conjugate boundary conditions. Gholami et al. [20] studied the existence of solutions for a coupled system of two-point nabla fractional difference BVPs. Recently, the author [26, 27] obtained sufficient conditions on existence and uniqueness of solutions for nonlinear nabla fractional difference equations associated with different classes of boundary conditions. In spite of the

Key words and phrases. Nabla fractional difference equation, anti-periodic boundary conditions, fixed point, existence, uniqueness, Ulam-Hyers stability.

2010 *Mathematics Subject Classification.* Primary: 39A12. Secondary: 39A70.

DOI 10.46793/KgJMat2305.739J

Received: March 10, 2020.

Accepted: December 12, 2020.

existence of a substantial mathematical theory of the continuous fractional anti-periodic BVPs [5–7, 13, 16, 36, 42], there has been no progress in developing the theory of discrete fractional anti-periodic BVPs in nabla perspective.

On the other hand, Hyers responses to Ulam's questions have initiated the study of stability of functional equations [23, 38]. Rassias [35] generalized the Hyers result for linear mappings. Later, several mathematicians have extended Ulam's problem in different directions [28]. There were significant contributions towards the study of Ulam-Hyers stability of ordinary as well as fractional differential equations [33, 41]. The study of Ulam-Hyers stability enriched the qualitative theory of fractional difference equations [17, 18, 25].

Motivated by these facts, in this article, we consider the following coupled system of nabla fractional difference equations with anti-periodic boundary conditions:

$$\begin{cases} \Big(\nabla_0^{\alpha_1-1}\big(\nabla u_1\big)\Big)(t) + f_1(t, u_1(t), u_2(t)) = 0, \quad t \in \mathbb{N}_2^T, \\ \Big(\nabla_0^{\alpha_2-1}\big(\nabla u_2\big)\Big)(t) + f_2(t, u_1(t), u_2(t)) = 0, \quad t \in \mathbb{N}_2^T, \\ u_1(0) + u_1(T) = 0, \quad \big(\nabla u_1\big)(1) + \big(\nabla u_1\big)(T) = 0, \\ u_2(0) + u_2(T) = 0, \quad \big(\nabla u_2\big)(1) + \big(\nabla u_2\big)(T) = 0. \end{cases} \tag{1.1}$$

Here $T \in \mathbb{N}_2$, $1 < \alpha_1, \alpha_2 < 2$, $f_1, f_2 : \mathbb{N}_0^T \times \mathbb{R}^2 \to \mathbb{R}$ are continuous, ∇_0^ν denotes the ν^{th}-th order Riemann-Liouville type backward (nabla) difference operator where $\nu \in \{\alpha_1 - 1, \alpha_2 - 1\}$ and ∇ denotes the first order nabla difference operator.

The present paper is organized as follows. Section 2 contains preliminaries. In Section 3, we establish sufficient conditions on existence, uniqueness and Ulam-Hyers stability of solutions of the BVP (1.1). We present a few examples in Section 4.

2. Preliminaries

For our convenience, in this section, we present a few useful definitions and fundamental facts of nabla fractional calculus, which can be found in [21].

Denote by $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$ and $\mathbb{N}_a^b = \{a, a+1, a+2, \ldots, b\}$ for any a, $b \in \mathbb{R}$ such that $b - a \in \mathbb{N}_1$. The backward jump operator $\rho : \mathbb{N}_a \to \mathbb{N}_a$ is defined by $\rho(t) = \max\{a, t-1\}$ for all $t \in \mathbb{N}_a$.

Definition 2.1 ([21])**.** Define the μ^{th}-order nabla fractional Taylor monomial by

$$H_\mu(t, a) = \frac{(t-a)^{\overline{\mu}}}{\Gamma(\mu+1)} = \frac{\Gamma(t-a+\mu)}{\Gamma(t-a)\Gamma(\mu+1)}, \quad \mu \in \mathbb{R} \setminus \{\ldots, -2, -1\}.$$

Here $\Gamma(\cdot)$ denotes the Euler gamma function. Observe that

$$H_\mu(a, a) = 0$$

and

$$H_\mu(t, a) = 0, \quad \text{for all } \mu \in \{\ldots, -2, -1\} \text{ and } t \in \mathbb{N}_a.$$

The first order backward (nabla) difference of $u : \mathbb{N}_a \to \mathbb{R}$ is defined by $\Big(\nabla u\Big)(t) = u(t) - u(t-1)$ for $t \in \mathbb{N}_{a+1}$.

Definition 2.2 ([21]). Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $\nu > 0$. The ν^{th}-order nabla sum of u based at a is given by

$$\Big(\nabla_a^{-\nu} u\Big)(t) = \sum_{s=a+1}^{t} H_{\nu-1}(t, \rho(s))u(s), \quad t \in \mathbb{N}_a,$$

where by convention $\Big(\nabla_a^{-\nu} u\Big)(a) = 0$.

Definition 2.3 ([21]). Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $0 < \nu \leq 1$. The ν^{th}-order nabla difference of u is given by

$$\Big(\nabla_a^{\nu} u\Big)(t) = \Big(\nabla\Big(\nabla_a^{-(1-\nu)} u\Big)\Big)(t), \quad t \in \mathbb{N}_{a+1}.$$

Lemma 2.1 ([21]). *We observe the following properties of nabla fractional Taylor monomials:*

(a) $\nabla H_\mu(t, a) = H_{\mu-1}(t, a)$, $t \in \mathbb{N}_a$;
(b) $\sum_{s=a+1}^{t} H_\mu(s, a) = H_{\mu+1}(t, a)$, $t \in \mathbb{N}_a$;
(c) $\sum_{s=a+1}^{t} H_\mu(t, \rho(s)) = H_{\mu+1}(t, a)$, $t \in \mathbb{N}_a$.

Proposition 2.1 ([24]). *Let $s \in \mathbb{N}_a$ and $-1 < \mu$. The following properties hold.*

(a) $H_\mu(t, \rho(s)) \geq 0$ *for* $t \in \mathbb{N}_{\rho(s)}$ *and* $H_\mu(t, \rho(s)) > 0$ *for* $t \in \mathbb{N}_s$.
(b) $H_\mu(t, \rho(s))$ *is a decreasing function with respect to* s *for* $t \in \mathbb{N}_{\rho(s)}$ *and* $\mu \in (0, \infty)$.
(c) *If* $t \in \mathbb{N}_s$ *and* $\mu \in (-1, 0)$, *then* $H_\mu(t, \rho(s))$ *is an increasing function of* s.
(d) $H_\mu(t, \rho(s))$ *is a non-decreasing function with respect to* t *for* $t \in \mathbb{N}_{\rho(s)}$ *and* $\mu \in [0, \infty)$.
(e) *If* $t \in \mathbb{N}_s$ *and* $\mu \in (0, \infty)$, *then* $H_\mu(t, \rho(s))$ *is an increasing function of* t.
(f) $H_\mu(t, \rho(s))$ *is a decreasing function with respect to* t *for* $t \in \mathbb{N}_{s+1}$ *and* $\mu \in (-1, 0)$.

Proposition 2.2 ([24]). *Let u and v be two nonnegative real-valued functions defined on a set S. Further, assume u and v achieve their maximum values in S. Then,*

$$|u(t) - v(t)| \leq \max\{u(t), v(t)\} \leq \max\Big\{\max_{t\in S} u(t), \max_{t\in S} v(t)\Big\},$$

for every fixed t in S.

3. Green's Function and Its Property

Assume $T \in \mathbb{N}_2$, $1 < \alpha < 2$ and $h : \mathbb{N}_2^T \to \mathbb{R}$. Consider the boundary value problem

$$\begin{cases} \Big(\nabla_0^{\alpha-1}\Big(\nabla u\Big)\Big)(t) + h(t) = 0, \quad t \in \mathbb{N}_2^T, \\ u(0) + u(T) = 0, \quad \Big(\nabla u\Big)(1) + \Big(\nabla u\Big)(T) = 0. \end{cases} \tag{3.1}$$

First, we construct the Green's function, $G(t,s)$ corresponding to (3.1), and obtain an expression for its unique solution. Denote by

$$D_1 = \{(t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T : t \geq s\}, \quad D_2 = \{(t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T : t \leq \rho(s)\}$$

and

$$\xi_\alpha = 2\left[1 + H_{\alpha-2}(T,0)\right]. \tag{3.2}$$

Theorem 3.1. *The unique solution of the nabla fractional boundary value problem* (3.1) *is given by*

$$u(t) = \sum_{s=2}^{T} G_\alpha(t,s)h(s), \quad t \in \mathbb{N}_0^T, \tag{3.3}$$

where

$$G_\alpha(t,s) = \begin{cases} K_\alpha(t,s) - H_{\alpha-1}(t,\rho(s)), & (t,s) \in D_1, \\ K_\alpha(t,s), & (t,s) \in D_2. \end{cases} \tag{3.4}$$

Here

$$\begin{aligned} K_\alpha(t,s) =& \frac{1}{\xi_\alpha}\Big[H_{\alpha-1}(T,\rho(s)) + 2H_{\alpha-1}(t,0)H_{\alpha-2}(T,\rho(s)) \\ &+ H_{\alpha-1}(T,\rho(s))H_{\alpha-2}(T,0) - H_{\alpha-1}(T,0)H_{\alpha-2}(T,\rho(s))\Big]. \end{aligned}$$

Proof. Denote by

$$\left(\nabla u\right)(t) = v(t), \quad t \in \mathbb{N}_1^T.$$

Subsequently, the difference equation in (3.1) takes the form

$$\left(\nabla_0^{\alpha-1} v\right)(t) + h(t) = 0, \quad t \in \mathbb{N}_2^T. \tag{3.5}$$

Let $v(1) = c_2$. Then, by Lemma 5.1 of [4], the unique solution of (3.5) is given by

$$v(t) = H_{\alpha-2}(t,0)c_2 - \left(\nabla_1^{-(\alpha-1)} h\right)(t), \quad t \in \mathbb{N}_1^T.$$

That is,

$$\left(\nabla u\right)(t) = H_{\alpha-2}(t,0)c_2 - \left(\nabla_1^{-(\alpha-1)} h\right)(t), \quad t \in \mathbb{N}_1^T. \tag{3.6}$$

Applying the first order nabla sum operator, ∇_0^{-1} on both sides of (3.6), we obtain

$$u(t) = c_1 + H_{\alpha-1}(t,0)c_2 - \left(\nabla_1^{-\alpha} h\right)(t), \quad t \in \mathbb{N}_0^T, \tag{3.7}$$

where $c_1 = u(0)$. We use the pair of anti-periodic boundary conditions considered in (3.1) to eliminate the constants c_1 and c_2 in (3.7). It follows from the first boundary condition $u(0) + u(T) = 0$ that

$$2c_1 + H_{\alpha-1}(T,0)c_2 = \left(\nabla_1^{-\alpha} h\right)(T). \tag{3.8}$$

The second boundary condition $\left(\nabla u\right)(1) + \left(\nabla u\right)(T) = 0$ yields

$$\left[1 + H_{\alpha-2}(T,0)\right]c_2 = \left(\nabla_1^{-(\alpha-1)} h\right)(T). \tag{3.9}$$

Solving (3.8) and (3.9) for c_1 and c_2, we obtain

$$c_1 = \frac{1}{2}\left[\sum_{s=2}^{T} H_{\alpha-1}(T,\rho(s))h(s) - \frac{2H_{\alpha-1}(T,0)}{\xi_\alpha}\sum_{s=2}^{T} H_{\alpha-2}(T,\rho(s))h(s)\right], \tag{3.10}$$

$$c_2 = \frac{2}{\xi_\alpha}\sum_{s=2}^{T} H_{\alpha-2}(T,\rho(s))h(s). \tag{3.11}$$

Substituting these expressions in (3.7), we achieve (3.4). □

Lemma 3.1. *Observe that*

$$|K_\alpha(t,s)| \le \frac{1}{\xi_\alpha}\Big[H_{\alpha-1}(T,1) + 2H_{\alpha-1}(T,0) + H_{\alpha-2}(T,0)H_{\alpha-1}(T,1)\Big], \tag{3.12}$$

for all $(t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T$.

Proof. Denote by

$$\begin{aligned} K'_\alpha(t,s) =& \frac{1}{\xi_\alpha}\Big[H_{\alpha-1}(T,\rho(s)) + 2H_{\alpha-1}(t,0)H_{\alpha-2}(T,\rho(s)) \\ &+ H_{\alpha-1}(T,\rho(s))H_{\alpha-2}(T,0)\Big] \end{aligned} \tag{3.13}$$

and

$$K''_\alpha(t,s) = \frac{1}{\xi_\alpha}\Big[H_{\alpha-1}(T,0)H_{\alpha-2}(T,\rho(s))\Big], \tag{3.14}$$

so that

$$K_\alpha(t,s) = K'_\alpha(t,s) - K''_\alpha(t,s), \quad (t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T.$$

Clearly, from Proposition 2.1,

$$K'_\alpha(t,s) \ge 0, \quad K''_\alpha(t,s) > 0, \quad \text{for all } (t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T.$$

From Proposition 2.2, it is obvious that

$$|K_\alpha(t,s)| \le \left\{\max_{(t,s)\in\mathbb{N}_0^T\times\mathbb{N}_2^T} K'_\alpha(t,s), \max_{(t,s)\in\mathbb{N}_0^T\times\mathbb{N}_2^T} K''_\alpha(t,s)\right\}. \tag{3.15}$$

First, we evaluate the first backward difference of $K'_\alpha(t,s)$ with respect to t for a fixed s. Consider

$$\nabla K'_\alpha(t,s) = \frac{1}{\xi_\alpha}\Big[2H_{\alpha-2}(t,0)H_{\alpha-2}(T,\rho(s))\Big] > 0,$$

for all $(t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T$, implying that $K'_\alpha(t,s)$ is an increasing function of t for a fixed s. Thus, we have

$$K'_\alpha(t,s) \le K'_\alpha(T,s), \quad (t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T. \tag{3.16}$$

It follows from (3.13)–(3.16) that

$$
\begin{aligned}
&|K_\alpha(t,s)|\\
\leq&\left\{\max_{(t,s)\in\mathbb{N}_0^T\times\mathbb{N}_2^T} K'_\alpha(t,s), \max_{(t,s)\in\mathbb{N}_0^T\times\mathbb{N}_2^T} K''_\alpha(t,s)\right\}\\
\leq&\left\{\max_{s\in\mathbb{N}_2^T} K'_\alpha(T,s), \max_{s\in\mathbb{N}_2^T} K''_\alpha(t,s)\right\}\\
=&\max_{s\in\mathbb{N}_2^T} K'_\alpha(T,s)\\
=&\frac{1}{\xi_\alpha}\max_{s\in\mathbb{N}_2^T}\Big[H_{\alpha-1}(T,\rho(s)) + 2H_{\alpha-1}(T,0)H_{\alpha-2}(T,\rho(s))\\
&+ H_{\alpha-1}(T,\rho(s))H_{\alpha-2}(T,0)\Big]\\
\leq&\frac{1}{\xi_\alpha}\Big[\max_{s\in\mathbb{N}_2^T} H_{\alpha-1}(T,\rho(s)) + 2H_{\alpha-1}(T,0)\max_{s\in\mathbb{N}_2^T} H_{\alpha-2}(T,\rho(s))\\
&+ H_{\alpha-2}(T,0)\max_{s\in\mathbb{N}_2^T} H_{\alpha-1}(T,\rho(s))\Big]\\
=&\frac{1}{\xi_\alpha}\Big[H_{\alpha-1}(T,\rho(2)) + 2H_{\alpha-1}(T,0)H_{\alpha-2}(T,\rho(T)) + H_{\alpha-2}(T,0)H_{\alpha-1}(T,\rho(2))\Big]\\
=&\frac{1}{\xi_\alpha}\Big[H_{\alpha-1}(T,1) + 2H_{\alpha-1}(T,0) + H_{\alpha-2}(T,0)H_{\alpha-1}(T,1)\Big].
\end{aligned}
$$

The proof is complete. □

4. Existence and Uniqueness of Solutions of (1.1)

Let $X = \mathbb{R}^{T+1}$ be the Banach space of all real $(T+1)$-tuples equipped with the maximum norm

$$\|u\|_X = \max_{t\in\mathbb{N}_0^T} |u(t)|.$$

Obviously, the product space $\Big(X\times X, \|\cdot\|_{X\times X}\Big)$ is also a Banach space with the norm

$$\|(u_1,u_2)\|_{X\times X} = \|u_1\|_X + \|u_2\|_X.$$

A closed ball with radius R centred on the zero function in $X\times X$ is defined by

$$\mathcal{B}_R = \{(u_1,u_2)\in X\times X : \|(u_1,u_2)\|_{X\times X} \leq R\}.$$

Define the operator $T : X\times X \to X\times X$ by

$$T(u_1,u_2)(t) = \begin{pmatrix} T_1(u_1,u_2)(t)\\ T_2(u_1,u_2)(t)\end{pmatrix}, \quad t\in\mathbb{N}_0^T, \tag{4.1}$$

where

$$T_1(u_1,u_2)(t) = \sum_{s=2}^{T} G_{\alpha_1}(t,s) f_1(s,u_1(s),u_2(s))$$

$$= \sum_{s=2}^{T} K_{\alpha_1}(t,s) f_1(s,u_1(s),u_2(s)) - \sum_{s=2}^{t} H_{\alpha_1-1}(t,s) f_1(s,u_1(s),u_2(s)) \tag{4.2}$$

and

$$T_2(u_1,u_2)(t) = \sum_{s=2}^{T} G_{\alpha_2}(t,s) f_2(s,u_1(s),u_2(s))$$

$$= \sum_{s=2}^{T} K_{\alpha_2}(t,s) f_2(s,u_1(s),u_2(s)) - \sum_{s=2}^{t} H_{\alpha_2-1}(t,s) f_2(s,u_1(s),u_2(s)). \tag{4.3}$$

Clearly, (u_1,u_2) is a fixed point of T if and only if (u_1,u_2) is a solution of (1.1). For our convenience, denote by

$$\Lambda_i = \frac{1}{\xi_{\alpha_i}} \Big[H_{\alpha_i-1}(T,1) + 2H_{\alpha_i-1}(T,0) + H_{\alpha_i-2}(T,0) H_{\alpha_i-1}(T,1) \Big], \tag{4.4}$$

$$a_i = l_i \left[\Lambda_i (T-1) + H_{\alpha_i}(T,1) \right], \tag{4.5}$$

$$b_i = m_i \left[\Lambda_i (T-1) + H_{\alpha_i}(T,1) \right], \tag{4.6}$$

$$c_i = n_i \left[\Lambda_i (T-1) + H_{\alpha_i}(T,1) \right], \tag{4.7}$$

$$d_i = M_i \left[\Lambda_i (T-1) + H_{\alpha_i}(T,1) \right], \tag{4.8}$$

for $i=1,2$. Assume

(H1)' for each $i \in \{1,2\}$, there exist nonnegative numbers l_i and m_i such that

$$|f_i(t,u_1,u_2) - f_i(t,v_1,v_2)| \le l_i \|u_1 - v_1\|_X + m_i \|u_2 - v_2\|_X,$$

for all (t,u_1,u_2), $(t,v_1,v_2) \in \mathbb{N}_0^T \times X \times X$;

(H1) for each $i \in \{1,2\}$, there exist nonnegative numbers l_i and m_i such that

$$|f_i(t,u_1,u_2) - f_i(t,v_1,v_2)| \le l_i \|u_1 - v_1\|_X + m_i \|u_2 - v_2\|_X,$$

for all (t,u_1,u_2), $(t,v_1,v_2) \in \mathbb{N}_0^T \times \mathcal{B}_R$;

(H2)' for each $i \in \{1,2\}$, there exist nonnegative numbers L_i such that

$$|f_i(t,u_1,u_2)| \le L_i,$$

for all $(t,u_1,u_2) \in \mathbb{N}_0^T \times X \times X$;

(H2) for each $i \in \{1,2\}$, there exist nonnegative numbers l_i, m_i, and n_i such that

$$|f_i(t,u_1,u_2)| \le l_i \|u_1\|_X + m_i \|u_2\|_X + n_i,$$

for all $(t,u_1,u_2) \in \mathbb{N}_0^T \times \mathcal{B}_R$;

(H3) for each $i \in \{1,2\}$,

$$\max_{t \in \mathbb{N}_0^T} |f_i(t,0,0)| = M_i;$$

(H4) $\lambda = (a_1 + a_2) + (b_1 + b_2) \in (0,1)$.

We apply Banach's fixed point theorem to establish existence and uniqueness of solutions of (1.1).

Theorem 4.1 ([37])**.** *Let S be a closed subset of a Banach space X. Then, any contraction mapping T of X into itself has a unique fixed point.*

Theorem 4.2. *Assume* (H1)*,* (H3) *and* (H4) *hold. If we choose*

$$R \geq \frac{(d_1 + d_2)}{1 - [(a_1 + a_2) + (b_1 + b_2)]},$$

then the system (1.1) *has a unique solution* $(u_1, u_2) \in \mathcal{B}_R$*.*

Proof. Clearly, $T : \mathcal{B}_R \to X \times X$. First, we show that T is a contraction mapping. To see this, let (u_1, u_2), $(v_1, v_2) \in \mathcal{B}_R$, and $t \in \mathbb{N}_0^T$. For each $i \in \{1, 2\}$, consider

$$\begin{aligned}
&|T_i(u_1, u_2)(t) - T_i(v_1, v_2)(t)| \\
\leq &\sum_{s=2}^{T} |K_{\alpha_i}(t, s)|\, |f_i(s, u_1(s), u_2(s)) - f_i(s, v_1(s), v_2(s))| \\
&+ \sum_{s=2}^{t} H_{\alpha_i - 1}(t, s)\, |f_i(s, u_1(s), u_2(s)) - f_i(s, v_1(s), v_2(s))| \\
\leq &[l_i \|u_1 - v_1\|_X + m_i \|u_2 - v_2\|_X] \left[\sum_{s=2}^{T} |K_{\alpha_i}(t, s)| + \sum_{s=2}^{t} H_{\alpha_i - 1}(t, s)\right] \\
\leq &[l_i \|u_1 - v_1\|_X + m_i \|u_2 - v_2\|_X] [\Lambda_i(T - 1) + H_{\alpha_i}(t, 1)] \\
\leq &[l_i \|u_1 - v_1\|_X + m_i \|u_2 - v_2\|_X] [\Lambda_i(T - 1) + H_{\alpha_i}(T, 1)] \\
\leq &a_i \|u_1 - v_1\|_X + b_i \|u_2 - v_2\|_X,
\end{aligned}$$

implying that, for each $i \in \{1, 2\}$,

$$\Big\|T_i(u_1, u_2) - T_i(v_1, v_2)\Big\|_X \leq \Big[a_i \|u_1 - v_1\|_X + b_i \|u_2 - v_2\|_X\Big]. \tag{4.9}$$

Thus, we have

$$\begin{aligned}
&\|T(u_1, u_2) - T(v_1, v_2)\|_{X \times X} \\
= &\Big\|T_1(u_1, u_2) - T_1(v_1, v_2)\Big\|_X + \Big\|T_2(u_1, u_2) - T_2(v_1, v_2)\Big\|_X \\
\leq &\Big[(a_1 + a_2)\|u_1 - v_1\|_X + (b_1 + b_2)\|u_2 - v_2\|_X\Big] \\
\leq &\lambda\Big[(\|u_1 - v_1\|_X + \|u_2 - v_2\|_X\Big] \\
= &\lambda \|(u_1, u_2) - (v_1, v_2)\|_{X \times X}.
\end{aligned}$$

Since $\lambda < 1$, T is a contraction mapping with contraction constant λ. Next, we show that

$$T : \mathcal{B}_R \to \mathcal{B}_R. \tag{4.10}$$

To see this, let $(u_1, u_2) \in \mathcal{B}_R$, and $t \in \mathbb{N}_0^T$. For each $i \in \{1, 2\}$, consider

$$\begin{aligned}
&\Big|T_i(u_1, u_2)(t)\Big| \\
\leq& \sum_{s=2}^{T} |K_{\alpha_i}(t,s)|\,|f_i(s, u_1(s), u_2(s))| + \sum_{s=2}^{t} H_{\alpha_i-1}(t,s)\,|f_i(s, u_1(s), u_2(s))| \\
\leq& \sum_{s=2}^{T} |K_{\alpha_i}(t,s)|\,|f_i(s, u_1(s), u_2(s)) - f_i(s,0,0)| + \sum_{s=2}^{T} |K_{\alpha_i}(t,s)|\,|f_i(s,0,0)| \\
&+ \sum_{s=2}^{t} H_{\alpha_i-1}(t,s)\,|f_i(s, u_1(s), u_2(s)) - f_i(s,0,0)| + \sum_{s=2}^{t} H_{\alpha_i-1}(t,s)\,|f_i(s,0,0)| \\
\leq& \Big[l_i\|u_1\|_X + m_i\|u_2\|_X\Big] \sum_{s=2}^{T} |K_{\alpha_i}(t,s)| + M_i \sum_{s=2}^{T} |K_{\alpha_i}(t,s)| \\
&+ \Big[l_i\|u_1\|_X + m_i\|u_2\|_X\Big] \sum_{s=2}^{t} H_{\alpha_i-1}(t,s) + M_i \sum_{s=2}^{t} H_{\alpha_i-1}(t,s) \\
\leq& \Big[l_i\|u_1\|_X + m_i\|u_2\|_X + M_i\Big]\,[\Lambda_i(T-1) + H_{\alpha_i}(t,1)] \\
\leq& \Big[l_i\|u_1\|_X + m_i\|u_2\|_X + M_i\Big]\,[\Lambda_i(T-1) + H_{\alpha_i}(T,1)] \\
\leq& a_i\|u_1\|_X + b_i\|u_2\|_X + d_i,
\end{aligned}$$

implying that, for each $i \in \{1, 2\}$,

$$\Big\|T_i(u_1, u_2)\Big\|_X \leq a_i\|u_1\|_X + b_i\|u_2\|_X + d_i. \tag{4.11}$$

Thus, we have

$$\begin{aligned}
\|T(u_1, u_2)\|_{X\times X} &= \Big\|T_1(u_1, u_2)\Big\|_X + \Big\|T_2(u_1, u_2)\Big\|_X \\
&\leq (a_1 + a_2)R + (b_1 + b_2)R + (d_1 + d_2) \leq R,
\end{aligned}$$

implying that (4.10) holds. Therefore, by Theorem 4.1, T has a unique fixed point $(u_1, u_2) \in \mathcal{B}_R$. The proof is complete. □

Corollary 4.1. *Assume* (H1)' *and* (H4) *hold. Then, the system* (1.1) *has a unique solution* $(u_1, u_2) \in X \times X$.

We apply Brouwer's fixed point theorem to establish existence of solutions of (1.1).

Theorem 4.3 ([37])**.** *Let C be a non-empty bounded closed convex subset of $\mathbb{R}^n$ and $T : C \to C$ be a continuous mapping. Then, T has a fixed point in C.*

Theorem 4.4. *Assume* (H2) *and* (H4) *hold. If we choose*

$$R \geq \frac{(c_1 + c_2)}{1 - [(a_1 + a_2) + (b_1 + b_2)]},$$

then the system (1.1) *has at least one solution* $(u_1, u_2) \in \mathcal{B}_R$.

Proof. We claim that $T : B_R \to B_R$. To see this, let $(u_1, u_2) \in \mathcal{B}_R$ and $t \in \mathbb{N}_0^T$. For each $i \in \{1, 2\}$, consider

$$\begin{aligned}
&\Big|T_i(u_1, u_2)(t)\Big| \\
\leq &\sum_{s=2}^{T} |K_{\alpha_i}(t,s)| \, |f_i(s, u_1(s), u_2(s))| + \sum_{s=2}^{t} H_{\alpha_i - 1}(t,s) \, |f_i(s, u_1(s), u_2(s))| \\
\leq &\Big[l_i\|u_1\|_X + m_i\|u_2\|_X + n_i\Big] \sum_{s=2}^{T} |K_{\alpha_i}(t,s)| \\
&+ \Big[l_i\|u_1\|_X + m_i\|u_2\|_X + n_i\Big] \sum_{s=2}^{t} H_{\alpha_i - 1}(t,s) \\
\leq &\Big[l_i\|u_1\|_X + m_i\|u_2\|_X + n_i\Big] \left[\Lambda_i(T-1) + H_{\alpha_i}(t,1)\right] \\
\leq &\Big[l_i\|u_1\|_X + m_i\|u_2\|_X + n_i\Big] \left[\Lambda_i(T-1) + H_{\alpha_i}(T,1)\right] \\
\leq &a_i\|u_1\|_X + b_i\|u_2\|_X + c_i,
\end{aligned}$$

implying that, for each $i \in \{1, 2\}$,

$$\Big\|T_i(u_1, u_2)\Big\|_X \leq a_i\|u_1\|_X + b_i\|u_2\|_X + c_i. \tag{4.12}$$

Thus, we have

$$\begin{aligned}
\|T(u_1, u_2)\|_{X \times X} &= \Big\|T_1(u_1, u_2)\Big\|_X + \Big\|T_2(u_1, u_2)\Big\|_X \\
&\leq (a_1 + a_2)R + (b_1 + b_2)R + (c_1 + c_2) \leq R,
\end{aligned}$$

implying that $T : B_R \to B_R$. Therefore, by Brouwer's fixed point theorem, T has a fixed point $(u_1, u_2) \in \mathcal{B}_R$. The proof is complete. □

Corollary 4.2. *Assume* (H2)' *hold. Then, the system* (1.1) *has at least one solution* $(u_1, u_2) \in X \times X$.

Urs [39] presented some Ulam-Hyers stability results for the coupled fixed point of a pair of contractive type operators on complete metric spaces. We use Urs's [39] approach to establish Ulam-Hyers stability of solutions of (1.1).

Definition 4.1 ([39])**.** Let X be a Banach space and T_1, $T_2 : X \times X \to X$ be two operators. Then, the operational equations system

$$\begin{cases} u_1 = T_1(u_1, u_2), \\ u_2 = T_2(u_1, u_2), \end{cases} \tag{4.13}$$

is said to be Ulam-Hyers stable if there exist C_1, C_2, C_3, $C_4 > 0$ such that for each ε_1, $\varepsilon_2 > 0$ and each solution-pair $(u_1^*, u_2^*) \in X \times X$ of the in-equations:

$$\begin{cases} \|u_1 - T_1(u_1, u_2)\|_X \leq \varepsilon_1, \\ \|u_2 - T_2(u_1, u_2)\|_X \leq \varepsilon_2, \end{cases} \tag{4.14}$$

there exists a solution $(v_1^*, v_2^*) \in X \times X$ of (4.13) such that

$$\begin{cases} \|u_1^* - v_1^*\|_X \leq C_1\varepsilon_1 + C_2\varepsilon_2, \\ \|u_2^* - v_2^*\|_X \leq C_3\varepsilon_1 + C_4\varepsilon_2. \end{cases} \tag{4.15}$$

Theorem 4.5 ([39])**.** *Let X be a Banach space, T_1, $T_2 : X \times X \to X$ be two operators such that*

$$\begin{cases} \|T_1(u_1, u_2) - T_1(v_1, v_2)\|_X \leq k_1\|u_1 - v_1\|_X + k_2\|u_2 - v_2\|_X, \\ \|T_2(u_1, u_2) - T_2(v_1, v_2)\|_X \leq k_3\|u_1 - v_1\|_X + k_4\|u_2 - v_2\|_X, \end{cases} \tag{4.16}$$

for all (u_1, u_2), $(v_1, v_2) \in X \times X$. Suppose

$$H = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}$$

converges to zero. Then, the operational equations system (4.13) *is Ulam-Hyers stable.*

Set

$$H = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}. \tag{4.17}$$

Theorem 4.6. *Assume the hypothesis of Theorem* 4.2 *holds. Further, assume the spectral radius of H is less than one. Then, the unique solution of the system* (1.1) *is Ulam-Hyers stable.*

Proof. In view of Theorem 4.2, we have

$$\begin{cases} \Big\|T_1(u_1, u_2) - T_1(v_1, v_2)\Big\|_X \leq a_1\|u_1 - v_1\|_X + b_1\|u_2 - v_2\|_X, \\ \Big\|T_2(u_1, u_2) - T_2(v_1, v_2)\Big\|_X \leq a_2\|u_1 - v_1\|_X + b_2\|u_2 - v_2\|_X, \end{cases} \tag{4.18}$$

which implies that

$$\|T(u_1, u_2) - T(v_1, v_2)\|_{X\times X} \leq H \begin{pmatrix} \|u_1 - v_1\|_X \\ \|u_2 - v_2\|_X \end{pmatrix}. \tag{4.19}$$

Since the spectral radius of H is less than one, the unique solution of (1.1) is Ulam-Hyers stable. The proof is complete. □

5. Examples

In this section, we provide two examples to illustrate the applicability of Theorem 4.2, Theorem 4.4 and Theorem 4.6.

Example 5.1. Consider the following coupled system of two-point nabla fractional difference boundary value problems

(5.1)
$$\begin{cases} \Big(\nabla_0^{0.5}\big(\nabla u_1\big)\Big)(t) + (0.001)e^{-t}\left[1 + \tan^{-1} u_1(t) + \tan^{-1} u_2(t)\right] = 0, \quad t \in \mathbb{N}_2^9, \\ \Big(\nabla_0^{0.5}\big(\nabla u_2\big)\Big)(t) + (0.002)\left[e^{-t} + \sin u_1(t) + \sin u_2(t)\right] = 0, \quad t \in \mathbb{N}_2^9, \\ u_1(0) + u_1(9) = 0, \quad \big(\nabla u_1\big)(1) + \big(\nabla u_1\big)(9) = 0, \\ u_2(0) + u_2(9) = 0, \quad \big(\nabla u_2\big)(1) + \big(\nabla u_2\big)(9) = 0. \end{cases}$$

Comparing (1.1) and (5.1), we have $T = 9$, $\alpha_1 = \alpha_2 = 1.5$,

$$f_1(t, u_1, u_2) = (0.001)e^{-t}\left[1 + \tan^{-1} u_1 + \tan^{-1} u_2\right]$$

and

$$f_2(t, u_1, u_2) = (0.002)\left[e^{-t} + \sin u_1 + \sin u_2\right],$$

for all $(t, u_1, u_2) \in \mathbb{N}_0^9 \times \mathbb{R}^2$. Clearly, f_1 and f_2 are continuous on $\mathbb{N}_0^9 \times \mathbb{R}^2$. Next, f_1 and f_2 satisfy assumption (H1) with $l_1 = 0.001$, $m_1 = 0.001$, $l_2 = 0.002$ and $m_2 = 0.002$. We have

$$\begin{aligned} M_1 &= \max_{t \in \mathbb{N}_0^9} |f_1(t, 0, 0)| = 0.001, \\ M_2 &= \max_{t \in \mathbb{N}_0^9} |f_2(t, 0, 0)| = 0.002, \\ a_1 &= l_1\left[\Lambda_1(T-1) + H_{\alpha_1}(T, 1)\right] = 0.0527, \\ a_2 &= l_2\left[\Lambda_2(T-1) + H_{\alpha_2}(T, 1)\right] = 0.1053, \\ b_1 &= m_1\left[\Lambda_1(T-1) + H_{\alpha_1}(T, 1)\right] = 0.0527, \\ b_2 &= m_2\left[\Lambda_2(T-1) + H_{\alpha_2}(T, 1)\right] = 0.1053, \\ d_1 &= M_1\left[\Lambda_1(T-1) + H_{\alpha_1}(T, 1)\right] = 0.0527, \\ d_2 &= M_2\left[\Lambda_1(T-1) + H_{\alpha_1}(T, 1)\right] = 0.1053. \end{aligned}$$

Also, $\lambda = (a_1 + a_2) + (b_1 + b_2) = 0.316 \in (0, 1)$, implying that assumptions (H3) and (H4) hold. Choose

$$R \geq \frac{(d_1 + d_2)}{1 - [(a_1 + a_2) + (b_1 + b_2)]} = 0.231.$$

Hence, by Theorem 4.2, the system (5.1) has a unique solution $(u_1, u_2) \in \mathcal{B}_R$. Further,

$$M = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} 0.0527 & 0.0527 \\ 0.1053 & 0.1053 \end{pmatrix}.$$

The spectral radius of M is 0.158, which is less than one, implying that M converges to zero. Thus, by Theorem 4.6, the unique solution of (5.1) is Ulam-Hyers stable.

Example 5.2. Consider the following coupled system of two-point nabla fractional difference boundary value problems

$$(5.2)\qquad \begin{cases} \Big(\nabla_0^{0.5}\big(\nabla u_1\big)\Big)(t) + (0.01)\left[e^{-t} + \frac{1}{\sqrt{1+u_1^2(t)}} + u_2(t)\right] = 0, & t \in \mathbb{N}_2^4, \\ \Big(\nabla_0^{0.5}\big(\nabla u_2\big)\Big)(t) + (0.02)\left[e^{-t} + u_1(t) + \frac{1}{\sqrt{1+u_2^2(t)}}\right] = 0, & t \in \mathbb{N}_2^4, \\ u_1(0) + u_1(4) = 0, \quad \big(\nabla u_1\big)(1) + \big(\nabla u_1\big)(4) = 0, \\ u_2(0) + u_2(4) = 0, \quad \big(\nabla u_2\big)(1) + \big(\nabla u_2\big)(4) = 0. \end{cases}$$

Comparing (1.1) and (5.2), we have $T = 4$, $\alpha_1 = \alpha_2 = 1.5$,

$$f_1(t, u_1, u_2) = (0.01)\left[e^{-t} + \frac{1}{\sqrt{1+u_1^2}} + u_2\right]$$

and

$$f_2(t, u_1, u_2) = (0.02)\left[e^{-t} + u_1 + \frac{1}{\sqrt{1+u_2^2}}\right],$$

for all $(t, u_1, u_2) \in \mathbb{N}_0^4 \times \mathbb{R}^2$. Clearly, f_1 and f_2 are continuous on $\mathbb{N}_0^4 \times \mathbb{R}^2$. Next, f_1 and f_2 satisfy assumption (H2) with $l_1 = 0.01$, $m_1 = 0.01$, $l_2 = 0.02$, $m_2 = 0.02$, $n_1 = 0.01$ and $n_2 = 0.02$. We have

$$\begin{aligned} a_1 &= l_1\left[\Lambda_1(T-1) + H_{\alpha_1}(T,1)\right] = 0.1219, \\ a_2 &= l_2\left[\Lambda_2(T-1) + H_{\alpha_2}(T,1)\right] = 0.2438, \\ b_1 &= m_1\left[\Lambda_1(T-1) + H_{\alpha_1}(T,1)\right] = 0.1219, \\ b_2 &= m_2\left[\Lambda_2(T-1) + H_{\alpha_2}(T,1)\right] = 0.2438, \\ c_1 &= n_1\left[\Lambda_1(T-1) + H_{\alpha_1}(T,1)\right] = 0.1219, \\ c_2 &= n_2\left[\Lambda_2(T-1) + H_{\alpha_2}(T,1)\right] = 0.2438. \end{aligned}$$

Also, $\lambda = (a_1 + a_2) + (b_1 + b_2) = 0.7314 \in (0, 1)$, implying that assumption (H4) hold. Choose

$$R \geq \frac{(c_1 + c_2)}{1 - [(a_1 + a_2) + (b_1 + b_2)]} = 1.3615.$$

Hence, by Theorem 4.2, the system (5.1) has at least one solution $(u_1, u_2) \in \mathcal{B}_R$.

Acknowledgements. We thank the referees for their careful review and constructive comments on the manuscript. Authors also thankful to DST New Delhi, Government of India, for providing DST-FIST grant with Reference No. 337 to Department of Mathematics, Birla Institute of Technology and Science Pilani, Hyderabad Campus.

References

[1] T. Abdeljawad, *On Riemann and Caputo fractional differences*, Comput. Math. Appl. **62**(3) (2011), 1602–1611. https://doi.org/10.1016/j.camwa.2011.03.036

[2] T. Abdeljawad and J. Alzabut, *On Riemann-Liouville fractional q-difference equations and their application to retarded logistic type model*, Math. Methods Appl. Sci. **41**(18) (2018), 8953–8962. https://doi.org/10.1002/mma.4743

[3] T. Abdeljawad, J. Alzabut and H. Zhou, *A Krasnoselskii existence result for nonlinear delay Caputo q-fractional difference equations with applications to Lotka-Volterra competition model*, Appl. Math. E-Notes **17** (2017), 307–318.

[4] T. Abdeljawad and F. M. Atici, *On the definitions of nabla fractional operators*, Abstr. Appl. Anal. **2012** (2012), Article ID 406757, 13 pages. https://doi.org/10.1155/2012/406757

[5] R. P. Agarwal, B. Ahmad and J. J. Nieto, *Fractional differential equations with nonlocal (parametric type) anti-periodic boundary conditions*, Filomat **31**(5) (2017), 1207–1214. https://doi.org/10.2298/FIL1705207A

[6] B. Ahmad and J. J. Nieto, *Existence of solutions for anti-periodic boundary value problems involving fractional differential equations via Leray-Schauder degree theory*, Topol. Methods Nonlinear Anal. **35**(2) (2010), 295–304.

[7] B. Ahmad and J. J. Nieto, *Anti-periodic fractional boundary value problems*, Comput. Math. Appl. **62**(3) (2011), 1150–1156. https://doi.org/10.1016/j.camwa.2011.02.034

[8] K. Ahrendt, L. Castle, M. Holm and K. Yochman, *Laplace transforms for the nabla-difference operator and a fractional variation of parameters formula*, Commun. Appl. Anal. **16**(3) (2012), 317–347.

[9] J. Alzabut, T. Abdeljawad and D. Baleanu, *Nonlinear delay fractional difference equations with applications on discrete fractional Lotka-Volterra competition model*, J. Comput. Anal. Appl. **25**(5) (2018), 889–898.

[10] J. Alzabut, T. Abdeljawad and H. Alrabaiah, *Oscillation criteria for forced and damped nabla fractional difference equations*, J. Comput. Anal. Appl. **24**(8) (2018), 1387–1394.

[11] F. M. Atici and P. W. Eloe, *Discrete fractional calculus with the nabla operator*, Electron. J. Qual. Theory Differ. Equ. Special Edition I (3) (2009), 1–12. https://doi.org/10.14232/EJQTDE.2009.4.3

[12] H. Baghani, J. Alzabut and J. J. Nieto, *A coupled system of Langevin differential equations of fractional order and associated to antiperiodic boundary conditions*, Math. Methods Appl. Sci. (2020), 1–11. https://doi.org/10.1002/mma.6639

[13] M. Benchohra, N. Hamidi and J. Henderson, *Fractional differential equations with anti-periodic boundary conditions*, Numer. Funct. Anal. Optim. **34**(4) (2013), 404–414. https://doi.org/10.1080/01630563.2012.763140

[14] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, MA, 2001.

[15] A. Brackins, *Boundary value problems of nabla fractional difference equations*, Ph. D. Thesis, The University of Nebraska-Lincoln, Nebraska-Lincoln, 2014.

[16] G. Chai, *Anti-periodic boundary value problems of fractional differential equations with the Riemann-Liouville fractional derivative*, Adv. Difference Equ. **306** (2013), 1–24. https://doi.org/10.1186/1687-1847-2013-306

[17] C. Chen, M. Bohner and B. Jia, *Ulam-Hyers stability of Caputo fractional difference equations*, Math. Methods Appl. Sci. (2019), 1–10. https://doi.org/10.1002/mma.5869

[18] F. Chen and Y. Zhou, *Existence and Ulam stability of solutions for discrete fractional boundary value problem*, Discrete Dyn. Nat. Soc. (2013), Article ID 459161, 7 pages. https://doi.org/10.1155/2013/459161

[19] Y. Chen, J. J. Nieto and D. O'Regan, *Anti-periodic solutions for fully nonlinear first-order differential equations*, Math. Comput. Model. **46**(9–10) (2007), 1183–1190. https://doi.org/10.1016/j.mcm.2006.12.006

[20] Y. Gholami and K. Ghanbari, *Coupled systems of fractional* ∇*-difference boundary value problems*, Differ. Equ. Appl. **8**(4) (2016), 459–470. https://dx.doi.org/10.7153/dea-08-26

[21] C. Goodrich and A. C. Peterson, *Discrete Fractional Calculus*, Springer, Cham, 2015.

[22] J. Hein, S. McCarthy, N. Gaswick, B. McKain and K. Speer, *Laplace transforms for the nabla difference operator*, PanAmer. Math. J. **21** (2011), 79–96.

[23] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. USA **27** (1941), 222–224. https://doi.org/10.1073/pnas.27.4.222

[24] A. Ikram, *Lyapunov inequalities for nabla Caputo boundary value problems*, J. Difference Equ. Appl. **25**(6) (2019), 757–775. https://doi.org/10.1080/10236198.2018.1560433

[25] J. J. Mohan, *Hyers-Ulam stability of fractional nabla difference equations*, Int. J. Anal. (2016), Article ID 7265307, 1–5. https://doi.org/10.1155/2016/7265307

[26] J. M. Jonnalagadda, *On two-point Riemann-Liouville type nabla fractional boundary value problems*, Adv. Dyn. Syst. Appl. **13**(2) (2018), 141–166.

[27] J. M. Jonnalagadda, *Existence results for solutions of nabla fractional boundary value problems with general boundary conditions*, Advances in the Theory of Nonlinear Analysis and its Application **4**(1) (2020), 29–42. https://doi.org/10.31197/atnaa.634557

[28] S. M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer, New York, 2011.

[29] W. G. Kelley and A. C. Peterson, *Difference Equations: An Introduction with Applications*, Second edition, Harcourt/Academic Press, San Diego, CA, 2001.

[30] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier Science B. V., Amsterdam, 2006.

[31] J. Liu and Z. Liu, *On the existence of anti-periodic solutions for implicit differential equations*, Acta Math. Hungar. **132**(3) (2011), 294–305. https://doi.org/10.1007/s10474-010-0054-2

[32] J. W. Lyons and J. T. Neugebauer, *A difference equation with anti-periodic boundary conditions*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **22**(1) (2015), 47–60.

[33] M. M. Matar, J. Alzabut and J. M. Jonnalagadda, *A coupled system of nonlinear Caputo-Hadamard Langevin equations associated with nonperiodic boundary conditions*, Math. Methods Appl. Sci. (2020), 1–21. https://doi.org/10.1002/mma.6711

[34] I. Podlubny, *Fractional Differential Equations*, Mathematics in Science and Engineering **198**, Academic Press, San Diego, CA, 1999.

[35] T. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72**(2) (1978), 297–300. https://doi.org/10.2307/2042795

[36] J. Sun, Y. Liu and G. Liu, *Existence of solutions for fractional differential systems with anti-periodic boundary conditions*, Comput. Math. Appl. **64**(6) (2012), 1557–1566. https://doi.org/10.1016/j.camwa.2011.12.083

[37] D. R. Smart, *Fixed Point Theorems*, Cambridge Tracts in Mathematics **66**, Cambridge University Press, London-New York, 1974.

[38] S. A. Ulam, *Collection of Mathematical Problems*, Interscience Tracts in Pure and Applied Mathematics **8**, New York, London, Interscience Publishers, 1960.

[39] C. Urs, *Coupled fixed point theorems and applications to periodic boundary value problems*, Miskolc Math. Notes **14**(1) (2013), 323–333. https://doi.org/10.18514/MMN.2013.598

[40] Y. Wang and Y. Shi, *Eigenvalues of second-order difference equations with periodic and anti-periodic boundary conditions*, J. Math. Anal. Appl. **309**(1) (2005), 56–69. https://doi.org/10.1016/j.jmaa.2004.12.010

[41] A. Zada, H. Waheed, J. Alzabut and X. Wang, *Existence and stability of impulsive coupled system of fractional integrodifferential equations*, Demonstr. Math. **52**(1) (2019), 296–335. https://doi.org/10.1515/dema-2019-0035

[42] H. Zhang and W. Gao, *Existence and uniqueness results for a coupled system of nonlinear fractional differential equations with anti-periodic boundary conditions*, Abstr. Appl. Anal. (2014), Article ID 463517, 7 pages. https://doi.org/10.1155/2014/463517

[1]Department of Mathematics,
Birla Institute of Technology and Science Pilani,
Hyderabad - 500078, Telangana, India.
Email address: j.jaganmohan@hotmail.com