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PICTURE FUZZY SUBSPACE OF A CRISP VECTOR SPACE

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Abstract. In this paper, the notion of picture fuzzy subspace of a crisp vector space is established and some related properties are explored on the basis of some basic operations (intersection, Cartesian product, union, (θ, ϕ, ψ)-cut etc.) on picture fuzzy sets. Direct sum of two picture fuzzy subspaces is initiated here over the direct sum of two crisp vector spaces. Also, the concepts of picture fuzzy linear transformation and picture fuzzy linearly independent set of vectors are introduced and some corresponding results are presented. Isomorphism between two picture fuzzy subspaces is developed here as an extension of isomorphism between two fuzzy subspaces.

1. Introduction

Vector space and subspace of a vector space are two pioneer concepts in the field of algebra. Rosenfeld [13] applied the notion of fuzzy set to group theory and established the idea of fuzzy group after the initiation of fuzzy set by Zadeh [15]. After that many researchers worked on different topics of algebra in the environment of fuzzy set. The concept of fuzzy subspace was initiated by Katsaras and Liu [10]. Vector space in fuzzy sense under triangular norm was studied by Das [5]. Kumar [11] enriched the idea of Das. The concept of picture fuzzy set, a generalization of the concepts of fuzzy set and intuitionistic fuzzy set, was introduced by Cuong [4]. With the advancement of time, different kinds of research works under picture fuzzy environment were performed by several researchers [6–9, 12, 14].

Key words and phrases. Picture fuzzy subspace, direct sum of two picture fuzzy subspaces, isomorphism between two picture fuzzy subspaces, picture fuzzy linear transformation, picture fuzzy linearly independent set of vectors.

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In this paper, we will introduce the notion of picture fuzzy subspace of a crisp vector space and study some basic results related to it on the basis of some basic operations on picture fuzzy sets. Also, we will establish the concepts of direct sum of two picture fuzzy subspaces, isomorphism between two picture fuzzy subspaces, picture fuzzy linear transformation and picture fuzzy linearly independent set of vectors and explore some properties connected to these.

2. Preliminaries

In the current section, we will call again some basic concepts about fuzzy set (FS), fuzzy subspace (FSS) of a crisp vector space (crisp VS), intuitionistic fuzzy set (IFS), picture fuzzy set (PFS) and some basic operations on picture fuzzy sets (PFSs).

Definition 2.1 ([15])**.** Let A be the set of universe. Then a FS P over A is defined as $P = \{(a, \mu_P(a)) : a \in A\}$, where $\mu_P(a) \in [0,1]$ is the measure of membership of a in P.

Togethering the concepts of FS and subspace of a crisp VS, Kumar defined FSS of a crisp VS as follows.

Definition 2.2 ([11])**.** Let V be a crisp VS over the field F and $P = \{(a, \mu_P(a)) : a \in V\}$ be a FS in V. Then P is said to be FSS of V if the below stated conditions are meet.
(i) $\mu_P(a_1 - a_2) \geqslant \mu_P(a_1) \wedge \mu_P(a_2)$.
(ii) $\mu_P(ra_1) \geqslant \mu_P(a)$ for all $a_1, a_2 \in V$ and for all $r \in F$.

The measure of non-membership was missing in FS. Including this type of uncertainty, Atanassov [1] defined IFS.

Definition 2.3 ([1])**.** Let A be the set of universe. An IFS P over A is defined as $P = \{(a, \mu_P(a), v_P(a)) : a \in A\}$, where $\mu_P(a) \in [0,1]$ is the measure of membership of a in P and $v_P(a) \in [0,1]$ is the measure of non-membership of a in P with the condition $0 \leqslant \mu_P(a) + v_P(a) \leqslant 1$ for all $a \in A$.

It can be noted that $s_P(a)= 1 - (\mu_P(a) + v_P(a))$ is the measure of suspicion of a in P, which excludes the measure of membership and the measure of non-membership.
Including more possible types of uncertainty, Cuong [4] defined PFS generalizing the concepts of FS and IFS.

Definition 2.4 ([4])**.** Let A be the set of universe. Then a PFS P over the universe A is defined as $P = \{(a, \mu_P(a), \eta_P(a), v_P(a)) : a \in A\}$, where $\mu_P(a) \in [0,1]$ is the measure of positive membership of a in P, $\eta_P(a) \in [0,1]$ is the measure of neutral membership of a in P and $v_P(a) \in [0,1]$ is the measure of negative membership of a in P with the condition $0 \leqslant \mu_P(a) + \eta_P(a) + v_P(a) \leqslant 1$ for all $a \in A$. For all $a \in A$, $1 - (\mu_P(a) + \eta_P(a) + v_P(a))$ is the measure of denial membership a in P.

The basic operations on PFSs consist of equality, union and intersection are defined below.

Definition 2.5 ([4])**.** Let $P = \{(a, \mu_P(a), \eta_P(a), v_P(a)) : a \in A\}$ and $Q = \{(a, \mu_Q(a), \eta_Q(a), v_Q(a)) : a \in A\}$ be two PFSs over the universe A. Then

(i) $P \subseteq Q$ if and only if $\mu_P(a) \leqslant \mu_Q(a)$, $\eta_P(a) \leqslant \eta_Q(a)$, $v_P(a) \geqslant v_Q(a)$ for all $a \in A$;

(ii) $P = Q$ if and only if $\mu_P(a) = \mu_Q(a), \eta_P(a) = \eta_Q(a), v_P(a) = v_Q(a)$ for all $a \in A$;

(iii) $P \cup Q = \{(a, \max(\mu_P(a), \mu_Q(a)), \min(\eta_P(a), \eta_Q(a)), \min(v_P(a), v_Q(a))) : a \in A\}$;

(iv) $P \cap Q = \{(a, \min(\mu_P(a), \mu_Q(a)), \min(\eta_P(a), \eta_Q(a)), \max(v_P(a), v_Q(a))) : a \in A\}$.

Definition 2.6. Let $P = \{(a, \mu_P, \eta_P, v_P) : a \in A\}$ be a PFS over the universe A. Then (θ, ϕ, ψ)-cut of P is the crisp set in A denoted by $C_{\theta,\phi,\psi}(P)$ and is defined as $C_{\theta,\phi,\psi}(P) = \{a \in A : \mu_P(a) \geqslant \theta, \eta_P(a) \geqslant \phi, v_P(a) \leqslant \psi\}$, where $\theta, \phi, \psi \in [0, 1]$ with the condition $0 \leqslant \theta + \phi + \psi \leqslant 1$.

Definition 2.7. Let A_1 and A_2 be two sets of universe. Let $h : A_1 \to A_2$ be a surjective mapping and $P = \{(a_1, \mu_P(a_1), \eta_P(a_1), v_P(a_1)) : a_1 \in A_1\}$ be a PFS in A_1. Then image of P under the map h is the PFS $h(P) = \{(a_2, \mu_{h(P)}(a_2), \eta_{h(P)}(a_2), v_{h(P)}(a_2)) : a_2 \in A_2\}$, where

$$\mu_{h(P)}(a_2) = \bigvee_{a_1 \in h^{-1}(a_2)} \mu_P(a_1), \quad \eta_{h(P)}(a_2) = \bigwedge_{a_1 \in h^{-1}(a_2)} \eta_P(a_1)$$

and

$$v_{h(P)}(a_2) = \bigwedge_{a_1 \in h^{-1}(a_2)} v_P(a_1),$$

for all $a_2 \in A_2$.

Definition 2.8. Let A_1 and A_2 be two sets of universe. Let $h : A_1 \to A_2$ be a mapping and $Q = \{(a_2, \mu_Q(a_2), \eta_Q(a_2), v_Q(a_2)) : a_2 \in A_2\}$ be a PFS in A_2. Then inverse image of Q under the map h is the PFS $h^{-1}(Q) = \{(a_1, \mu_{h^{-1}(Q)}(a_1), \eta_{h^{-1}(Q)}(a_1), v_{h^{-1}(Q)}(a_1)) : a_1 \in A_1\}$, where $\mu_{h^{-1}(Q)}(a_1) = \mu_Q(h(a_1))$, $\eta_{h^{-1}(Q)}(a_1) = \eta_Q(h(a_1))$ and $v_{h^{-1}(Q)}(a_1) = v_Q(h(a_1))$ for all $a_1 \in A_1$.

Definition 2.9. Let $P = \{(a_1, \mu_P(a_1), \eta_P(a_1), v_P(a_1)) : a_1 \in A_1\}$ and $Q = \{(a_2, \mu_Q(a_2), \eta_Q(a_2), v_Q(a_2)) : a_2 \in A_2\}$ be two PFSs over A_1 and A_2 respectively, where A_1, A_2 be two sets of universe. Then the Cartesian product of P and Q is the PFS $P \times Q$=$\{((a, b), \mu_{P\times Q}((a, b)), \eta_{P\times Q}((a, b)), v_{P\times Q}((a, b))) : (a, b) \in A_1 \times A_2\}$, where $\mu_{P\times Q}((a, b)) = \mu_P(a) \wedge \mu_Q(b)$, $\eta_{P\times Q}((a, b)) = \eta_P(a) \wedge \eta_Q(b)$ and $v_{P\times Q}((a, b)) = v_P(a) \vee v_Q(b)$ for all $(a, b) \in A_1 \times A_2$.

Throughout the paper, we write PFS $P = \{(a, \mu_P(a), \eta_P(a), v_P(a)) : a \in A\}$ as $P = (\mu_P, \eta_P, v_P)$.

Now, it is time to introduce picture fuzzy subspace (PFSS) of a crisp vector space (crisp VS).

3. Picture Fuzzy Subspace

In the current section, the concept of PFSS is initiated and some basic results on PFSS are explored on the basis of intersection, union, Cartesian product and (θ, ϕ, ψ)-cut on PFSs. Also, some properties of PFSS under image and inverse image of PFS are studied when the map is a linear map in crisp sense.

Now, let us define PFSS of a crisp VS.

Definition 3.1. Let V be a crisp VS over the field F and $P = (\mu_P, \eta_P, \upsilon_P)$ be a PFS in V. Then P is said to be a PFSS of V if

(i) $\mu_P(a-b) \geqslant \mu_P(a) \wedge \mu_P(b)$, $\eta_P(a-b) \geqslant \eta_P(a) \wedge \eta_P(b)$ and $\upsilon_P(a-b) \leqslant \upsilon_P(a) \vee \upsilon_P(b)$;

(ii) $\mu_P(ra) \geqslant \mu_P(a)$, $\eta_P(ra) \geqslant \eta_P(a) \wedge \eta_P(b)$ and $\upsilon_P(ra) \leqslant \upsilon_P(a)$ for all $a, b \in V$ and for all $r \in F$.

Proposition 3.1. *Let V be a crisp VS over the field F and $P = (\mu_P, \eta_P, \upsilon_P)$ be a PFSS of V. Also, let ρ be the null vector in V. Then*

(i) $\mu_P(\rho) \geqslant \mu_P(a)$, $\eta_P(\rho) \geqslant \eta_P(a)$, $\upsilon_P(\rho) \leqslant \upsilon_P(a)$;

(ii) $\mu_P(ra) = \mu_P(a)$, $\eta_P(ra) = \eta_P(a)$ *and* $\upsilon_P(ra) = \upsilon_P(a)$ *for all* $a \in V$ *and for any non-zero* $r \in F$.

Proof. (i) Since P is a PFSS of V, therefore

$$\mu_P(\rho) = \mu_P(a-a) \geqslant \mu_P(a) \wedge \mu_P(a) = \mu_P(a),$$
$$\eta_P(\rho) = \eta_P(a-a) \geqslant \eta_P(a) \wedge \eta_P(a) = \eta_P(a),$$
$$\upsilon_P(\rho) = \upsilon_P(a-a) \leqslant \upsilon_P(a) \vee \upsilon_P(a) = \upsilon_P(a).$$

Thus, it is obtained that $\mu_P(\rho) \geqslant \mu_P(a)$, $\eta_P(\rho) \geqslant \eta_P(a)$ and $\upsilon_P(\rho) \leqslant \upsilon_P(a)$ for all $a \in V$.

(ii) Since P is a PFSS of V therefore

$$\mu_P(ra) \geqslant \mu_P(a), \quad \eta_P(ra) \geqslant \eta_P(a) \quad \text{and} \quad \upsilon_P(ra) \leqslant \upsilon_P(a),$$

for all $a \in V$ and for all $r \in F$. Let r be a non-zero scalar. Then

$$\mu_P(a) = \mu_P(r^{-1}(ra)) \geqslant \mu_P(ra) \quad [\text{because } P \text{ is a PFSS of } V],$$
$$\eta_P(a) = \eta_P(r^{-1}(ra)) \geqslant \eta_P(ra) \quad [\text{because } P \text{ is a PFSS of } V],$$
$$\upsilon_P(a) = \upsilon_P(r^{-1}(ra)) \leqslant \upsilon_P(ra) \quad [\text{because } P \text{ is a PFSS of } V],$$

for all $a \in V$. Consequently, $\mu_P(ra) = \mu_P(a)$, $\eta_P(ra) = \eta_P(a)$ and $\upsilon_P(ra) = \upsilon_P(a)$ for all $a \in V$ and for any non-zero $r \in F$. □

Proposition 3.2. *Let V be a crisp VS over the field F and $P = (\mu_P, \eta_P, \upsilon_P)$ be a PFS in V. Then P is a PFSS of V if and only if* $\mu_P(ra+sb) \geqslant \mu_P(a) \wedge \mu_P(b)$, $\eta_P(ra+sb) \geqslant \eta_P(a) \wedge \eta_P(b)$ *and* $\upsilon_P(ra+sb) \leqslant \upsilon_P(a) \vee \upsilon_P(b)$ *for all* $a, b \in V$ *and for all* $r, s \in F$.

Proof. Let us suppose that P is a PFSS of V. Therefore,

$$\mu_P(ra+sb) = \mu_P(ra-(-sb)) \geqslant \mu_P(ra) \wedge \mu_P((-s)b) \geqslant \mu_P(a) \wedge \mu_P(b),$$

$$\eta_P(ra+sb) = \eta_P(ra-(-sb)) \geqslant \eta_P(ra) \wedge \eta_P((-s)b) \geqslant \eta_P(a) \wedge \eta_P(b),$$
$$v_P(ra+sb) = v_P(ra-(-sb)) \leqslant v_P(ra) \vee v_P((-s)b) \leqslant v_P(a) \vee v_P(b),$$

for all $a, b \in V$ and for all $r, s \in F$. Thus, it is obtained that $\mu_P(ra+sb) \geqslant \mu_P(a) \wedge \mu_P(b)$, $\eta_P(ra+sb) \geqslant \eta_P(a) \wedge \eta_P(b)$ and $v_P(ra+sb) \leqslant v_P(a) \vee v_P(b)$ for all $a, b \in V$ and for all $r, s \in F$.

Conversely, let $\mu_P(ra+sb) \geqslant \mu_P(a) \wedge \mu_P(b)$, $\eta_P(ra+sb) \geqslant \eta_P(a) \wedge \eta_P(b)$ and $v_P(ra+sb) \leqslant v_P(a) \vee v_P(b)$ for all $a, b \in V$ and for all $r, s \in F$. Let us suppose that ρ be the null vector in V.

Now, setting $r = 1$ and $s = -1$, it is obtained that $\mu_P(a-b) \geqslant \mu_P(a) \wedge \mu_P(b)$, $\eta_P(a-b) \geqslant \eta_P(a) \wedge \eta_P(b)$ and $v_P(a-b) \leqslant v_P(a) \vee v_P(b)$ for all $a, b \in V$. Now, setting $a = b$, it is obtained that

$$\begin{aligned}
&\mu_P(a-a) \geqslant \mu_P(a) \wedge \mu_P(a) \quad \text{i.e. } \mu_P(\rho) \geqslant \mu_P(a),\\
&\eta_P(a-a) \geqslant \eta_P(a) \wedge \eta_P(a) \quad \text{i.e. } \eta_P(\rho) \geqslant \eta_P(a),\\
&v_P(a-a) \leqslant v_P(a) \vee v_P(a) \quad \text{i.e. } v_P(\rho) \leqslant v_P(a), \quad \text{for all } a \in V.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mu_P(ra) = \mu_P(ra+s\rho) \geqslant \mu_P(a) \wedge \mu_P(\rho) = \mu_P(a),\\
&\eta_P(ra) = \eta_P(ra+s\rho) \geqslant \eta_P(a) \wedge \eta_P(\rho) = \eta_P(a),\\
&v_P(ra) = v_P(ra+s\rho) \leqslant v_P(a) \vee v_P(\rho) = v_P(a), \quad \text{for all } a \in V \text{ and for all } r \in F.
\end{aligned}$$

Consequently, P is a PFSS of V. □

Example 3.1. Let us consider a crisp VS $V = \mathbb{R}^3$ over the field $F = \mathbb{R}$ and a PFS $P = (\mu_P, \eta_P, v_P)$ in V defined below.

$$\mu_P(a) = \begin{cases} 0.3, & \text{when } a \in \{(a_1, a_2, 0) : a_1, a_2 \in \mathbb{R}\},\\ 0.2, & \text{otherwise,}\end{cases}$$

$$\eta_P(a) = \begin{cases} 0.35, & \text{when } a \in \{(a_1, a_2, 0) : a_1, a_2 \in \mathbb{R}\},\\ 0.15, & \text{otherwise,}\end{cases}$$

$$v_P(a) = \begin{cases} 0.15, & \text{when } a \in \{(a_1, a_2, 0) : a_1, a_2 \in \mathbb{R}\},\\ 0.45, & \text{otherwise.}\end{cases}$$

Clearly, P is a PFSS of V.

Proposition 3.3. *Let V be a crisp VS over the field F and $P = (\mu_P, \eta_P, v_P)$ be a PFSS of V. If for $a, b \in V$, $\mu_P(a-b) = \mu_P(\rho)$, $\eta_P(a-b) = \eta_P(\rho)$ and $v_P(a-b) = v_P(\rho)$ then $\mu_P(a) = \mu_P(b)$, $\eta_P(a) = \eta_P(b)$ and $v_P(a) = v_P(b)$, where ρ be the null vector in V.*

Proof. Here it is observed that

$$\begin{aligned}
\mu_P(a) = \mu_P((a-b)+b) &\geqslant \mu_P(a-b) \wedge \mu_P(b) \quad [\text{because } P \text{ is a PFSS of } V]\\
&= \mu_P(\rho) \wedge \mu_P(b)
\end{aligned}$$

$$
\begin{aligned}
&= \mu_P(b) \quad \text{[by Proposition 3.1]},\\
\eta_P(a) = \eta_P((a-b)+b) &\geqslant \eta_P(a-b) \wedge \eta_P(b) \quad \text{[because } P \text{ is a PFSS of } V]\\
&= \eta_P(\rho) \wedge \eta_P(b)\\
&= \eta_P(b) \quad \text{[by Proposition 3.1]},\\
\upsilon_P(a) = \upsilon_P((a-b)+b) &\leqslant \upsilon_P(a-b) \vee \upsilon_P(b) \quad \text{[because } P \text{ is a PFSS of } V]\\
&= \upsilon_P(\rho) \vee \upsilon_P(b)\\
&= \upsilon_P(b) \quad \text{[by Proposition 3.1]}.
\end{aligned}
$$

Thus, $\mu_P(a) \geqslant \mu_P(b)$, $\eta_P(a) \geqslant \eta_P(b)$ and $\upsilon_P(a) \leqslant \upsilon_P(b)$.

Also,

$$
\begin{aligned}
\mu_P(b) = \mu_P(a-(a-b))) &= \mu_P(a+(-1)(a-b))\\
&\geqslant \mu_P(a) \wedge \mu_P(a-b) \quad \text{[because } P \text{ is a PFSS of } V]\\
&= \mu_P(a) \wedge \mu_P(\rho)\\
&= \mu_P(a) \quad \text{[by Proposition 3.1]},\\
\eta_P(b) = \eta_P(a-(a-b)) &= \eta_P(a+(-1)(a-b))\\
&\geqslant \eta_P(a) \wedge \eta_P(a-b) \quad \text{[because } P \text{ is a PFSS of } V]\\
&= \eta_P(a) \wedge \eta_P(\rho)\\
&= \eta_P(a) \quad \text{[by Proposition 3.1]},\\
\upsilon_P(b) = \upsilon_P(a-(a-b)) &= \upsilon_P(a+(-1)(a-b))\\
&\leqslant \upsilon_P(a) \vee \upsilon_P(a-b) \quad \text{[becuase } P \text{ is a PFSS of } V]\\
&= \upsilon_P(a) \vee \upsilon_P(\rho)\\
&= \upsilon_P(a) \quad \text{[by Proposition 3.1]}.
\end{aligned}
$$

Thus, $\mu_P(b) \geqslant \mu_P(a)$, $\eta_P(b) \geqslant \eta_P(a)$ and $\upsilon_P(b) \leqslant \upsilon_P(a)$.

Consequently, it is obtained that $\mu_P(a) = \mu_P(b)$, $\eta_P(a) = \eta_P(b)$ and $\upsilon_P(a) = \upsilon_P(b)$. □

Proposition 3.4. *Let V be a crisp VS over the field F and $P = (\mu_P, \eta_P, \upsilon_P)$ be a PFSS of V. If for $a, b \in V$, $\mu_P(a) < \mu_P(b)$, $\eta_P(a) < \eta_P(b)$ and $\upsilon_P(a) > \upsilon_P(b)$ hold then $\mu_P(a-b) = \mu_P(a) = \mu_P(b-a)$, $\eta_P(a-b) = \eta_P(a) = \eta_P(b-a)$ and $\upsilon_P(a-b) = \upsilon_P(a) = \upsilon_P(b-a)$.*

Proof. It is observed that

$$
\begin{aligned}
\mu_P(a-b) &\geqslant \mu_P(a) \wedge \mu_P(b) \quad \text{[because } P \text{ is a PFSS of } V]\\
&= \mu_P(a) \quad [\text{as } \mu_P(a) < \mu_P(b)],\\
\eta_P(a-b) &\geqslant \eta_P(a) \wedge \eta_P(b) \quad \text{[because } P \text{ is a PFSS of } V]\\
&= \eta_P(a) \quad [\text{as } \eta_P(a) < \eta_P(b)],\\
\upsilon_P(a-b) &\leqslant \upsilon_P(a) \vee \upsilon_P(b) \quad \text{[because } P \text{ is a PFSS of } V]
\end{aligned}
$$

$$= v_P(a) \quad [\text{as } v_P(a) > v_P(b)].$$

Thus, it is obtained that $\mu_P(a-b) \geqslant \mu_P(a)$, $\eta_P(a-b) \geqslant \eta_P(a)$ and $v_P(a-b) \leqslant v_P(a)$. Also,

$$\begin{aligned}
\mu_P(a) &\geqslant \mu_P(a-b) \wedge \mu_P(b) \quad [\text{because } P \text{ is a PFSS of } V] \\
&= \mu_P(a-b) \quad \text{or} \quad \mu_P(b), \\
\eta_P(a) &\geqslant \eta_P(a-b) \wedge \eta_P(b) \quad [\text{because } P \text{ is a PFSS of } V] \\
&= \eta_P(a-b) \quad \text{or} \quad \eta_P(b), \\
v_P(a) &\leqslant v_P(a-b) \vee v_P(b) \quad [\text{because } P \text{ is a PFSS of } V] \\
&= v_P(a-b) \quad \text{or} \quad v_P(b).
\end{aligned}$$

If $\mu_P(a) \geqslant \mu_P(b)$, $\eta_P(a) \geqslant \eta_P(b)$ and $v_P(a) \leqslant v_P(b)$ then they contradict the given conditions $\mu_P(a) < \mu_P(b)$, $\eta_P(a) < \eta_P(b)$ and $v_P(a) > v_P(b)$. So, it follows that $\mu_P(a) \geqslant \mu_P(a-b)$, $\eta_P(a) \geqslant \eta_P(a-b)$ and $v_P(a) \leqslant v_P(a-b)$.

Consequently, it is obtained that $\mu_P(a) = \mu_P(a-b)$, $\eta_P(a) = \eta_P(a-b)$ and $v_P(a) = v_P(a-b)$.

Moreover, it is clear that $\mu_P(a-b) = \mu_P(-(b-a)) = \mu_P(b-a)$, $\eta_P(a-b) = \eta_P(-(b-a)) = \eta_P(b-a)$ and $v_P(a-b) = v_P(-(b-a)) = v_P(b-a)$ [by Proposition 3.1].

Consequently, $\mu_P(a-b) = \mu_P(b-a) = \mu_P(a)$, $\eta_P(a-b) = \eta_P(b-a) = \eta_P(a)$ and $v_P(a-b) = v_P(b-a) = v_P(a)$. $\square$

Proposition 3.5. *Let V be a crisp VS over the field F and $P = (\mu_P, \eta_P, v_P)$, $Q = (\mu_Q, \eta_Q, v_Q)$ be two PFSS of V. Then $P \cap Q$ is a PFSS of V.*

Proof. Let $P \cap Q = R = (\mu_R, \eta_R, v_R)$, where $\mu_R(a) = \mu_P(a) \wedge \mu_Q(a)$, $\eta_R(a) = \eta_P(a) \wedge \eta_Q(a)$ and $v_R(a) = v_P(a) \vee v_Q(a)$ for all $a \in V$.

Now,

$$\begin{aligned}
\mu_R(ra+sb) &= \mu_P(ra+sb) \wedge \mu_Q(ra+sb) \\
&\geqslant (\mu_P(a) \wedge \mu_P(b)) \wedge (\mu_Q(a) \wedge \mu_Q(b)) \quad [\text{because } P, Q \text{ are PFSSs of } V] \\
&= (\mu_P(a) \wedge \mu_Q(a)) \wedge (\mu_P(b) \wedge \mu_Q(b)) \\
&= \mu_R(a) \wedge \mu_R(b), \\
\eta_R(ra+sb) &= \eta_P(ra+sb) \wedge \eta_Q(ra+sb) \\
&\geqslant (\eta_P(a) \wedge \eta_P(b)) \wedge (\eta_Q(a) \wedge \eta_Q(b)) \quad [\text{because } P, Q \text{ are PFSSs of } V] \\
&= (\eta_P(a) \wedge \eta_Q(a)) \wedge (\eta_P(b) \wedge \eta_Q(b)) \\
&= \eta_R(a) \wedge \eta_R(b), \\
v_R(ra+sb) &= v_P(ra+sb) \vee v_Q(ra+sb) \\
&\leqslant (v_P(a) \vee v_P(b)) \vee (v_Q(a) \vee v_Q(b)) \quad [\text{because } P, Q \text{ are PFSSs of } V] \\
&= (v_P(a) \vee v_Q(a)) \vee (v_P(b) \vee v_Q(b))
\end{aligned}$$

$$= \upsilon_R(a) \vee \upsilon_R(b), \quad \text{for all } a, b \in V \text{ and all } r, s \in F.$$

Consequently, $R = P \cap Q$ is a PFSS of V.

Thus, we have proved that the intersection of two PFSSs is a PFSS. But union of two PFSSs is not necessarily a PFSS. This can be proved by two examples. If P, Q be two PFSSs of a crisp VS V over the field F then Example 3.2 shows that $P \cup Q$ is not a PFSS of V while Example 3.3 shows that $P \cup Q$ is a PFSS of V. □

Example 3.2. Let us consider a crisp VS $V = \mathbb{R}^2$ over the field $F = \mathbb{R}$ and two PFSs $P = (\mu_P, \eta_P, \upsilon_P)$, $Q = (\mu_Q, \eta_Q, \upsilon_Q)$ in V defined below.

$$\mu_P(a) = \begin{cases} 0.45, & \text{when } a = (k, 0) \text{ for some } k \neq 0 \text{ or } a = (0, 0), \\ 0.1, & \text{otherwise,} \end{cases}$$

$$\eta_P(a) = \begin{cases} 0.35, & \text{when } a = (k, 0) \text{ for some } k \neq 0 \text{ or } a = (0, 0), \\ 0.15, & \text{otherwise,} \end{cases}$$

$$\upsilon_P(a) = \begin{cases} 0.1, & \text{when } a = (k, 0) \text{ for some } k \neq 0 \text{ or } a = (0, 0), \\ 0.4, & \text{otherwise,} \end{cases}$$

and

$$\mu_Q(a) = \begin{cases} 0.4, & \text{when } a = (0, k) \text{ for some } k \neq 0 \text{ or } a = (0, 0), \\ 0.2, & \text{otherwise,} \end{cases}$$

$$\eta_Q(a) = \begin{cases} 0.25, & \text{when } a = (0, k) \text{ for some } k \neq 0 \text{ or } a = (0, 0), \\ 0.2, & \text{otherwise,} \end{cases}$$

$$\upsilon_Q(a) = \begin{cases} 0.2, & \text{when } a = (0, k) \text{ for some } k \neq 0 \text{ or } a = (0, 0), \\ 0.35, & \text{otherwise.} \end{cases}$$

Thus, $P \cup Q$ is given by

$$\mu_{P\cup Q}(a) = \begin{cases} 0.45, & \text{when } a = (0, 0), \\ 0.45, & \text{when } a = (k, 0) \text{ for some } k \neq 0, \\ 0.4, & \text{when } a = (0, k) \text{ for some } k \neq 0, \\ 0.2, & \text{otherwise,} \end{cases}$$

$$\eta_{P\cup Q}(a) = \begin{cases} 0.25, & \text{when } a = (0, 0), \\ 0.2, & \text{when } a = (k, 0) \text{ for some } k \neq 0, \\ 0.15, & \text{when } a = (0, k) \text{ for some } k \neq 0, \\ 0.15, & \text{otherwise,} \end{cases}$$

$$\upsilon_{P\cup Q}(a) = \begin{cases} 0.1, & \text{when } a = (0, 0), \\ 0.1, & \text{when } a = (k, 0) \text{ for some } k \neq 0, \\ 0.2, & \text{when } a = (0, k) \text{ for some } k \neq 0, \\ 0.35, & \text{otherwise.} \end{cases}$$

It is observed that

$$0.2 = \mu_{P\cup Q}((2, 2)) \ngeq \mu_{P\cup Q}((2, 0)) \wedge \mu_{P\cup Q}(0, 2) = 0.45 \wedge 0.4 = 0.4,$$
$$0.35 = \upsilon_{P\cup Q}((2, 2)) \nleq \upsilon_P((2, 0)) \vee \upsilon_P((0, 2)) = 0.1 \vee 0.2 = 0.2,$$

but

$$0.15 = \eta_{P\cup Q}((2,2)) \geqslant \eta_{P\cup Q}((2,0)) \wedge \eta_{P\cup Q}(0,2) = 0.2 \wedge 0.15 = 0.15.$$

Hence, $P \cup Q$ is not a PFSS of V.

Example 3.3. Let us consider a crisp VS $V = \mathbb{R}^2$ over the field $F = \mathbb{R}$ and two PFSs $P = (\mu_P, \eta_P, \upsilon_P)$, $Q = (\mu_Q, \eta_Q, \upsilon_Q)$ in V defined below.

$$\mu_P(a) = \begin{cases} 0.3, & \text{when } a = (0,0), \\ 0.1, & \text{otherwise,} \end{cases}$$

$$\eta_P(a) = \begin{cases} 0.35, & \text{when } a = (0,0), \\ 0.1, & \text{otherwise,} \end{cases}$$

$$\upsilon_P(a) = \begin{cases} 0.15, & \text{when } a = (0,0), \\ 0.4, & \text{otherwise,} \end{cases}$$

and

$$\mu_Q(a) = \begin{cases} 0.25, & \text{when } a = (0,0), \\ 0.2, & \text{otherwise,} \end{cases}$$

$$\eta_Q(a) = \begin{cases} 0.25, & \text{when } a = (0,0), \\ 0.2, & \text{otherwise,} \end{cases}$$

$$\upsilon_Q(a) = \begin{cases} 0.2, & \text{when } a = (0,0), \\ 0.3, & \text{otherwise.} \end{cases}$$

Thus, $P \cup Q$ is given by

$$\mu_{P\cup Q}(a) = \begin{cases} 0.3, & \text{when } a = (0,0), \\ 0.2, & \text{otherwise,} \end{cases}$$

$$\eta_{P\cup Q}(a) = \begin{cases} 0.25, & \text{when } a = (0,0), \\ 0.1, & \text{otherwise,} \end{cases}$$

$$\upsilon_{P\cup Q}(a) = \begin{cases} 0.15, & \text{when } a = (0,0), \\ 0.3, & \text{otherwise.} \end{cases}$$

Here, $P \cup Q$ is a PFSS of V.

Proposition 3.6. *Let V be a crisp VS over the field F and $P = (\mu_P, \eta_P, \upsilon_P)$, $Q = (\mu_Q, \eta_Q, \upsilon_Q)$ be two PFSSs of V. Then $P \cup Q$ is a PFSS of V if either $P \subseteq Q$ or $Q \subseteq P$.*

Proof. Case 1. Let $P \subseteq Q$. Then $\mu_P(a) \leqslant \mu_Q(a)$, $\eta_P(a) \leqslant \eta_Q(a)$ and $\upsilon_P(a) \geqslant \upsilon_Q(a)$ for all $a \in A$. Then $\mu_{P\cup Q}(a) = \mu_P(a) \vee \mu_Q(a) = \mu_Q(a)$, $\eta_{P\cup Q}(a) = \eta_P(a) \wedge \eta_Q(a) = \eta_P(a)$ and $\upsilon_{P\cup Q}(a) = \upsilon_P(a) \wedge \upsilon_Q(a) = \upsilon_Q(a)$ for all $a \in V$.

Now,

$$\begin{aligned} \mu_{P\cup Q}(ra + sb) &= \mu_Q(ra + sb) \\ &\geqslant \mu_Q(a) \wedge \mu_Q(b) \quad [\text{because } Q \text{ is a PFSS of } V] \end{aligned}$$

$$
\begin{aligned}
&= \mu_{P\cup Q}(a) \wedge \mu_{P\cup Q}(b),\\
\eta_{P\cup Q}(ra+sb) &= \eta_P(ra+sb)\\
&\geqslant \eta_P(a) \wedge \eta_P(b) \quad [\text{because } P \text{ is a PFSS of } V]\\
&= \eta_{P\cup Q}(a) \wedge \eta_{P\cup Q}(b),\\
\upsilon_{P\cup Q}(ra+sb) &= \upsilon_Q(ra+sb)\\
&\leqslant \upsilon_Q(a) \vee \upsilon_Q(b) \quad [\text{because } Q \text{ is a PFSS of } V]\\
&= \upsilon_{P\cup Q}(a) \vee \mu_{P\cup Q}(b) \text{ for all } a,b \in V.
\end{aligned}
$$

Thus, $P \cup Q$ is a PFSS of V whenever $P \subseteq Q$.

Case 2. Let $Q \subseteq P$. Then $\mu_Q(a) \leqslant \mu_P(a)$, $\eta_Q(a) \leqslant \eta_P(a)$ and $\upsilon_Q(a) \geqslant \upsilon_P(a)$ for all $a \in V$. Then $\mu_{P\cup Q}(a) = \mu_P(a) \vee \mu_Q(a) = \mu_P(a)$, $\eta_{P\cup Q}(a) = \eta_P(a) \wedge \eta_Q(a) = \eta_Q(a)$ and $\upsilon_{P\cup Q}(a) = \upsilon_P(a) \vee \upsilon_Q(a) = \upsilon_P(a)$ for all $a \in V$. Proceeding in the similar way like case 1, it is obtained that $P \cup Q$ is a PFSS of V whenever $Q \subseteq P$. □

Proposition 3.7. *Let V be a crisp VS over the field F and $P = (\mu_P, \eta_P, \upsilon_P)$, $Q = (\mu_Q, \eta_Q, \upsilon_Q)$ be two PFSSs of V. Then $P \times Q$ is a PFSS of $V \times V$.*

Proof. Let $P \times Q = (\mu_{P\times Q}, \eta_{P\times Q}, \upsilon_{P\times Q})$, where $\mu_{P\times Q}((a,b)) = \mu_P(a) \wedge \mu_Q(b)$, $\eta_{P\times Q}((a,b)) = \eta_P(a) \vee \eta_Q(b)$ and $\upsilon_{P\times Q}((a,b)) = \upsilon_P(a) \vee \upsilon_Q(b)$ for all $(a,b) \in V \times V$. Now,

$$
\begin{aligned}
\mu_{P\times Q}(r(a,b)+s(c,d)) &= \mu_P(ra+sc) \wedge \mu_Q(rb+sd)\\
&\geqslant (\mu_P(a) \wedge \mu_P(c)) \wedge (\mu_Q(b) \wedge \mu_Q(d))\\
&\quad [\text{because } P, Q \text{ are PFSSs of } V]\\
&= (\mu_P(a) \wedge \mu_Q(b)) \wedge (\mu_P(c) \wedge \mu_Q(d))\\
&= \mu_{P\times Q}((a,b)) \wedge \mu_{P\times Q}((c,d)),\\
\eta_{P\times Q}(r(a,b)+s(c,d)) &= \eta_P(ra+sc) \wedge \eta_Q(rb+sd)\\
&\geqslant (\eta_P(a) \wedge \eta_P(c)) \wedge (\eta_Q(b) \wedge \eta_Q(d))\\
&\quad [\text{because } P, Q \text{ are PFSSs of } V]\\
&= (\eta_P(a) \wedge \eta_Q(b)) \wedge (\eta_P(c) \wedge \eta_Q(d))\\
&= \eta_{P\times Q}((a,b)) \wedge \eta_{P\times Q}((c,d)),\\
\upsilon_{P\times Q}(r(a,b)+s(c,d)) &= \upsilon_P(ra+sc) \wedge \upsilon_Q(rb+sd)\\
&\leqslant (\upsilon_P(a) \vee \upsilon_P(c)) \vee (\upsilon_Q(b) \vee \upsilon_Q(d))\\
&\quad [\text{because } P, Q \text{ are PFSSs of } V]\\
&= (\upsilon_P(a) \vee \upsilon_Q(b)) \vee (\upsilon_P(c) \vee \upsilon_Q(d))\\
&= \upsilon_{P\times Q}((a,b)) \vee \upsilon_{P\times Q}((c,d)),
\end{aligned}
$$

for all $(a,b), (c,d) \in V \times V$ and for all $r, s \in F$. Consequently, $P \times Q$ is a PFSS of $V \times V$. □

Proposition 3.8. *Let V_1 and V_2 be two crisp VSs over the field F and $P = (\mu_P, \eta_P, v_P)$, $Q = (\mu_Q, \eta_Q, v_Q)$ be two PFSSs of V_1 and V_2 respectively. Also, let ρ_1 and ρ_2 be two null vectors in V_1 and V_2 respectively. Then $\mu_{P\times Q}((\rho_1,\rho_2)) \geqslant \mu_{P\times Q}((a,b))$, $\eta_{P\times Q}((\rho_1,\rho_2)) \geqslant \eta_P((a,b))$ and $v_{P\times Q}((\rho_1,\rho_2)) \leqslant v_P((a,b))$ for all $(a,b) \in V_1 \times V_2$.*

Proof. Here, it is observed that

$$\begin{aligned}
\mu_{P\times Q}((\rho_1,\rho_2)) &= \mu_P(\rho_1) \wedge \mu_Q(\rho_2)\\
&\geqslant \mu_P(a) \wedge \mu_Q(b), \quad \text{for all } a \in V_1 \text{ and for all } b \in V_2\\
&= \mu_{P\times Q}((a,b)), \quad \text{for all } (a,b) \in V_1 \times V_2,\\
\eta_{P\times Q}((\rho_1,\rho_2)) &= \eta_P(\rho_1) \wedge \eta_Q(\rho_2)\\
&\geqslant \eta_P(a) \wedge \eta_Q(b), \quad \text{for all } a \in V_1 \text{ and for all } b \in V_2\\
&= \eta_{P\times Q}((a,b)), \quad \text{for all } (a,b) \in V_1 \times V_2,\\
\text{and } v_{P\times Q}((\rho_1,\rho_2)) &= v_P(\rho_1) \wedge \eta_Q(\rho_2)\\
&\leqslant v_P(a) \vee v_Q(b), \quad \text{for all } a \in V_1 \text{ and for all } b \in V_2\\
&= v_{P\times Q}((a,b)), \quad \text{for all } (a,b) \in V_1 \times V_2.
\end{aligned}$$

Consequently, it is obtained that $\mu_{P\times Q}((\rho_1,\rho_2)) \geqslant \mu_{P\times Q}((a,b))$, $\eta_{P\times Q}((\rho_1,\rho_2)) \geqslant \eta_{P\times Q}((a,b))$ and $v_{P\times Q}((\rho_1,\rho_2)) \leqslant v_{P\times Q}((a,b))$ for all $(a,b) \in V_1 \times V_2$. ☐

Proposition 3.9. *Let V_1 and V_2 be two crisp VSs over the field F and $P = (\mu_P, \eta_P, v_P)$, $Q = (\mu_Q, \eta_Q, v_Q)$ be two PFSSs of V_1 and V_2 respectively. Then one of the below stated conditions must hold.*

(i) $\mu_P(a) \leqslant \mu_Q(\rho_2)$, $\eta_P(a) \leqslant \eta_Q(\rho_2)$ and $v_P(a) \geqslant v_Q(\rho_2)$.

(ii) $\mu_Q(b) \leqslant \mu_P(\rho_1)$, $\eta_Q(b) \leqslant \eta_P(\rho_1)$ and $v_Q(b) \geqslant v_P(\rho_1)$ for all $a \in V_1$ and for all $b \in V_2$, where ρ_1 and ρ_2 be two null vectors in V_1 and V_2, respectively.

Proof. Let none of the stated conditions be hold. Then there exist $a \in V_1$ and $b \in V_2$ such that $\mu_P(a) > \mu_Q(\rho_2)$, $\eta_P(a) > \eta_Q(\rho_2)$, $v_P(a) < v_Q(\rho_2)$ and $\mu_Q(b) > \mu_P(\rho_1)$, $\eta_Q(b) > \eta_P(\rho_1)$, $v_Q(b) < v_P(\rho_1)$. Now,

$$\begin{aligned}
\mu_{P\times Q}((a,b)) &= \mu_P(a) \wedge \mu_Q(b) > \mu_Q(\rho_2) \wedge \mu_P(\rho_1) = \mu_{P\times Q}((\rho_1,\rho_2)),\\
\eta_{P\times Q}((a,b)) &= \eta_P(a) \wedge \eta_Q(b) > \eta_Q(\rho_2) \wedge \eta_P(\rho_1) = \eta_{P\times Q}((\rho_1,\rho_2)),\\
v_{P\times Q}((a,b)) &= v_P(a) \vee v_Q(b) < v_Q(\rho_2) \vee v_P(\rho_1) = v_{P\times Q}((\rho_1,\rho_2)).
\end{aligned}$$

Thus, it is obtained that $\mu_{P\times Q}((a,b)) > \mu_P((\rho_1,\rho_2))$, $\eta_{P\times Q}((a,b)) > \eta_{P\times Q}((\rho_1,\rho_2))$ and $v_P((a,b)) < v_{P\times Q}((\rho_1,\rho_2))$. But it is known from Proposition 3.8 that $\mu_{P\times Q}((a,b)) \leqslant \mu_P((\rho_1,\rho_2))$, $\eta_{P\times Q}((a,b)) \leqslant \eta_{P\times Q}((\rho_1,\rho_2))$ and $v_{P\times Q}((a,b)) \geqslant v_{P\times Q}((\rho_1,\rho_2))$. Hence, one of the stated conditions must hold. ☐

Proposition 3.10. *Let V be a crisp VS over the field F and $P = (\mu_P, \eta_P, v_P)$ be a PFSS of V. Then $C_{\theta,\phi,\psi}(P)$ is a crisp subspace of V, provided that $\mu_P(\rho) \geqslant \theta$, $\eta_P(\rho) \leqslant \phi$ and $v_P(\rho) \leqslant \psi$, where ρ be the null vector in V.*

Proof. Clearly, $C_{\theta,\phi,\psi}(P)$ is non-empty. Let $a, b \in C_{\theta,\phi,\psi}(P)$ and $r, s \in F$. Then $\mu_P(a) \geqslant \theta$, $\eta_P(a) \geqslant \phi$, $v_P(a) \leqslant \psi$ and $\mu_P(b) \geqslant \theta$, $\eta_P(b) \geqslant \phi$, $v_P(b) \leqslant \psi$. Now,

$$\begin{aligned}
\mu_P(ra+sb) &\geqslant \mu_P(a) \wedge \mu_P(b) \quad \text{[because } P \text{ is a PFSS of } V\text{]}\\
&\geqslant \theta \wedge \theta\\
&= \theta,\\
\eta_P(ra+sb) &\geqslant \eta_P(a) \wedge \eta_P(b) \quad \text{[because } P \text{ is a PFSS of } V\text{]}\\
&\geqslant \phi \wedge \phi\\
&= \phi,\\
v_P(ra+sb) &\leqslant v_P(a) \vee v_P(b) \quad \text{[because } P \text{ is a PFSS of } V\text{]}\\
&\leqslant \psi \vee \psi\\
&= \psi.
\end{aligned}$$

Thus, $a, b \in C_{\theta,\phi,\psi}(P)$ and $r, s \in F$ imply $ra+sb \in C_{\theta,\phi,\psi}(P)$. Consequently, $C_{\theta,\phi,\psi}(P)$ is a crisp subspace of V. □

Proposition 3.11. *Let V be a crisp VS over the field F and $P = (\mu_P, \eta_P, v_P)$ be a PFS in V. Then P is a PFSS of V if all (θ, ϕ, ψ)-cuts of P are crisp subspaces of V.*

Proof. Let $a, b \in V$. Take, $\mu_P(a) \wedge \mu_P(b) = \theta$, $\eta_P(a) \wedge \eta_P(b) = \phi$ and $v_P(a) \vee v_P(b) = \psi$. Clearly, $\theta \in [0,1]$, $\phi \in [0,1]$ and $\psi \in [0,1]$ with $0 \leqslant \theta + \phi + \psi \leqslant 1$.

Now,

$$\begin{aligned}
\mu_P(a) &\geqslant \mu_P(a) \wedge \mu_P(b) = \theta,\\
\eta_P(a) &\geqslant \eta_P(a) \wedge \eta_P(b) = \phi,\\
v_P(a) &\leqslant v_P(a) \vee v_P(b) = \psi.
\end{aligned}$$

Thus, $\mu_P(a) \geqslant \theta$, $\eta_P(a) \geqslant \phi$ and $v_P(a) \leqslant \psi$. So, $a \in C_{\theta,\phi,\psi}(P)$. Also,

$$\begin{aligned}
\mu_P(b) &\geqslant \mu_P(a) \wedge \mu_P(b) = \theta,\\
\eta_P(b) &\geqslant \eta_P(a) \wedge \eta_P(b) = \phi,\\
v_P(b) &\leqslant v_P(a) \vee v_P(b) = \psi.
\end{aligned}$$

Thus, $\mu_P(b) \geqslant \theta$, $\eta_P(b) \geqslant \phi$ and $v_P(b) \leqslant \psi$. So, $b \in C_{\theta,\phi,\psi}(P)$.

Since $C_{\theta,\phi,\psi}(P)$ is a crisp subspace of V therefore $ra+sb \in C_{\theta,\phi,\psi}(P)$ for all $r, s \in F$.

As a result,

$$\begin{aligned}
\mu_P(ra+sb) &\geqslant \theta = \mu_P(a) \wedge \mu_P(b),\\
\eta_P(ra+sb) &\geqslant \phi = \eta_P(a) \wedge \eta_P(b),\\
v_P(ra+sb) &\leqslant \psi = v_P(a) \vee v_P(b), \quad \text{for all } r, s \in F.
\end{aligned}$$

Since a, b are arbitrary elements of V therefore $\mu_P(ra+sb) \geqslant \mu_P(a) \wedge \mu_P(b)$, $\eta_P(ra+sb) \geqslant \eta_P(a) \wedge \eta_P(b)$ and $v_P(ra+sb) \leqslant v_P(a) \vee v_P(b)$ for all $a, b \in V$ and for all $r, s \in F$. Thus, P is a PFSS of V. □

Proposition 3.12. *Let V and W be two crisp VSs over the field F and Q be a PFSS of W. Then for a linear transformation (LT) $h : V \to W$, $h^{-1}(Q)$ is a PFSS of V.*

Proof. Let $h^{-1}(Q) = (\mu_{h^{-1}(Q)}, \eta_{h^{-1}(Q)}, \upsilon_{h^{-1}(Q)})$. Then $\mu_{h^{-1}(Q)}(a) = \mu_Q(h(a))$, $\eta_{h^{-1}(Q)}(a) = \eta_Q(h(a))$ and $\upsilon_{h^{-1}(Q)}(a) = \upsilon_Q(h(a))$ for all $a \in V$.

Now,

$$\begin{aligned}
\mu_{h^{-1}(Q)}(ra+sb) &= \mu_Q(h(ra+sb)) \\
&= \mu_Q(rh(a)+sh(b)) \quad [\text{because } h \text{ is a crisp LT from } V \text{ to } W] \\
&\geqslant \mu_Q(h(a)) \wedge \mu_Q(h(b)) \quad [\text{because } Q \text{ is a PFSS of } W] \\
&= \mu_{h^{-1}(Q)}(a) \wedge \mu_{h^{-1}(Q)}(b), \\
\eta_{h^{-1}(Q)}(ra+sb) &= \eta_Q(h(ra+sb)) \\
&= \eta_Q(rh(a)+sh(b)) \quad [\text{because } h \text{ is a crisp LT from } V \text{ to } W] \\
&\geqslant \eta_Q(h(a)) \wedge \eta_Q(h(b)) \quad [\text{because } Q \text{ is a PFSS of } W] \\
&= \eta_{h^{-1}(Q)}(a) \wedge \eta_{h^{-1}(Q)}(b), \\
\upsilon_{h^{-1}(Q)}(ra+sb) &= \upsilon_Q(h(ra+sb)) \\
&= \upsilon_Q(rh(a)+sh(b)) \quad [\text{because } h \text{ is a crisp LT from } V \text{ to } W] \\
&\leqslant \upsilon_Q(h(a)) \vee \upsilon_Q(h(b)) \quad [\text{because } Q \text{ is a PFSS of } W] \\
&= \upsilon_{h^{-1}(Q)}(a) \vee \upsilon_{h^{-1}(Q)}(b), \quad \text{for all } a,b \in V \text{ and for all } r,s \in F.
\end{aligned}$$

Consequently, $h^{-1}(Q)$ is a PFSS of V. □

Proposition 3.13. *Let V and W be two VSs over the same field F and $P = (\mu_P, \eta_P, \upsilon_P)$ be a PFSS of V. Then for a bijective LT $h : V \to W$, $h(P)$ is a PFSS of W.*

Proof. Let $h(P) = (\mu_{h(P)}, \eta_{h(P)}, \upsilon_{h(P)})$. Then

$$\begin{aligned}
\mu_{h(P)}(q) &= \bigvee_{p \in h^{-1}(q)} \mu_P(p), \\
\eta_{h(P)}(q) &= \bigwedge_{p \in h^{-1}(q)} \eta_P(p), \\
\upsilon_{h(P)}(q) &= \bigwedge_{p \in h^{-1}(q)} \upsilon_P(p).
\end{aligned}$$

Since h is bijective therefore $h^{-1}(q)$ must be a singleton set. So, for $q \in W$, there exists an unique $p \in V$ such that $p = h^{-1}(q)$, i.e., $h(p) = q$. Thus, in this case, $\mu_{h(P)}(q) = \mu_{h(P)}(h(p)) = \mu_P(p)$, $\eta_{h(P)}(q) = \eta_{h(P)}(h(p)) = \eta_P(p)$ and $\upsilon_{h(P)}(q) = \upsilon_{h(P)}(h(p)) = \upsilon_P(p)$. Now,

$$\begin{aligned}
\mu_{h(P)}(rc+sd) &= \mu_{h(P)}(rh(a)+sh(b)) \\
&\quad [\text{where } c = h(a) \text{ and } d = h(b) \text{ for unique } a,b \in V] \\
&= \mu_{h(P)}(h(ra+sb)) \quad [\text{because } h \text{ is a crisp LT}] \\
&= \mu_P(ra+sb)
\end{aligned}$$

$$
\begin{aligned}
&\geqslant \mu_P(a) \wedge \mu_P(b) \quad [\text{because } P \text{ is a PFSS of } V]\\
&= \mu_{h(P)}(h(a)) \wedge \mu_{h(P)}(h(b))\\
&= \mu_{h(P)}(c) \wedge \mu_{h(P)}(d),\\
\eta_{h(P)}(rc+sd) &= \eta_{h(P)}(rh(a)+sh(b))\\
&[\text{where } c = h(a) \text{ and } d = h(b) \text{ for unique } a, b \in V]\\
&= \eta_{h(P)}(h(ra+sb)) \quad [\text{because } h \text{ is a crisp LT}]\\
&= \eta_P(ra+sb)\\
&\geqslant \eta_P(a) \wedge \eta_P(b) \quad [\text{because } P \text{ is a PFSS of } V]\\
&= \eta_{h(P)}(h(a)) \wedge \eta_{h(P)}(h(b))\\
&= \eta_{h(P)}(c) \wedge \eta_{h(P)}(d),\\
\upsilon_{h(P)}(rc+sd) &= \upsilon_{h(P)}(rh(a)+sh(b))\\
&[\text{where } c = h(a) \text{ and } d = h(b) \text{ for unique } a, b \in V]\\
&= \upsilon_{h(P)}(h(ra+sb)) \quad [\text{because } h \text{ is a crisp LT}]\\
&= \upsilon_P(ra+sb)\\
&\leqslant \upsilon_P(a) \vee \upsilon_P(b) \quad [\text{because } P \text{ is a PFSS of } V]\\
&= \upsilon_{h(P)}(h(a)) \vee \upsilon_{h(P)}(h(b))\\
&= \upsilon_{h(P)}(c) \vee \upsilon_{h(P)}(d), \quad \text{for all } r, s \in F.
\end{aligned}
$$

Since, c, d are arbitrary elements of W therefore $\mu_{h(P)}(rc+sd) \geqslant \mu_{h(P)}(c) \wedge \mu_{h(P)}(d)$, $\eta_{h(P)}(rc+sd) \geqslant \eta_{h(P)}(c) \wedge \eta_{h(P)}(d)$ and $\upsilon_{h(P)}(rc+sd) \leqslant \upsilon_{h(P)}(c) \vee \upsilon_{h(P)}(d)$ for all $c, d \in W$ and for all $r, s \in F$. Consequently, $h(P)$ is a PFSS of W. ☐

4. Direct Sum of two Picture Fuzzy Subspaces

The current section introduces direct sum of two PFSSs over the direct sum of two crisp VSs and investigates some important results connected to it.

Definition 4.1. Let V and W be two crisp VSs over the same field F and $P = (\mu_P, \eta_P, \upsilon_P)$, $Q = (\mu_Q, \eta_Q, \upsilon_Q)$ be two PFSSs of V and W respectively. Then direct sum of P and Q is defined as the PFS $P \oplus Q = (\mu_{P\oplus Q}, \eta_{P\oplus Q}, \upsilon_{P\oplus Q})$ over the set of universe $V \oplus W$, where

$$
\begin{aligned}
\mu_{P\oplus Q}(c) &= \mu_P(a) \wedge \mu_Q(b),\\
\eta_{P\oplus Q}(c) &= \eta_P(a) \wedge \eta_Q(b),\\
\upsilon_{P\oplus Q}(c) &= \upsilon_P(a) \vee \upsilon_Q(b), \quad \text{for any } c \in V \oplus W,
\end{aligned}
$$

with $c = a + b$, where $a \in V$ and $b \in W$.

Example 4.1. Let us consider two crisp VSs $V_1 = \{(a_1, 0) : a_1 \in \mathbb{R}\}$ and $V_2 = \{(0, a_2) : a_2 \in \mathbb{R}\}$ over the field $F = \mathbb{R}$. Also, let us suppose two PFSSs P_1 and P_2 of V_1 and

V_2 respectively defined below.

$$\mu_P(a) = \begin{cases} 0.55, & \text{when } a = (0,0), \\ 0.2, & \text{otherwise,} \end{cases}$$

$$\eta_P(a) = \begin{cases} 0.45, & \text{when } a = (0,0), \\ 0.2, & \text{otherwise,} \end{cases}$$

$$\upsilon_P(a) = \begin{cases} 0.05, & \text{when } a = (0,0), \\ 0.37, & \text{otherwise,} \end{cases}$$

and

$$\mu_Q(a) = \begin{cases} 0.35, & \text{when } a = (0,0), \\ 0.3, & \text{otherwise,} \end{cases}$$

$$\eta_Q(a) = \begin{cases} 0.4, & \text{when } a = (0,0), \\ 0.3, & \text{otherwise,} \end{cases}$$

$$\upsilon_Q(a) = \begin{cases} 0.1, & \text{when } a = (0,0), \\ 0.3, & \text{otherwise.} \end{cases}$$

Thus, for any $a \in V_1 \oplus V_2$, $P \oplus Q$ is defined as follows.

$$\mu_{P\oplus Q}(a) = \begin{cases} 0.35, & \text{when } a = (0,0), \\ 0.3, & \text{when } a \in V_2 - \{(0,0)\}, \\ 0.2, & \text{otherwise,} \end{cases}$$

$$\eta_{P\oplus Q}(a) = \begin{cases} 0.4, & \text{when } a = (0,0), \\ 0.3, & \text{when } a \in V_2 - \{(0,0)\}, \\ 0.2, & \text{otherwise,} \end{cases}$$

$$\upsilon_{P\oplus Q}(a) = \begin{cases} 0.1, & \text{when } a = (0,0), \\ 0.3, & \text{when } a \in V_2 - \{(0,0)\}, \\ 0.37, & \text{otherwise.} \end{cases}$$

Proposition 4.1. *Let V and W be two crisp VSs over the same field F and $P = (\mu_P, \eta_P, \upsilon_P)$, $Q = (\mu_Q, \eta_Q, \upsilon_Q)$ be two PFSSs of V and W respectively. Then $P \oplus Q$ is a PFSS of $V \oplus W$.*

Proof. Let $c_1, c_2 \in V \oplus W$ with $c_1 = a_1 + b_1$ and $c_2 = a_2 + b_2$ where $a_1, a_2 \in V$ and $b_1, b_2 \in W$. Let $r, s \in F$. Now,

$$\begin{aligned} \mu_{P\oplus Q}(rc_1 + sc_2) &= \mu_{P\oplus Q}(r(a_1 + b_1) + s(a_2 + b_2)) \\ &= \mu_{P\oplus Q}((ra_1 + sa_2) + (rb_1 + sb_2)) \\ &= \mu_P(ra_1 + sa_2) \wedge \mu_Q(rb_1 + sb_2) \\ &\geqslant (\mu_P(a_1) \wedge \mu_P(a_2)) \wedge (\mu_Q(b_1) \wedge \mu_Q(b_2)) \\ &\quad [\text{because } P \text{ is a PFSS of } V \text{ and } Q \text{ is a PFSS of } W] \\ &= (\mu_P(a_1) \wedge \mu_Q(b_1)) \wedge (\mu_P(a_2) \wedge \mu_Q(b_2)) \\ &= \mu_{P\oplus Q}(c_1) \wedge \mu_{P\oplus Q}(c_2), \end{aligned}$$

$$\begin{aligned}
\eta_{P\oplus Q}(rc_1+sc_2) &= \eta_{P\oplus Q}(r(a_1+b_1)+s(a_2+b_2))\\
&= \eta_{P\oplus Q}((ra_1+sa_2)+(rb_1+sb_2))\\
&= \eta_P(ra_1+sa_2)\wedge\eta_Q(rb_1+sb_2)\\
&\geqslant (\eta_P(a_1)\wedge\eta_P(a_2))\wedge(\eta_Q(b_1)\wedge\eta_Q(b_2))\\
&\quad [\text{because } P \text{ is a PFSS of } V \text{ and } Q \text{ is a PFSS of } W]\\
&= (\eta_P(a_1)\wedge\eta_Q(b_1))\wedge(\eta_P(a_2)\wedge\eta_Q(b_2))\\
&= \eta_{P\oplus Q}(c_1)\wedge\eta_{P\oplus Q}(c_2),\\
v_{P\oplus Q}(rc_1+sc_2) &= v_{P\oplus Q}(r(a_1+b_1)+s(a_2+b_2))\\
&= v_{P\oplus Q}((ra_1+sa_2)+(rb_1+sb_2))\\
&= v_P(ra_1+sa_2)\vee v_Q(rb_1+sb_2)\\
&\leqslant (v_P(a_1)\vee v_P(a_2))\vee(v_Q(b_1)\vee v_Q(b_2))\\
&\quad [\text{because } P \text{ is a PFSS of } V \text{ and } Q \text{ is a PFSS of } W]\\
&= (v_P(a_1)\vee v_Q(b_1))\vee(v_P(a_2)\vee v_Q(b_2))\\
&= v_{P\oplus Q}(c_1)\vee v_{P\oplus Q}(c_2).
\end{aligned}$$

Since c_1, c_2 are arbitrary elements of $V\oplus W$ and r, s are arbitrary scalars of F therefore $\mu_{P\oplus Q}(rc_1+sc_2)\geqslant\mu_{P\oplus Q}(c_1)\wedge\mu_{P\oplus Q}(c_2)$, $\eta_{P\oplus Q}(rc_1+sa_2)\geqslant\eta_{P\oplus Q}(c_1)\wedge\eta_{P\oplus Q}(c_2)$ and $v_{P\oplus Q}(rc_1+rc_2)\leqslant v_{P\oplus Q}(c_1)\vee v_{P\oplus Q}(c_2)$ for all $c_1, c_2\in V\oplus W$ and for all $r, s\in F$. Consequently, $P\oplus Q$ is a PFSS of $V\oplus W$. □

Proposition 4.2. *Let V and W be two crisp VSs over the same field F and $P=(\mu_P,\eta_P,v_P)$, $Q=(\mu_Q,\eta_Q,v_Q)$ be two PFSSs of V and W respectively. Then $C_{\theta,\phi,\psi}(P\oplus Q)=C_{\theta,\phi,\psi}(P)\oplus C_{\theta,\phi,\psi}(Q)$.*

Proof. Let $c\in C_{\theta,\phi,\psi}(P\oplus Q)$. Then clearly, $c\in V\oplus W$. Say, $c=a+b$ with $a\in V$ and $b\in W$. Then

$$\begin{aligned}
\mu_{P\oplus Q}(c)&=\mu_P(a)\wedge\mu_Q(b)\geqslant\theta\Rightarrow\mu_P(a)\geqslant\theta \quad\text{and}\quad \mu_Q(b)\geqslant\theta,\\
\eta_{P\oplus Q}(c)&=\eta_P(a)\wedge\eta_Q(b)\geqslant\phi\Rightarrow\eta_P(a)\geqslant\phi \quad\text{and}\quad \eta_Q(b)\geqslant\phi,\\
v_{P\oplus Q}(c)&=v_P(a)\vee v_Q(b)\leqslant\psi\Rightarrow v_P(a)\leqslant\psi \quad\text{and}\quad \eta_Q(b)\leqslant\psi.
\end{aligned}$$

Thus, $a\in C_{\theta,\phi,\psi}(P)$ and $b\in C_{\theta,\phi,\psi}(Q)$. So, $c=a+b\in C_{\theta,\phi,\psi}(P)\oplus C_{\theta,\phi,\psi}(Q)$. Consequently, $C_{\theta,\phi,\psi}(P\oplus Q)\subseteq C_{\theta,\phi,\psi}(P)\oplus C_{\theta,\phi,\psi}(Q)$.

Conversely, let $c\in C_{\theta,\phi,\psi}(P)\oplus C_{\theta,\phi,\psi}(Q)$. Then there exists $a\in C_{\theta,\phi,\psi}(P)$ and $b\in C_{\theta,\phi,\psi}(Q)$ with $c=a+b$. Then $\mu_P(a)\geqslant\theta$, $\eta_P(a)\geqslant\phi$, $v_P(a)\leqslant\psi$ and $\mu_Q(b)\geqslant\theta$, $\eta_Q(b)\geqslant\phi$, $v_Q(b)\leqslant\psi$. Thus, $\mu_P(a)\wedge\mu_Q(b)\geqslant\theta$, $\eta_P(a)\wedge\eta_Q(b)\geqslant\phi$ and $v_P(a)\vee v_Q(b)\leqslant\psi$. Now, $a\in C_{\theta,\phi,\psi}(P)\Rightarrow a\in V$ and $b\in C_{\theta,\phi,\psi}(Q)\Rightarrow b\in W$. As a result, $c=a+b\in V\oplus W$. It follows that $\mu_{P\oplus Q}(c)\geqslant\theta$, $\eta_{P\oplus Q}(c)\geqslant\phi$ and $v_{P\oplus Q}(c)\leqslant\psi$. Therefore, $c\in C_{\theta,\phi,\psi}(P\oplus Q)$. Thus, it is obtained that $C_{\theta,\phi,\psi}(P)\oplus C_{\theta,\phi,\psi}(Q)\subseteq C_{\theta,\phi,\psi}(P\oplus Q)$.

Consequently, we get $C_{\theta,\phi,\psi}(P\oplus Q)=C_{\theta,\phi,\psi}\oplus C_{\theta,\phi,\psi}(Q)$. □

5. Isomorphism between two Picture Fuzzy Subspaces

Isomorphism is a pioneer concept in crisp sense. In this section, the notion of isomorphism is introduced between two PFSSs. An important result is established here through a proposition.

Definition 5.1. Let V and W be two crisp VSs over the same field F and $P = (\mu_P, \eta_P, v_P)$, $Q = (\mu_Q, \eta_Q, v_Q)$ be two PFSSs of V and W respectively. Then P is said to be isomorphic to Q if there exists an isomorphism $H : V \to W$ such that $\mu_Q(H(a)) = \mu_P(a)$, $\eta_Q(H(a)) = \eta_P(a)$ and $v_Q(H(a)) = \mu_P(a)$ for all $a \in V$.

Proposition 5.1. *Let V and W be two crisp VSs over the field F and $P = (\mu_P, \eta_P, v_P)$, $Q = (\mu_Q, \eta_Q, v_Q)$ be two PFSSs of V and W respectively. Let P is isomorphic to Q. Then $C_{\theta,\phi,\psi}(P)$ is isomorphic to $C_{\theta,\phi,\psi}(Q)$, provided that*
(i) $\mu_P(\rho_1) \geqslant \theta$, $\eta_P(\rho_1) \geqslant \phi$ and $v_P(\rho_1) \leqslant \psi$;
(ii) $\mu_Q(\rho_2) \geqslant \theta$, $\eta_Q(\rho_2) \geqslant \phi$ and $v_Q(\rho_2) \leqslant \psi$,
where ρ_1, ρ_2 be the null vectors in V and W, respectively.

Proof. Since P and Q are isomorphic therefore there exists an isomorphism $H : V \to W$ such that $\mu_Q(H(a)) = \mu_P(a)$, $\eta_Q(H(a)) = \eta_P(a)$ and $v_Q(H(a)) = v_P(a)$. Now, let us define $h : C_{\theta,\phi,\psi}(P) \to C_{\theta,\phi,\psi}(Q)$ such that $h(a) = H(a)$ for every $a \in C_{\theta,\phi,\psi}(P)$. Let $a \in C_{\theta,\phi,\psi}(P)$. Then $\mu_P(a) \geqslant \theta$, $\eta_P(a) \geqslant \phi$ and $v_P(a) \leqslant \psi$. Since P is isomorphic to Q therefore it follows that $\mu_Q(H(a)) \geqslant \theta$, $\eta_Q(H(a)) \geqslant \phi$ and $v_Q(H(a)) \leqslant \psi$. Thus, for $a \in C_{\theta,\phi,\psi}(P)$, it is obtained that $H(a) \in C_{\theta,\phi,\psi}(Q)$. So, h is well defined.

Since H is an isomorphism therefore H is one-one and onto. Due to injectivity of H, $\ker H = \{a \in V : H(a) = \rho_2\} = \{\rho_1\}$. It follows that $\{a \in C_{\theta,\phi,\psi}(P) : H(a) = \rho_2\} = \{\rho_1\}$ because $C_{\theta,\phi,\psi}(P) \subseteq V$. So, $\ker h$=$\{\rho_1\}$. Thus, h is one-one.

Let us suppose $b \in C_{\theta,\phi,\psi}(Q)$. Then $\mu_Q(b) \geqslant \theta$, $\eta_Q(b) \geqslant \phi$ and $v_Q(b) \leqslant \psi$. Since H is an isomorphism therefore there exists $a \in V$ such that $H(a) = b$. So, we can write $\mu_Q(H(a)) \geqslant \theta$, $\eta_Q(H(a)) \geqslant \phi$ and $v_Q(H(a)) \leqslant \psi$. Since P is isomorphic to Q therefore $\mu_P(a) \geqslant \theta$, $\eta_P(a) \geqslant \phi$ and $v_P(a) \leqslant \psi$. So, $a \in C_{\theta,\phi,\psi}(P)$. Thus, for each $b \in C_{\theta,\phi,\psi}(Q)$ there exists a pre-image $a \in C_{\theta,\phi,\psi}(P)$. So, h is onto.

Consequently, $C_{\theta,\phi,\psi}(P)$ is isomorphic to $C_{\theta,\phi,\psi}(Q)$. □

6. Picture Fuzzy Linear Transformation

In the current section, the notion of picture fuzzy linear transformation (PFLT) is initiated with a suitable example and some corresponding properties are studied.

Definition 6.1. Let V be a crisp VS over the field F and $P = (\mu_P, \eta_P, v_P)$ be a PFSS of V. Then a map $T : V \to V$ is said to be PFLT on V if
(*i*) T is a linear map in crisp sense;
(*ii*) $\mu_P(T(a)) \geqslant \mu_P(a)$, $\eta_P(T(a)) \geqslant \eta_P(a)$ and $v_P(T(a)) \leqslant v_P(a)$ for all $a \in V$.

Example 6.1. Consider the Example 3.1. Define a map $T : V \to V$ by $T((a_1, a_2, a_3)) = (a_1 + a_2, a_2 + a_3, 0)$. Clearly, T is a linear map in crisp sense. For any $(a_1, a_2, a_3) \in V$, it is observed that

$$\mu_P(T(a_1,a_2,a_3)) = \mu_P((a_1+a_2,a_2+a_3,0)) = 0.3 \geqslant \mu_P((a_1,a_2,a_3)),$$
$$\eta_P(T(a_1,a_2,a_3)) = \eta_P((a_1+a_2,a_2+a_3,0)) = 0.35 \geqslant \eta_P((a_1,a_2,a_3)),$$
$$v_P(T(a_1,a_2,a_3)) = v_P((a_1+a_2,a_2+a_3,0)) = 0.15 \leqslant v_P((a_1,a_2,a_3)).$$

Thus, T is a PFLT on V.

Proposition 6.1. *Let V be a crisp VS over the field F and P be a PFSS of V. If T_1 and T_2 are two PFLTs on V then so is T_1+T_2.*

Proof. Let $a \in V$. Now,

$$\begin{aligned}
\mu_P((T_1+T_2)(a)) &= \mu_P(T_1(a)+T_2(a)) \\
&\geqslant \mu_P(T_1(a)) \wedge \mu_P(T_2(a)) \quad \text{[because } P \text{ is a PFSS of } V] \\
&\geqslant \mu_P(a) \wedge \mu_P(a) \quad \text{[because } T \text{ is a PFLT on } V] \\
&= \mu_P(a), \\
\eta_P((T_1+T_2)(a)) &= \eta_P(T_1(a)+T_2(a)) \\
&\geqslant \eta_P(T_1(a)) \wedge \eta_P(T_2(a)) \quad \text{[because } P \text{ is a PFSS of } V] \\
&\geqslant \eta_P(a) \wedge \eta_P(a) \quad \text{[because } T \text{ is a PFLT on } V] \\
&= \eta_P(a), \\
\text{and } v_P((T_1+T_2)(a)) &= v_P(T_1(a)+T_2(a)) \\
&\leqslant v_P(T_1(a)) \vee v_P(T_2(a)) \quad \text{[because } P \text{ is a PFSS of } V] \\
&\leqslant v_P(a) \vee v_P(a) \quad \text{[because } T \text{ is a PFLT on } V] \\
&= v_P(a).
\end{aligned}$$

Since a is an arbitrary element of V therefore $\mu_P((T_1+T_2)(a)) \geqslant \mu_P(a)$, $\eta_P((T_1+T_2)(a)) \geqslant \eta_P(a)$ and $v_P((T_1+T_2)(a)) \leqslant v_P(a)$ for all $a \in V$. Consequently, T_1+T_2 is a PFLT on V. □

Proposition 6.2. *Let V be a crisp VS over the field F and P be a PFSS of V. If T is a PFLT on V then so is kT for some scalar $k \in F$.*

Proof. Let $a \in V$. Now,

$$\begin{aligned}
\mu_P((kT)(a)) &= \mu_P(kT(a)) \\
&\geqslant \mu_P(T(a)) \quad \text{[because } P \text{ is a PFSS of } V] \\
&\geqslant \mu_P(a) \quad \text{[because } T \text{ is a PFLT on } V], \\
\eta_P((kT)(a)) &= \eta_P(kT(a)) \\
&\geqslant \eta_P(T(a)) \quad \text{[because } P \text{ is a PFSS of } V] \\
&\geqslant \eta_P(a) \quad \text{[because } T \text{ is a PFLT on } V], \\
\text{and } v_P((kT)(a)) &= v_P(kT(a))
\end{aligned}$$

$$\leqslant v_P(T(a)) \quad \text{[because } P \text{ is a PFSS of } V\text{]}$$
$$\leqslant v_P(a) \quad \text{[because } T \text{ is a PFLT on } V\text{]}.$$

Since a is an arbitrary element of V therefore $\mu_P((kT)(a)) \geqslant \mu_P(a)$, $\eta_P((kT)(a)) \geqslant \eta_P(a)$ and $v_P((kT)(a)) \leqslant v_P(a)$ for all $a \in V$ and for some scalar $k \in F$. Consequently, kT is a PFLT on V. □

7. Linear Independency of a Finite Set of Vectors in Picture Fuzzy Sense

The current section introduces the concept of picture fuzzy linearly independent (PFLI) set of vectors with suitable example. An important result related to it is highlighted through a proposition.

Definition 7.1. Let V be a crisp VS over the field F and $P = (\mu_p, \eta_P, v_P)$ be a PFSS of V. A finite set of vectors $\{a_1, a_2, a_3, \ldots, a_n\}$ in V is said to be PFLI in V with respect to PFSS P if

(i) $\{a_1, a_2, a_3, \ldots, an\}$ is linearly independent set of vectors in V;
(ii)

$$\mu_P(c_1a_1 + c_2a_2 + \cdots + c_na_n) = \mu_P(c_1a_1) \wedge \mu_P(c_2a_2) \wedge \cdots \wedge \mu_P(c_na_n),$$
$$\eta_P(c_1a_1 + c_2a_2 + \cdots + c_na_n) = \eta_P(c_1a_1) \wedge \eta_P(c_2a_2) \wedge \cdots \wedge \eta_P(c_na_n),$$
$$v_P(c_1a_1 + c_2a_2 + \cdots + c_na_n) = v_P(c_1a_1) \vee v_P(c_2a_2) \vee \cdots \vee v_P(c_na_n),$$

where $c_i \in F$ for $i = 1, 2, \ldots, n$.

Proposition 7.1. *Let V be a crisp VS over the field F and $P = (\mu_P, \eta_P, v_P)$ be a PFSS of V. Also, let M be a finite set of vectors in V which is PFLI in V with respect to PFSS P. Then any subset of M is PFLI in V with respect to PFSS P.*

Proof. Let $M = \{a_1, a_2, \ldots, a_n\}$. Now, let $\{a_1, a_2, a_3, \ldots, a_r\}$ be any subset of M, where $r \leqslant n$. It is known that $\{a_1, a_2, a_3, \ldots, a_r\}$ is linearly independent in V. Let ρ be the null vector in V. Now, $c_1a_1 + c_2a_2 + \cdots + c_ra_r = c_1a_1 + c_2a_2 + \cdots + c_ra_r + c_{r+1}a_{r+1} + \cdots + c_na_n$, where $c_{r+1} = c_{r+2} = \cdots = c_n = 0$. Now,

$$\begin{aligned}
&\mu_P(c_1a_1 + c_2a_2 + \cdots + c_ra_r)\\
=&\mu_P(c_1a_1 + c_2a_2 + \cdots + c_ra_r + c_{r+1}a_{r+1} + \cdots + c_na_n)\\
=&\mu_P(c_1a_1) \wedge \mu_P(c_2a_2) \wedge \cdots \mu_P(c_ra_r) \wedge \mu_P(c_{r+1}a_{r+1}) \wedge \cdots \wedge \mu_P(c_na_n)\\
&\text{[since } \{a_1, a_2, \ldots, a_n\} \text{ is PFLI set of vectors in } V \text{ with respect to PFSS } P\text{]}\\
=&\mu_P(c_1a_1) \wedge \mu_P(c_2a_2) \wedge \cdots \wedge \mu_P(c_ra_r) \wedge \mu_P(\rho)\\
=&\mu_P(c_1a_1) \wedge \mu_P(c_2a_2) \wedge \cdots \wedge \mu_P(c_ra_r) \quad \text{[by Proposition 3.1]},\\
&\eta_P(c_1a_1 + c_2a_2 + \cdots + c_ra_r)\\
=&\eta_P(c_1a_1 + c_2a_2 + \cdots + c_ra_r + c_{r+1}a_{r+1} + \cdots + c_na_n)\\
=&\eta_P(c_1a_1) \wedge \mu_P(c_2a_2) \wedge \cdots \wedge \eta_P(c_ra_r) \wedge \eta_P(c_{r+1}a_{r+1}) \wedge \cdots \wedge \eta_P(c_na_n)
\end{aligned}$$

[since $\{a_1, a_2, \ldots, a_n\}$ is PFLI set of vectors in V with respect to PFSS P]

$=\eta_P(c_1a_1) \wedge \eta_P(c_2a_2) \wedge \cdots \wedge \eta_P(c_ra_r) \wedge \eta_P(\rho)$

$=\eta_P(c_1a_1) \wedge \eta_P(c_2a_2) \wedge \cdots \wedge \eta_P(c_ra_r)$ [by Proposition 3.1]

and

$$\begin{aligned}
&\upsilon_P(c_1a_1 + c_2a_2 + \cdots + c_ra_r)\\
=&\upsilon_P(c_1a_1 + c_2a_2 + \cdots + c_ra_r + c_{r+1}a_{r+1} + \cdots + c_na_n)\\
=&\upsilon_P(c_1a_1) \vee \upsilon_P(c_2a_2) \vee \cdots \vee \upsilon_P(c_ra_r) \vee \upsilon_P(c_{r+1}a_{r+1}) \vee \cdots \vee \upsilon_P(c_na_n)
\end{aligned}$$

[since $\{a_1, a_2, \ldots, a_n\}$ is PFLI set of vectors in V with respect to PFSS P]

$=\upsilon_P(c_1a_1) \vee \upsilon_P(c_2a_2) \vee \cdots \vee \upsilon_P(c_ra_r) \vee \upsilon_P(\rho)$

$=\upsilon_P(c_1a_1) \vee \upsilon_P(c_2a_2) \vee \cdots \vee \upsilon_P(c_ra_r)$ [by Proposition 3.1],

for $c_1, c_2, c_3, \ldots, c_n \in F$. Thus, $\{a_1, a_2, a_3, \ldots, a_r\}$ is PFLI in V with respect to PFSS P. Therefore, any subset of M is PFLI in V with respect to PFSS P. □

Example 7.1. Let us consider a crisp VS $V = \mathbb{R}^2$ and a PFSS P of V as follows:

$$\mu_P(a) = \begin{cases} 0.57, & \text{when } a \in \{(k,0) : k \in \mathbb{R}\},\\ 0.23, & \text{otherwise,}\end{cases}$$

$$\eta_P(a) = \begin{cases} 0.32, & \text{when } a \in \{(k,0) : k \in \mathbb{R}\},\\ 0.17, & \text{otherwise,}\end{cases}$$

$$\upsilon_P(a) = \begin{cases} 0.11, & \text{when } a \in \{(k,0) : k \in \mathbb{R}\},\\ 0.37, & \text{otherwise.}\end{cases}$$

It is clear that $\{(a_1, 0), (0, a_2)\}$ is a PFLI set of vectors in V with respect to PFSS P for $a_1 \neq 0$ and $a_2 \neq 0$.

8. Conclusion

In this paper, the notion of PFSS of a crisp VS is introduced. Some basic properties of PFSS in context of some basic operations on PFSs are studied. Here, we have shown that the intersection of two PFSSs is a PFSS. Similar type of result is also true for Cartesian product of two PFSSs, but not necessarily true in case of union which is highlighted with two suitable examples. Also, it is shown that (θ, ϕ, ψ)-cut of a PFSS is a crisp subspace. A result on (θ, ϕ, ψ)-cut of a PFS is established here which gives a condition under which a PFS will be a PFSS. The idea of direct sum of two PFSSs over the direct sum of two crisp VSs is established and related properties are studied. The concept of isomorphism between two PFSSs is initiated here and related result is investigated. Also, the notions of PFLT and PFLI set of vectors are introduced here. It is proved that the sum of two PFLTs is a PFLT and scalar multiplication with PFLT is a PFLT. Also, it is proved that any subset of a PFLI set of vectors is PFLI. We expect that our works will help the researchers to go through more advanced level of works on PFSS and it will also encourage them to explore the idea of subspace in the environment of some other kinds of sets.

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