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QUASILINEAR PARABOLIC PROBLEM WITH $p(x)$-LAPLACIAN OPERATOR BY TOPOLOGICAL DEGREE

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Abstract. We prove the existence of a weak solution for the quasilinear parabolic initial boundary value problem associated to the equation

$$u_t - \Delta_{p(x)} u = h,$$

by using the Topological degree theory for operators of the form $L+S$, where L is a linear densely defined maximal monotone map and S is a bounded demicontinuous map of class (S_+) with respect to the domain of L.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with a Lipschitz boundary denoted by $\partial\Omega$. Fixing a final time $T > 0$, we denote by Q the cylinder $\Omega\times]0,T[$ and $\Gamma = \partial\Omega\times]0,T[$ its lateral surface. We consider the following quasilinear parabolic initial-boundary problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_{p(x)} u = h, & \text{in } Q, \\ u(x,t) = 0, & \text{in } \Gamma, \\ u(x,0) = u_0(x), & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Delta_{p(x)} u := \operatorname{div}(|\nabla \mathrm{u}|^{\mathrm{p(x)}-2}\nabla \mathrm{u})$ is the $p(x)$-Laplacian applied on u, defined from

$$\mathcal{V} := \left\{ u \in L^{p^-}(0,T; W_0^{1,p(\cdot)}(\Omega)) : \ |\nabla u| \in L^{p(\cdot)}(Q) \right\}$$

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introduced and discussed in [7,22] (and that we think it is a reasonable framework to discuss our problem) to its dual $\mathcal{V}^*$. The variable exponent $p(\cdot):\overline{\Omega}\to[1,+\infty[$ is a Log-Hölder continuous function only dependent on the space variable x (see definitions below). The right-hand side h is assumed to belong to $\mathcal{V}^*$ and u_0 lies in $L^2(\Omega)$.

The importance of investigating these problems lies in their occurrence in modeling various physical problems involving strong anisotropic phenomena related to electrorheological fluids [19], the processes of filtration in complex media [4], image processing [10], mathematical biology [13], stratigraphy problems [15], and also elasticity [23].

Problems similar to the problem (1.1) are treated by several authors (see for instance [14, 22] and references therein) where the proved independently the existence of at least a weak solution for these problems.

In this work, we prove the existence of solutions for the quasilinear parabolic initial boundary value problem (1.1) using another approach: that of Topological degree theory.

The use of the theory of topological degrees is an efficient tool for solving some elliptical PDEs even in variable exponent spaces without resorting to variational methods (see [1–3]). This theory has recently been used also to solve some fractional differential equations (see [5, 18, 20, 21]). In this paper, we will use this approach to solve a parabolic problem in a space also with variable exponent.

The rest of this paper is organized as follows. In Section 2, we state some mathematical preliminaries about the functional framework where we will treat our problem. In Section 3, we introduce some classes of operators and then the associated topological degree. We will prove the main results in Section 4.

2. Preliminaries

We first recall some basic properties of variable exponent Lebesgue and Sobolev spaces (see [8, 11, 12, 16, 17] for more details).

Let
$$p^- := \operatorname{essinf}_{x\in\Omega} p(x) \quad \text{and} \quad p^+ := \operatorname{esssup}_{x\in\Omega} p(x).$$
We will make use of the following assumption
$$1 < p^- \le p(x) \le p^+ < +\infty. \tag{2.1}$$
An interesting feature of generalized variable exponent Sobolev space is that smooth functions are not dense in it without additional assumptions on the exponent $p(\cdot)$. However, when the exponent satisfies the following so-called log-Hölder condition
$$(\exists C>0)\, |p(x)-p(y)| \log\left(e+\frac{1}{|x-y|}\right) \le C, \qquad \text{for all } x,y\in\overline{\Omega},$$
then $C_0^\infty(\overline{\Omega})$ is dense in $L^{p(\cdot)}(\Omega)$ (see [8, Theorem 3.7] and [11, Section 6.5.3]) and we have the Poincaré inequality (see [12, Theorem 8.2.4] and [16, Theorem 4.3])
$$\|u\|_{L^{p(\cdot)}(\Omega)} \le C\|\nabla u\|_{L^{p(\cdot)}(\Omega)}, \qquad \text{for all } u\in W_0^{1,p(\cdot)}(\Omega),$$

where the constant $C > 0$ depends only on Ω and the function p.

In particular, the space $W_0^{1,p(\cdot)}(\Omega)$ has a norm $\|\cdot\|_{W_0^{1,p(\cdot)}(\Omega)}$ given by

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{L^{p(\cdot)}(\Omega)}, \qquad \text{for all } u \in W_0^{1,p(\cdot)}(\Omega),$$

which equivalent to $\|\cdot\|_{W^{1,p(\cdot)}(\Omega)}$. Moreover, the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is compact (see [17]).

We extend a variable exponent function $p : \overline{\Omega} \to [1, +\infty[$ to $\overline{Q} \to [1, +\infty[$ by setting $p(x,t) = p(x)$ for all $(x,t) \in \overline{Q}$.

As in [7], we consider the following functional space

$$\mathcal{V} := \Big\{ u \in L^{p^-}(0,T; W_0^{1,p(\cdot)}(\Omega)) : \ |\nabla u| \in L^{p(\cdot)}(Q) \Big\},$$

which is a separable and reflexive Banach space endowed with the norm

$$|u|_{\mathcal{V}} := \|u\|_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega))} + \|\nabla u\|_{L^{p(\cdot)}(Q)}$$

or the equivalent norm

$$\|u\|_{\mathcal{V}} := \|\nabla u\|_{L^{p(\cdot)}(Q)}.$$

Note that, under the assumption (2.1), we have

$$\|u\|_{L^{p(\cdot)}(Q)}^{p^-} - 1 \le \int_Q |u|^{p(x)} \, dxdt \le \|u\|_{L^{p(\cdot)}(Q)}^{p^+} + 1. \tag{2.2}$$

Remark 2.1. ([7, Lemma 3.1]). $\mathcal{C}_0^\infty(Q)$ is dense in $\mathcal{V}$. Moreover we have the following continuous dense embedding

$$L^{p^+}(0,T; W_0^{1,p(\cdot)}(\Omega)) \hookrightarrow_d \mathcal{V} \hookrightarrow_d L^{p^-}(0,T; W_0^{1,p(\cdot)}(\Omega)).$$

For the corresponding dual spaces, we have

$$L^{(p^-)'}(0,T; W^{-1,p'(\cdot)}(\Omega)) \hookrightarrow \mathcal{V}^* \hookrightarrow L^{(p^+)'}(0,T; W_0^{-1,p'(\cdot)}(\Omega)).$$

3. Classes of Mappings and Topological Degree

Let X be a real separable reflexive Banach space with dual X^* and with continuous pairing $\langle \cdot, \cdot \rangle$ and let Ω be a nonempty subset of X. The symbol $\to$ ($\rightharpoonup$) stands for strong (weak) convergence.

We consider a multi-values mapping T from X to 2^{X^*} (i.e., with values subsets of X^*). With each such map, we associate its graph

$$G(T) = \{(u,w) \in X \times X^* : w \in T(u)\}.$$

The multi-values mapping T is said to be *monotone* if for any pair of elements $(u_1, w_1), (u_2, w_2)$ in $G(T)$, we have the inequality

$$\langle w_1 - w_2, u_1 - u_2 \rangle \ge 0.$$

T is said to be *maximal monotone* if it is monotone and maximal in the sense of graph inclusion among monotone multi-values mappings from X to 2^{X^*}. An equivalent version of the last clause is that for any $(u_0, w_0) \in X \times X^*$ for which $\langle w_0 - w, u_0 - u\rangle \geq 0$, for all $(u, w) \in G(T)$, we have $(u_0, w_0) \in G(T)$.

We recall that a mapping $T : D(T) \subset X \to Y$ is *demicontinuous* if for any $(u_n) \subset \Omega$, $u_n \to u$ implies $T(u_n) \rightharpoonup T(u)$. T is said to be *of class* (S_+) if for any $(u_n) \subset D(T)$ with $u_n \rightharpoonup u$ and $\limsup\langle Tu_n, u_n - u\rangle \leq 0$, it follows that $u_n \to u$.

Let L be a linear maximal monotone map from $D(L) \subset X$ to X^* such that $D(L)$ is dense in X. For each open and bounded subset G on X, we consider the following classes of operators:

$$\begin{aligned}\mathcal{F}_G :=& \{L + S : \bar{G} \cap D(L) \to X^* \mid S \text{ is bounded, demicontinuous map}\\ &\quad \text{of class } (S_+) \text{ with respect to } D(L) \text{ from } \bar{G} \text{ to } X^*\},\\ \mathcal{H}_G :=& \{L + S(t) : \bar{G} \cap D(L) \to X^* \mid S(t) \text{ is a bounded homotopy of class}\\ &\quad (S_+) \text{ with respect to } D(L) \text{ from } \bar{G} \text{ to } X^*\}.\end{aligned}$$

Note that the class $\mathcal{H}_G$ (class of *admissible homotopies*) includes all affine homotopies $L + (1 - t)S_1 + tS_2$ with $(L + S_i) \in \mathcal{F}_G$, $i = 1, 2$.

We introduce the topological degree for the class $\mathcal{F}_G$ due to Berkovits and Mustonen [6].

Theorem 3.1. *Let L be a linear maximal monotone densely defined map from $D(L) \subset X$ to X^*. There exists a topological degree function*

$$d : \{(F, G, h) : F \in \mathcal{F}_G, G \text{ an open bounded subset in } X, h \notin F(\partial G \cap D(L))\} \to \mathbb{Z}$$

satisfying the following properties.

(a) (*Existence*) *If $d(F, G, h) \neq 0$, then the equation $Fu = h$ has a solution in $G \cap D(L)$.*

(b) (*Additivity*) *If G_1 and G_2 are two disjoint open subsets of G such that $h \notin F[(\bar{G} \setminus (G_1 \cup G_2)) \cap D(L)]$, then we have*

$$d(F, G, h) = d(F, G_1, h) + d(F, G_2, h).$$

(c) (*Invariance under homotopies*) *If $F(t) \in \mathcal{H}_G$ and $h(t) \notin F(t)(\partial G \cap D(L))$ for all $t \in [0, 1]$, where $h(t)$ is a continuous curve in X^*, then*

$$d(F(t), G, h(t)) = \text{constant}, \quad \text{for all } t \in [0, 1].$$

(d) (*Normalization*) *$L + J$ is a normalising map, where J is the duality mapping of X into X^*, that is,*

$$d(L + J, G, h) = 1, \quad \text{whenever } h \in (L + J)(G \cap D(L)).$$

Theorem 3.2. *Let $L + S \in \mathcal{F}_X$ and $h \in X^*$. Assume that there exists $R > 0$ such that*

$$\langle Lu + Su - h, u\rangle > 0, \tag{3.1}$$

for all $u \in \partial B_R(0) \cap D(L)$. Then $(L + S)(D(L)) = X^*$.

Proof. Let $\varepsilon > 0$, $t \in [0, 1]$ and

$$F_\varepsilon(t, u) = Lu + (1-t)Ju + t(Su + \varepsilon Ju - h).$$

Since $0 \in L(0)$ and by using the boundary condition (3.1), we see that

$$\begin{aligned}\langle F_\varepsilon(t, u), u\rangle &= \langle t(Lu + Su - h), u\rangle + \langle (1-t)Lu + (1-t+t\varepsilon)Ju, u\rangle \\ &\geq \langle (1-t)Lu + (1-t+t\varepsilon)Ju, u\rangle \\ &= (1-t)\langle Lu, u\rangle + (1-t+t\varepsilon)\langle Ju, u\rangle \\ &\geq (1-t+t\varepsilon)\|u\|^2 = (1-t+t\varepsilon)R^2 > 0.\end{aligned}$$

That is $0 \notin F_\varepsilon(t, u)$. Since J and $S + \varepsilon J$ are continuous, bounded and of type (S_+), $\{F_\varepsilon(t, \cdot)\}_{t\in[0,1]}$ is an admissible homotopy. Therefore, by invariance under homotopy and normalisation, we obtain

$$d(F_\varepsilon(t, \cdot), B_R(0), 0) = d(L + J, B_R(0), 0) = 1.$$

Hence, there exists $u_\varepsilon \in D(L)$ such that $0 \in F_\varepsilon(t, \cdot)$. Letting $\varepsilon \to 0^+$ and $t = 1$, we have $h \in Lu + Su$ for some $u \in D(L)$. Since $h \in X^*$ is arbitrary, we conclude that $(L + S)(D(L)) = X^*$. $\square$

4. The Main Result

Lemma 4.1. *The operator $S := -\Delta_{p(x)}$ defined from $\mathcal{V}$ to $\mathcal{V}^*$ by*

$$\langle Su, v\rangle = \int_Q |\nabla u|^{p(x)-2}\nabla u\nabla v dxdt, \quad \text{for all } u, v \in \mathcal{V},$$

is bounded, continuous and of class (S_+).

Proof. Let $t \in]0, T[$ and denote by A the operator defined from $W_0^{1,p(x)}(\Omega)$ to $W^{-1,p'(x)}(\Omega)$ by

$$\langle Au(x, t), v(x, t)\rangle = \int_\Omega |\nabla u(x, t)|^{p(x)-2}\nabla u(x, t)\nabla v(x, t)dx,$$

for all $u(\cdot, t), v(\cdot, t) \in W_0^{1,p(x)}(\Omega)$. Then

$$\langle Su, v\rangle = \int_0^T \langle Au(x, t), v(x, t)\rangle dt, \quad \text{for all } u, v \in \mathcal{V}.$$

It is known by [9, Theorem 3.1] that A is bounded, continuous and of class (S_+); then it is the same for S. $\square$

Our main result is the following existence theorem.

Theorem 4.1. *Let $h \in \mathcal{V}^*$ and $u_0 \in L^2(\Omega)$. There exists at least one weak solution $u \in D(L)$ of problem (1.1) in the following sense*

$$-\int_Q uv_t\, dxdt + \int_Q |\nabla u|^{p(x)-2}\nabla u\nabla v dxdt = \int_Q hv dxdt,$$

for all $v \in \mathcal{V}$.

Proof. Let L be the operator defined from $\mathcal{V} \supset D(L)$ to $\mathcal{V}^*$, where

$$D(L) = \{v \in \mathcal{V} : v' \in \mathcal{V}^*, v(0) = 0\},$$

by

$$\langle Lu, v\rangle = -\int_Q uv_t dxdt, \qquad \text{for all } u \in D(L), v \in \mathcal{V}.$$

The operator L is generated by $\partial/\partial t$ via the relation

$$\langle Lu, v\rangle = \int_0^T \langle u'(t), v(t)\rangle dt, \qquad \text{for all } u \in D(L), v \in \mathcal{V}.$$

One can verify, as in [24] that L is a densely defined maximal monotone operator.

By the monotonicity of L ($\langle Lu, u\rangle \geq 0$ for all $u \in D(L)$) and by (2.2), we get

$$\langle Lu + Su, u\rangle \geq \langle Su, u\rangle = \int_Q |\nabla u|^{p(x)} dxdt \geq \|\nabla u\|^{p^-}_{L^{p(\cdot)}(Q)} - 1 = \|u\|^{p^-}_{\mathcal{V}} - 1,$$

for all $u \in \mathcal{V}$.

Since the right side of the above inequality approaches ∞ as $\|u\|_{\mathcal{V}} \to \infty$, then for each $h \in \mathcal{V}^*$ there exists $R = R(h)$ such that $\langle Lu + Su - h, u\rangle > 0$ for all $u \in B_R(0) \cap D(L)$. By applying Theorem 3.1, we conclude that the equation $Lu + Su = h$ is solvable in $D(L)$, that is, (1.1) admits at least one-weak solution. □

Conclusion and future remark

So, we have used the theory of topological degree to solve a parabolic problem in a space with variable exponent. We hope in a future work to solve other parabolic problems by generalization of (1.1), by replacing, for example, the $p(x)$-Laplacian operator by a Leray-Lions type operator $-\mathrm{div}\, \mathrm{a}(\mathrm{x}, \mathrm{t}, \nabla \mathrm{u})$ under some suitable conditions.

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