Kragujevac Journal of Mathematics
Volume 47(5) (2023), Pages 691–700.

APPROXIMATION BY AN EXPONENTIAL-TYPE COMPLEX OPERATORS

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Abstract. In the present paper, we discuss the approximation properties of a complex exponential kind operator. Upper estimate, Voronovskaya-type formula and exact estimate are obtained.

1. Introduction

In the year 1978, Ismail [10] and Ismail and May [11] introduced and studied some exponential type operators. A type of the operators constructed in [11, (3.11)] is the following sequence

$$Q_n(f,x)=\int_0^\infty W(n,x,t)f(t)dt, \quad x\in(0,\infty),\, n\in\mathbb{N}, \tag{1.1}$$

where the kernel is given by

$$W(n,x,t)=\left(\frac{n}{2\pi}\right)^{1/2}\exp\left(n/x\right)t^{-3/2}\exp\left(-\frac{nt}{2x^2}-\frac{n}{2t}\right).$$

The kernel of these operators satisfies the partial differential equation

$$\frac{\partial}{\partial x}W(n,x,t)=\frac{n(t-x)}{x^3}W(n,x,t). \tag{1.2}$$

Due to its complicated behavior in integration, these operators were not previously much studied by researchers. Recently in case of real variables these operators were studied by Gupta [8], who established some direct results. The asymptotic formula for certain exponential type operators are discussed in [1].

Key words and phrases. Complex exponential kind operator, approximation properties, upper estimate, Voronovskaya-type formula, exact estimate.

2010 *Mathematics Subject Classification.* Primary: 30E10. Secondary: 41A36.

DOI 10.46793/KgJMat2305.691G

Received: April 28, 2020.

Accepted: November 02, 2020.

Also, in the recent years, the study of approximation by complex operators on compact disks is an active area of research, see for instance [2–4, 6, 7, 9] and [12] etc.

In this paper, we study the approximation properties of the complex variant in (1.1), obtained by replacing x with z in the formula (1.1). Section 2 contains some auxiliary results used in the next sections. Section 3 deals with upper estimate, while in Section 4, we study a Voronovskaya-type result and the exact estimate in approximation.

2. Auxiliary Results

The proofs of our main results require three additional lemmas, as follows.

Lemma 2.1. *If we denote $T_{n,m}(x) = Q_n(e_m, x)$, $e_m(t) = t^m$, then using Mapple, we find that $T_{n,0}(x) = 1$ and there holds the following recurrence relation:*

$$nT_{n,m+1}(x) = x^3[T_{n,m}(x)]' + nxT_{n,m}(x), \quad n, m \in \mathbb{N}.$$

In particular

$$\begin{aligned}
T_{n,0}(x) =&1,\\
T_{n,1}(x) =&x,\\
T_{n,2}(x) =&x^2 + \frac{x^3}{n},\\
T_{n,3}(x) =&x^3 + \frac{3x^4}{n} + \frac{3x^5}{n^2},\\
T_{n,4}(x) =&x^4 + \frac{6x^5}{n} + \frac{15x^6}{n^2},\\
T_{n,5}(x) =&x^5 + \frac{10x^6}{n} + \frac{45x^7}{n^2} + \frac{105x^8}{n^3} + \frac{105x^9}{n^4},\\
T_{n,6}(x) =&x^6 + \frac{15x^7}{n} + \frac{105x^8}{n^2} + \frac{420x^9}{n^3} + \frac{945x^{10}}{n^4} + \frac{945x^{11}}{n^5}.
\end{aligned}$$

Proof. By definition

$$T_{n,m}(x) \quad = \quad \left(\frac{n}{2\pi}\right)^{1/2} \exp(n/x) \int_0^\infty t^{-3/2} \exp\left(-\frac{nt}{2x^2} - \frac{n}{2t}\right) t^m dt.$$

Thus, differentiating w.r.t x both the sides and using (1.2), we have

$$\begin{aligned}
x^3[T_{n,m}(x)]' =& \int_0^\infty x^3[W(n,x,t)]' t^m dt\\
=& \int_0^\infty n(t-x)W(n,x,t)t^m dt\\
=& nT_{n,m+1}(x) - nxT_{n,m}(x).
\end{aligned}$$

This completes the proof of lemma, other consequences follow from the recurrence relation. ☐

Lemma 2.2. *Suppose that* $f : \mathbb{C} \to \mathbb{C}$, $f(z) = \sum_{k=0}^{\infty} c_k z^k$, *is an entire function satisfying the condition* $|c_k| \leq M \frac{A^k}{k!}$, $k = 0, 1, \dots$, *with* $M > 0$ *and* $A \in (0, 1/2)$ (*which implies that* f *is of exponential growth since* $|f(z)| \leq M \exp(A|z|)$ *for all* $z \in \mathbb{C}$). *Then* $Q_n(f, z)$ *is well defined for any* $n \in \mathbb{N}$ *and any* $z \in \mathbb{C}$ *satisfying*

$$\mathrm{Re}\,(z^2) > 0 \quad \text{and} \quad \frac{|z|^2}{\mathrm{Re}\,(z^2)} < \frac{1}{2A}. \tag{2.1}$$

Proof. Since $|\exp(z)| = \exp(\mathrm{Re}\,(z))$, $\mathrm{Re}\,(1/z) = \mathrm{Re}\,(z)/|z|$ and $\mathrm{Re}\,(1/z^2) = \mathrm{Re}\,(z^2)/|z|^2$, we get

$$\begin{aligned}
&|Q_n(f, z)| \\
\leq& M \left(\frac{n}{2\pi}\right)^{1/2} |e(n/z)| \int_0^\infty t^{-3/2} \exp(-n/(2t) + At) |\exp(-nt/(2z^2))| dt \\
=& M \exp(n\mathrm{Re}\,(z)/|z|) \int_0^\infty t^{-3/2} \exp(-n/(2t)) \exp(-t[n\mathrm{Re}\,(z^2)/(2|z|^2) - A]) dt.
\end{aligned}$$

By the hypothesis on z, we easily seen that $n\mathrm{Re}\,(z^2)/(2|z|^2) - A > 0$ for all $n \geq 1$. Therefore, for fixed z as in the hypothesis and denoting $n\mathrm{Re}\,(z^2)/(2|z|^2) - A$ with $C > 0$, we have to deal with the existence of the integral

$$I := \int_0^\infty t^{-3/2} \exp(-n/(2t)) \exp(-Ct) dt.$$

Changing the variable $t = \frac{1}{v}$, we easily obtain

$$I = \int_0^\infty v^{-1/2} \exp(-nv/2) \exp(-C/v) dv < \infty.$$

Indeed, for $K > 0$ an arbitrary fixed constant, we have

$$\begin{aligned}
I =& \int_0^K v^{-1/2} \exp(-nv/2) \exp(-C/v) dv + \int_K^\infty v^{-1/2} \exp(-nv/2) \exp(-C/v) dv \\
:=& I_1 + I_2,
\end{aligned}$$

where

$$I_1 \leq \int_0^K \exp(-nv/2) v^{-1/2} \frac{v}{C} dv \leq \frac{1}{C} \int_0^K v^{1/2} \exp(-nv/2) dv < \infty$$

and $I_2 \leq \frac{1}{\sqrt{K}} \int_K^\infty e(-nv/2) dv < \infty$. □

Lemma 2.3. *Suppose that* f *is an entire function, i.e.,* $f(z) = \sum_{k=0}^{\infty} c_k z^k$ *for all* $z \in \mathbb{C}$ *such that there exist* $M > 0$ *and* $A \in (0, 1)$, *with the property* $|c_k| \leq M \frac{A^k}{k!}$ *for all* $k = 0, 1, \dots$ (*which implies* $|f(z)| \leq M \exp(A|z|)$ *for all* $z \in \mathbb{C}$).

Then for all $n \in \mathbb{N}$ *and* z *satisfying* (2.1), *we have*

$$Q_n(f, z) = \sum_{k=0}^{\infty} c_k Q_n(e_k, z).$$

Proof. Since we can write

$$Q_n(f;z)=\left(\frac{n}{2\pi}\right)^{1/2}\exp(n/z)\int_0^\infty t^{-3/2}\exp\left(-\frac{nt}{2z^2}-\frac{n}{2t}\right)\left(\sum_{k=0}^\infty c_k t^k\right)dt,$$

if above the integral would commute with the infinite sum, then we would obtain

$$\begin{aligned}Q_n(f,z)=&\sum_{k=0}^\infty c_k\left(\frac{n}{2\pi}\right)^{1/2}\exp(n/z)\int_0^\infty t^{-3/2}\exp\left(-\frac{nt}{2z^2}-\frac{n}{2t}\right)t^k dt\\ =&\sum_{k=0}^\infty c_k Q_n(e_k,z).\end{aligned}$$

It is well-known by the Fubini type result that a sufficient condition for the commutativity is that

$$\int_0^\infty t^{-3/2}\left|\exp\left(-\frac{nt}{2z^2}-\frac{n}{2t}\right)\right|\left(\sum_{k=0}^\infty |c_k|t^k\right)dt<\infty.$$

Applied to our case, for $n\in\mathbb{N}$ and z satisfying (2.1), we get

$$\begin{aligned}&\int_0^\infty t^{-3/2}\left|\exp\left(-\frac{nt}{2z^2}-\frac{n}{2t}\right)\right|\left(\sum_{k=0}^\infty |c_k|t^k\right)dt\\ \leq&M\int_0^\infty t^{-3/2}\exp\left(-\frac{n}{2t}\right)\exp\left(-nt\mathrm{Re}\,(z^2)/(2|z|^2)\right)\left(\sum_{k=0}^\infty\frac{A^k t^k}{k!}\right)dt\\ =&M\int_0^\infty t^{-3/2}\exp\left(-\frac{n}{2t}\right)\exp\left(-nt\mathrm{Re}\,(z^2)/(2|z|^2)\right)e^{At}dt\\ =&M\int_0^\infty t^{-3/2}\exp\left(-\frac{n}{2t}\right)\exp\left(-nt\mathrm{Re}\,(z^2)/(2|z|^2)+At\right)dt<\infty,\end{aligned}$$

by the proof of Lemma 2.2. □

Remark 2.1. It is easy to see that from geometric point of view, the conditions on z in (2.1) means that z belongs to two symmetric cones with respect to origin (but without containing the origin) containing the x axis, which are included in the two symmetric cones with respect to origin between the first and second bisectrix, containing the x axis. Indeed, since $|z|^2=x^2+y^2$ and $\mathrm{Re}\,(z^2)=x^2-y^2$, simple calculations show that the condition (2.1) satisfied by $z=x+iy$ can easily be written under the form

$$\sqrt{\left(1+\frac{1}{2A}\right)}|y|<\sqrt{\left(\frac{1}{2A}-1\right)}|x|,$$

that is

$$\frac{|y|}{|x|}<\frac{\sqrt{1/(2A)-1}}{\sqrt{1/(2A)+1}}<1.$$

3. Upper Estimate

The first main result concerns an upper estimate in approximation by $Q_n(f, z)$.

Theorem 3.1. *Suppose that f is an entire function, i.e., $f(z) = \sum_{k=0}^{\infty} c_k z^k$ for all $z \in \mathbb{C}$ such that there exist $M > 0$ and $A \in (0, 1/2)$, with the property $|c_k| \leq M\frac{A^k}{k!}$, for all $k = 0, 1, \ldots$ (which implies $|f(z)| \leq Me^{A|z|}$ for all $z \in \mathbb{C}$). Consider $1 \leq r < \frac{1}{A}$.*

Then for all $n \geq r^2$, $|z| \leq r$ and z satisfying (2.1), the following estimate hold:

$$|Q_n(f, z) - f(z)| \leq \frac{C_{r,M,A}}{n},$$

where $C_{r,M,A} = Mr\sum_{k=2}^{\infty}(k+1)(Ar)^k < \infty$.

Proof. By Lemma 2.1 written with x replaced by z, we easily obtain

$$n[T_{n,m+1}(z) - z^{m+1}] = z^3[T_{n,m}(z) - z^m]' + nz[T_{m,n}(z) - z^m] + mz^{m+2}.$$

Applying the Bernstein's inequality on $|z| \leq r$ to the polynomial of degree m, $T_{n,m}(z) - z^m$, we get $\|[T_{n,m}(z) - z^m]'\|_r \leq \frac{m}{r}\|T_{n,m}(z) - z^m\|_r$, where $\|P\|_r = \sup_{|z|\leq r}|P(z)|$. Then, denoting $e_m = z^m$, from the above recurrence we immediately obtain

$$\|T_{n,m+1} - e_{m+1}\|_r \leq \left(r + \frac{mr^2}{n}\right)\|T_{m,n} - e_m\|_r + \frac{mr^{m+2}}{n}.$$

In what follows we prove by mathematical induction with respect to m that for $n \geq r^2$, this recurrence implies

$$||T_{n,m} - e_m||_r \leq \frac{(m+1)!}{n}r^{m+1}, \quad \text{for all } m \geq 0.$$

Indeed for $m = 0$ and $m = 1$ it is trivial, as the left-hand side is zero. Suppose that it is valid for m, the above recurrence relation implies that

$$||T_{n,m+1} - e_{m+1}||_r \leq \left(r + \frac{r^2m}{n}\right)\frac{(m+1)!}{n}r^{m+1} + \frac{m}{n}r^{m+2}.$$

It remains to prove that

$$\left(r + \frac{r^2m}{n}\right)\frac{(m+1)!}{n}r^{m+1} + \frac{m}{n}r^{m+2} \leq \frac{(m+2)!}{n}r^{m+2},$$

or after simplifications, equivalently to

$$\left(r + \frac{r^2m}{n}\right)(m+1)! + rm \leq (m+2)!r,$$

for all $m \in \mathbb{N}$ and $r \geq 1$.

Since $n \geq r^2$, we get

$$\left(r + \frac{r^2m}{n}\right)(m+1)! + rm \leq (r+m)(m+1)! + rm,$$

it is good enough if we prove that

$$(r+m)\,(m+1)!+rm \le (m+2)!r.$$

But this last inequality is obviously equivalent with

$$m(m+1)!+rm \le rm(m+1)!+r(m+1)!,$$

which is clearly valid for all $m \ge 1$ (and fixed $r \ge 1$).

Finally, taking into account Lemma 2.3, for all $n \ge r^2$, we obtain

$$\begin{aligned}|Q_n(f,z)-f(z)| &\le \sum_{k=0}^{\infty} |c_k| \cdot |Q_n(e_k,z)-e_k(z)| \\ &\le \frac{M}{n} \cdot \sum_{k=2}^{\infty} \frac{A^k}{k!} \cdot (k+1)!r^{k+1} = \frac{C_{r,M,A}}{n},\end{aligned}$$

where $C_{r,M,A} = Mr\sum_{k=2}^{\infty}(k+1)(Ar)^k < \infty$. □

Remark 3.1. The smaller A is, the larger is the portion of the symmetrical cones where the estimation in Theorem 3.1 takes place. This happens because of the intersection between the symmetrical cones and the disk $\{|z| \le r\}$ with $1 \le r < \frac{1}{A}$, where if $A \searrow 0$ then $r \nearrow \infty$.

4. Voronovskaya Type Formula and Exact Estimate

The following estimate is a Voronovskaja-kind quantitative result.

Theorem 4.1. *Suppose that f is an entire function, i.e., $f(z) = \sum_{k=0}^{\infty} c_k z^k$ for all $z \in \mathbb{C}$ such that there exist $M > 0$ and $A \in (0, 1/2)$, with the property $|c_k| \le M\frac{A^k}{k!}$, for all $k = 0, 1, \dots$ (which implies $|f(z)| \le M\exp(A|z|)$ for all $z \in \mathbb{C}$). Consider $1 \le r < \frac{1}{A}$.*

Then for all $n \ge r^2$, $|z| \le r$ and z satisfying (2.1), the following estimate holds:

$$\left|Q_n(f,z)-f(z)-\frac{z^3 f''(z)}{2n}\right| \le \frac{E_{r,M,A}(f)}{n^2},$$

where

$$E_{r,M,A}(f) = 3Mr^2\sum_{k=2}^{\infty}(k+1)^2(Ar)^k < \infty.$$

Proof. Everywhere in the proof consider z and n as in hypothesis.

By the proof of Lemma 2.3, we can write $Q_n(f,z) = \sum_{k=0}^{\infty} c_k\, Q_n(e_k,z)$. Also, since

$$\frac{z^3 f''(z)}{2n} = \frac{z^3}{2n}\sum_{k=2}^{\infty} c_k k(k-1)z^{k-2} = \frac{1}{2n}\sum_{k=2}^{\infty} c_k\, k(k-1)\, z^{k+1},$$

we get

$$\left|Q_n(f,z)-f(z)-\frac{z^3\, f''(z)}{2n}\right| \le \sum_{k=2}^{\infty} |c_k| \left|T_{n,k}(z)-e_k(z)-\frac{k(k-1)z^{k+1}}{2n}\right|.$$

By Lemma 2.1, we have

$$T_{n,k}(z) = \frac{z^3}{n}\, T'_{n,k-1}(z) + z\, T_{n,k-1}(z).$$

If we denote

$$J_{n,k}(z) = T_{n,k}(z) - e_k(z) - \frac{k(k-1)z^{k+1}}{2n},$$

then it is obvious that $J_{n,k}(z)$ is a polynomial of degree less than or equal to $k+2$ and by simple computation and the use of above recurrence relation, we are led to

$$J_{n,k}(z) \quad = \quad \frac{z^3}{n} J'_{n,k-1}(z) + zJ_{n,k-1}(z) + X_{n,k}(z),$$

where after simple computation, we have

$$X_{n,k}(z) \quad = \quad \frac{k(k-1)(k-2)z^{k+2}}{2n^2}.$$

Using the estimate in the proof of Theorem 3.1, we have

$$|T_{n,k}(z) - e_k(z)| \leq \frac{(k+1)!}{n} \cdot r^{k+1}.$$

It follows

$$|J_{n,k}(z)| \leq \frac{r^3}{n}\, |J'_{n,k-1}(z)| + r\, |J_{n,k-1}(z)| + |X_{n,k}(z)|,$$

where

$$|X_{n,k}(z)| \leq \frac{k(k-1)(k-2)r^{k+2}}{2n^2}.$$

Now we shall find the estimation of $|J'_{n,k-1}(z)|$. Taking into account the fact that $J_{n,k-1}(z)$ is a polynomial of degree $\leq k+1$, we have

$$\begin{aligned}
|J'_{n,k-1}(z)| \leq & \frac{k}{r}||J_{n,k-1}(z)||_r \\
\leq & \frac{k}{r}\left[||T_{n,k-1}(z) - e_{k-1}(z)||_r + \frac{(k-1)(k-2)r^k}{2n}\right] \\
\leq & \frac{(k+1)!}{n} \cdot r^{k-1} + \frac{k(k-1)(k-2)r^{k-1}}{2n}.
\end{aligned}$$

Thus,

$$\frac{r^3}{n}|J'_{n,k-1}(z)| \leq \frac{1}{n}\left[\frac{(k+1)!}{n}r^{k+2} + \frac{k(k-1)(k-2)r^{k+2}}{2n}\right]$$

and

$$\begin{aligned}
|J_{n,k}(z)| \leq & r|J_{n,k-1}(z)| + \frac{1}{n}\left[\frac{(k+1)!}{n}r^{k+2} + \frac{k(k-1)(k-2)r^{k+2}}{2n}\right] \\
& + \frac{k(k-1)(k-2)r^{k+2}}{2n^2}.
\end{aligned}$$

This immediately implies

$$|J_{n,k}(z)| \le r|J_{n,k-1}(z)| + \frac{3}{n^2}(k+1)!r^{k+2}.$$

By writing this inequality for $k = 1, 2, 3, \dots$, we easily obtain step by step the following

$$|J_{n,k}(z)| \le \frac{3}{n^2} r^{k+2} \left[\sum_{j=1}^{k+1} j!\right] \le \frac{3}{n^2} r^{k+2}(k+1)!(k+1).$$

In conclusion,

$$\begin{aligned}\left|Q_n(f,z) - f(z) - \frac{z^3 f''(z)}{2n}\right| \le& \frac{3}{n^2} \cdot \sum_{k=2}^{\infty} |c_k| r^{k+2} \cdot (k+1)!(k+1) \\ \le& \frac{3Mr^2}{n^2} \cdot \sum_{k=2}^{\infty} (k+1)^2 (Ar)^k.\end{aligned}$$

This completes the proof of theorem. □

Using the above Voronovskaja's theorem, we obtain the following lower order in approximation.

Theorem 4.2. *Under the hypothesis in Theorem* 4.1*, if f is not a polynomial of degree ≤ 1, then for all $n \ge r^2$ we have*

$$||Q_n(f,\cdot) - f||_r^* \ge \frac{K_{r,M,A}(f)}{n},$$

where $\|F\|_r^* = \sup\{|F(z)| : |z| \le r$ and z satisfies (2.1)$\}$ *and $K_{r,M,A}(f)$ is a constant which depends only on f, M, A and r.*

Proof. For all $n \ge r^2$, $|z| \le r$ and z satisfying (2.1), we have

$$Q_n(f,z) - f(z) = \frac{1}{n}\left[0.5\, z^3 f''(z) + \frac{1}{n}\left\{n^2\left(Q_n(f,z) - f(z) - \frac{z^3 f''(z)}{2n}\right)\right\}\right].$$

Also, we have

$$||F + G||_r^* \ge |||F||_r^* - ||G||_r^*| \ge ||F||_r^* - ||G||_r^*.$$

It follows

$$||Q_n(f,\cdot) - f||_r^* \ge \frac{1}{n}\left[||\,0.5\, e_3\, f''\,||_r^* - \frac{1}{n}\left\{n^2 \left\|Q_n(f,\cdot) - f - \frac{e_3\, f''}{2n}\right\|_r^*\right\}\right].$$

Taking into account that by hypothesis, f is not a polynomial of degree ≤ 1, we get $||0.5\, e_3\, f''||_r^* > 0$. Indeed, supposing the contrary it follows that $z^3 f''(z) = 0$, which by the fact that f is entire function, clearly implies $f''(z) = 0$, i.e., f is a polynomial of degree ≤ 1, a contradiction with the hypothesis.

Now by Theorem 4.1, we have

$$n^2 \left\|Q_n(f,z) - f(z) - \frac{z^3 f''(z)}{2n}\right\|_r^* \le E_{r,M,A}(f).$$

Therefore, there exists an index n_0 depending only on f and r, such that for all $n \geq n_0$, we have

$$||\,0.5\,e_3\,f''\,||_r^* - \frac{1}{n}\left\{n^2 \left\|Q_n(f,z) - f(z) - \frac{0.5\,z^3\,f''(z)}{n}\right\|_r^*\right\} \geq \frac{1}{2}\;||\,0.5e_3\,f''\,||_r^*\,,$$

which immediately implies

$$||Q_n(f,\cdot) - f||_r^* \geq \frac{1}{2n}||\,0.5\,e_3\,f''\,||_r^*, \quad \text{for all } n \geq n_0.$$

For $n \in \{1, 2, \ldots, n_0 - 1\}$ we obviously have

$$||Q_n(f,\cdot) - f||_r^* \geq \frac{M_{r,n}(f)}{n},$$

with $M_{r,n}(f) = n||Q_n(f,\cdot) - f||_r^* > 0$. Indeed, if we would have $||Q_n(f,\cdot) - f||_r^* = 0$, then would follow $Q_n(f,z) = f(z)$ for all $|z| \leq r$, z satisfying (2.1), which is valid only for f a polynomial of degree ≤ 1, contradicting the hypothesis on f. Hence, we obtain $||Q_n(f,\cdot) - f||_r^* \geq \frac{K_{r,M,A}(f)}{n}$ for all n, where

$$K_{r,M,A}(f) = \min\left\{M_{r,1}(f), M_{r,2}(f), \ldots, M_{r,n_0-1}(f),\; \frac{1}{2}\;||\,0.5\,e_3\,f''\,||_r^*\right\},$$

which completes the proof. □

Combining Theorem 3.1 with Theorem 4.2, we immediately get the following exact estimate.

Corollary 4.1. *Under the hypothesis in Theorem* 4.1*, if f is not a polynomial of degree ≤ 1, then we have*

$$||Q_n(f,\cdot) - f||_r^* \sim \frac{1}{n}, \quad n \in \mathbb{N},$$

where the symbol $\sim$ represents the well-known equivalence between the orders of approximation.

Remark 4.1. Particular cases of the exponential-type operators studied in the real case in [11], are the Bernstein polynomials, the operators of Szász, of Post-Widder, of Gauss-Weierstrass, of Baskakov, to mention only a few. In the complex variable case, only the approximation properties of the operators of Bernstein, Szász, Baskakov and Post-Widder were already studied, see, e.g., [5, 7, 9]. It remains as open question to use the method in this paper for other complex exponential-type operators, too.

Acknowledgements. The authors are thankful to the reviewers for helpful remarks and suggestions which lead to essential improvement of the whole manuscript.

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