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SKEW HURWITZ SERIES RINGS AND MODULES WITH BEACHY-BLIAR CONDITIONS

RAJENDRA KUMAR SHARMA[1] AND AMIT BHOOSHAN SINGH [2]

Abstract. A ring R satisfies the right Beachy-Blair condition if for every faithful right ideal J of a ring R (that is, a right ideal J of a ring R is faithful if $r_R(J) = 0$) is co-faithful (that is, a right ideal J of a ring R is called co-faithful if there exists a finite subset $J_1 \subseteq J$ such that $r_R(J_1) = 0$). In this note, we prove two main results.

(a) Let R be a ring which is skew Hurwitz series-wise Armendariz, ω-compatible and torsion-free as a $\mathbb{Z}$-module, and ω be an automorphism of R. If R satisfies the right Beachy-Blair condition then the skew Hurwitz series ring (HR, ω) satisfies the right Beachy-Blair condition.

(b) Let M_R be a right R-module which is ω-Armendariz of skew Hurwitz series type and torsion-free as a $\mathbb{Z}$-module, and ω be an automorphism of R. If M_R satisfies the right Beachy-Blair condition then the skew Hurwitz series module $HM_{(HR,\omega)}$ satisfies the right Beachy-Blair condition.

1. Introduction

Throughout this article, R and M_R denote an associative ring with identity and a unitary module, respectively. For any subset P of a ring R, $r_R(P)$ denotes the right annihilator of P in R. In fact, for any subset Y of a right R-module M_R, $r_R(Y)$ denotes the right annihilator of Y in M_R. In 1975, Beachy and Blair [4] discovered rings that satisfy the condition in which every faithful right ideal of R is co-faithful. On the other hand, Zelmanowitz [39] proved that any ring which satisfies the descending chain condition on right annihilators is right zip. The converse however does not

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hold. The term right zip was coined by Faith [10]. A ring R is right zip if the right annihilator $r_R(X)$ of a subset X of R is zero then there exists a finite subset $Y \subseteq X$ such that $r_R(Y) = 0$. Similarly, a left zip ring can be defined. A ring R is called zip if it is both right and left zip. From the above discussion it is clear that every right zip ring satisfies the right Beachy-Blair condition. Faith [10] also asked the following questions.

(a) Does R being a right zip ring imply $R[x]$ is right zip?
(b) Does R being a right zip imply $M_{n\times n}(R)$ is right zip?
(c) Does R being a right zip ring imply the group ring $R[G]$ is right zip when G is a finite group?

Cedó [7] answered all these questions negatively and positively question-2 for commutative rings. Above questions and their extensions have been studied by several authors, see [8, 10, 16, 18, 26, 29, 36, 37, 40], using some conditions. Motivated by above questions of Faith [10], any one can ask similar questions for rings with the Beachy-Blair condition. We have no idea of question-3 being answered so for. However, question-2 has been answered positively by Beachy and Blair [4]. In particular, they proved that a ring R satisfies the right Beachy-Blair condition if and only if $M_{n\times n}(R)$ satisfies the right Beachy-Blair condition. We now discuss question-1. Again, Beachy and Blair [4] have answered affirmatively in case of commutative rings. Desale and Varadarajan [9] attempted question-1 for non-commutative rings. In particular, they proved that if R is ω-reduced and satisfies the right Beachy-Blair condition then the ω-twisted power series ring $R[[x;\omega]]$ satisfies the right Beachy-Blair condition. Here, $\omega : R \to R$ is a ring automorphism of R. Recall that a ring R is called reduced if R has no nonzero nilpotent element. A reduced ring with condition $ab = 0$ if and only if $a\omega(b) = 0$ if and only if $\omega(a)b = 0$ (that is, ω-compatible) is called ω-reduced. Rodríguez- Jorge [34] continued the study of rings with the right Beachy-Blair condition. She gave a counterexample that a ring with the right Beachy-Blair condition need not be right zip. She also generalized the result of Desale and Varadarajan [9]. Moreover, they proved that if R satisfies the right Beachy-Blair condition then the ω-twisted power series ring $R[[x;\omega]]$ satisfies the right Beachy-Blair condition when R is strongly ω-skew Armendariz. A ring R is said to be strongly ω-skew Armendariz if for every $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and every $g(x) = \sum_{j=0}^{\infty} b_j x^j$ in $R[[x;\omega]]$, $f(x)g(x) = 0$ then $a_i\omega^i(b_j) = 0$ for all i, j, where $\omega : R \to R$ is a ring automorphism of R. This result of Rodríguez-Jorge [34] is also similar to the result of Cortes [8]. In [8], Cortes proved that if R is strongly ω-skew Armendariz then R is right zip if and only if the ω-twisted power series ring $R[[x;\omega]]$ is right zip. Recently, Ouyang et al. [28] generalized further the concept of rings with the right Beachy-Blair condition to a right R-module with the right Beachy-Blair condition. A right R-module satisfies the right Beachy-Blair condition if every faithful submodules of a right R-module is co-faithful. A right R-module M_R is called faithful if $r_R(M_R) = 0$. A right R-module M_R is called co-faithful if there exists a finite subset F of M_R such that $r_R(F) = 0$.

Moreover, Ouyang et al. [28] proved the relationship between a right R-module M_R with the right Beachy-Blair condition and its skew polynomial, skew monoid and skew generalized power series extensions. Recently, Sharma and Singh [36] studied the behavior of zip property of skew Hurwitz series rings and modules for non-commutative ring R. In this note, we prove two main results.

(a) Let R be a ring which is skew Hurwitz series-wise Armendariz, ω-compatible and torsion-free as a $\mathbb{Z}$-module, and ω be an automorphism of R. If R satisfies the right Beachy-Blair condition then the skew Hurwitz series ring (HR, ω) satisfies the right Beachy-Blair condition.

(b) Let M_R be a right R-module which is ω-Armendariz of skew Hurwitz series type and torsion-free as a $\mathbb{Z}$-module, and ω be an automorphism of R. If M_R satisfies the right Beachy-Blair condition then the skew Hurwitz series module $HM_{(HR,\omega)}$ satisfies the right Beachy-Blair condition.

2. Construction of Skew Hurwitz Series Rings and Modules

Rings of formal power series have been interesting. These have important applications. One of these is differential algebra. Keigher [20] considered a variant of the ring of formal power series and studied some of its properties. In [21], he extended the study of this type of rings and introduced the ring of Hurwitz series over a commutative ring with identity. Moreover, he showed that the Hurwitz series ring HR is very closely connected to the base ring R itself if R is of positive characteristic. Recall the construction of Hurwitz series ring from [21]. The elements of the Hurwitz series HR are sequences of the form $a = (a_n) = (a_1, a_2, a_3, \ldots)$, where $a_n \in R$ for each $n \in \mathbb{N} \cup \{0\}$. Addition in HR is point-wise, while the multiplication of two elements (a_n) and (b_n) in HR is defined by $(a_n)(b_n) = (c_n)$, where

$$c_n = \sum_{k=0}^{n} C_k^n a_k b_{n-k}.$$

Here, C_k^n is a binomial symbol $\frac{n!}{k!(n-k)!}$ for all $n \geq k$, where $n, k \in \mathbb{N} \cup \{0\}$. This product is similar to the usual product of formal power series, except the binomial coefficients C_k^n. This type of product was considered first by Hurwitz [19], and then by Bochner and Martin [6], Fliess [12] and Taft [38] also. Inspired by the contribution of Hurwitz, Keigher [21] coined the term ring of Hurwitz series over commutative rings. After that, Hassenin [14] extended this construction to the skew Hurwitz series rings (HR, ω), where $\omega : R \to R$ is an automorphism of R. Here, the ring R is not necessarily commutative. Recall from [14], the elements of (HR, ω) are functions $f : \mathbb{N} \cup \{0\} \to R$. Addition in (HR, ω) is component wise. Multiplication is defined for every $f, g \in (HR, \omega)$, by

$$fg(p) = \sum_{k=0}^{p} C_k^p f(k) \omega^k (g(p-k)),$$

for all $p, k \in \mathbb{N} \cup \{0\}$.

It can be easily shown that (HR, ω) is a ring with identity h_1, defined by

$$h_1(n) = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{if } n \geq 1, \end{cases}$$

where $n \in \mathbb{N} \cup \{0\}$. It is clear that R is canonically embedded as a subring of (HR, ω) via $a \rightarrow\ h_a \in (HR, \omega)$, where

$$h_a(n) = \begin{cases} a, & \text{if } n = 0, \\ 0, & \text{if } n \geq 1. \end{cases}$$

Further, Kamal [31, 32] gave the construction of the skew Hurwitz series ring by taking $\omega : R \rightarrow R$ to be an endomorphism of R and $\omega(1) = 1$ instead of $\omega : R \rightarrow R$ to be an automorphism of R. A number of authors, see for example [1, 14, 15, 31–33], have studied the properties of abstract ring structures of the skew Hurwitz series ring (HR, ω).

For any function $f \in (HR, \omega)$, $\operatorname{supp}(f) = \{n \in \mathbb{N} \cup \{0\} \mid f(n) \neq 0\}$ denote the support of f and $\pi(f)$ denote the minimal element of $\operatorname{supp}(f)$. For any nonempty subset X of R, we denote:

$$(HX, \omega) = \{f \in (HR, \omega) \mid f(n) \in X \cup \{0\}, n \in \mathbb{N} \cup \{0\}\}.$$

In [33], Kamal generalized the construction of the skew Hurwitz series rings and proposed the concept of the skew Hurwitz series modules. He extended the properties of the simple and semisimple modules to the skew Hurwitz series module $HM_{(HR,\omega)}$. Let M_R be a right R-module and HM be the set of all maps $\phi : \mathbb{N} \cup \{0\} \rightarrow M$. With pointwise addition, $HM_{(HR,\omega)}$ is an abelian additive group. Moreover, $HM_{(HR,\omega)}$ becomes a module over the skew Hurwitz series ring (HR, ω), by the scalar multiplication for each $\phi \in HM_{(HR,\omega)}$ and each $g \in (HR, \omega)$ is defined by:

$$\phi g(p) = \sum_{k=0}^{p} C_k^p \phi(k) \omega^k(g(p-k)),$$

for each $p, k \in \mathbb{N} \cup \{0\}$.

For any $m \in M$ and any $n \in \mathbb{N} \cup \{0\}$, we define $h_m \in HM_{(HR,\omega)}$ by

$$h_m(p) = \begin{cases} m, & \text{if } p = 0, \\ 0, & \text{if } p \geq 1. \end{cases}$$

Then it is clear that $m \rightarrow h_m$ is a module embedding of M into $HM_{(HR,\omega)}$.

3. Skew Hurwitz Series Rings With The Right Beachy-Blair Condition

Beachy and Blair [4] proved that if R is commutative and satisfies the right Beachy-Blair condition, then $R[x]$ satisfies the right Beachy-Blair condition. Afterwards, Desale and Varadarajan [9] generalized above result. In particular, they proved that if R is ω-reduced and satisfies the right Beachy-Blair condition then the ω-twisted power series ring $R[[x; \omega]]$ satisfies the right Beachy-Blair condition. In [34], Rodríguez-Jorge gave the following example in which the Beachy-Blair condition passes to the power series ring $R[[x]]$.

Example 3.1 ([34, Example 3.1]). For any field $\mathbb{F}$, there exists a right zip $\mathbb{F}$-algebra R such that $R[[x]]$ is not zip but the power series ring $R[[x]]$ satisfies the right Beachy-Blair condition.

While in general it remains as an open problem whether or not the Beachy-Blair condition passes to the power series ring $R[[x]]$. Further, Rodríguez-Jorge [34] proved that if R is a strongly ω-Armendariz ring then the Beachy-Blair condition passes to the skew power series ring $R[[x;\omega]]$. This result of Rodríguez-Jorge [34] is a generalization of Desale and Varadarajan [9]. Motivated by this result, in this section, we prove that the right Beachy-Blair condition passes to the skew Hurwitz series ring (HR,ω) under certain conditions. To prove our main result of this section, we need some definitions and results.

Due to Krempa [23], a monomorphism ω of a ring R is said to be rigid if $a\omega(a)=0$ implies $a=0$, for $a\in R$. A ring R is called ω-rigid if there exists a rigid endomorphism ω of R. In [3], Annin said a ring R is ω-compatible if for each $a,b\in R$, $ab=0$ if and only if $a\omega(b)=0$. Hashemi and Moussavi [13] gave some examples of non-rigid ω-compatible rings and proved following lemma.

Lemma 3.1. *Let ω be an endomorphism of a ring R. Then*

(a) *if ω is compatible, then ω is injective;*
(b) *ω is compatible if and only if for all $a,b\in R$,*

$$\omega(a)b=0 \Leftrightarrow ab=0;$$

(c) *the following conditions are equivalent:*
 (i) *ω is rigid;*
 (ii) *ω is compatible and R is reduced;*
 (iii) *for every $a\in R$, $\omega(a)a=0$ implies that $a=0$.*

In [1], Ahmadi et al. introduced the concept of skew Hurwitz series-wise Armendariz by considering R as a commutative ring and defined as follows.

Definition 3.1. Let R be a commutative ring and $\omega: R\to R$ be an endomorphism of R. The ring R is said to be skew Hurwitz series-wise Armendariz, if for every skew Hurwitz series $f,g\in(HR,\omega)$, $fg=0$ if and only if $f(n)g(m)=0$ for all n, m.

The concept of skew Hurwitz series-wise Armendariz in case of non-commutative ring was introduced by Sharma and Singh [36] and defined as follows.

Definition 3.2. Let R be a ring and $\omega: R\to R$ be an endomorphism of R. The ring R is said to be skew Hurwitz series-wise Armendariz, if for every skew Hurwitz series $f,g\in(HR,\omega)$, $fg=0$ implies $f(n)\omega^n g(m)=0$ for all n,m.

The following theorem shows that every reduced is skew Hurwitz series-wise Armendariz under some additional conditions.

Theorem 3.1. *Let R be a ring and ω be an automorphism of R. If R is reduced, ω-compatible and torsion-free as a $\mathbb{Z}$-module then R is skew Hurwitz series-wise Armendariz.*

Proof. Following the proof of Sharma and Singh [36, Theorem 3.5], we get the result. □

Now, we prove our main result.

Theorem 3.2. *Let R be a ring which is skew Hurwitz series-wise Armendariz, ω-compatible and torsion-free as a $\mathbb{Z}$-module, and ω be an automorphism of R. If R satisfies the right Beachy-Blair condition then the skew Hurwitz series ring (HR, ω) satisfies the right Beachy-Blair condition.*

Proof. Suppose R satisfies the right Beachy-Blair condition and U be a right ideal of (HR, ω) such that $r_{(HR,\omega)}(U) = 0$. Then the ideal generated by U is the two-sided ideal and $V = (HR, \omega)U$. Let $C_V = \cup_{f \in V} \{f(n) | f \in V, n \in \text{supp}(f)\}$ which is a nonempty subset of R. Now, we show $r_R(C_V) = 0$. Let $a \in r_R(C_V)$, $f(n)a = 0$ for all $n \in \text{supp}(f)$. Which gives $0 = f(n)a = f(n)h_a(0) = f(n)\omega^n(h_a(0))$ since R is ω-compatible and torsion-free as a $\mathbb{Z}$-module. It follows that $h_a \in r_{(HR,\omega)}(V)$. Thus, $h_a = 0$ which implies $a = 0$. Therefore, $r_R(C_V) = 0$.

Now, we show C_V is an ideal of R. Since ω is an automorphism of R, for any $r \in R$, $h_r, \omega^{-n}(h_r) \in (HR, \omega)$. Then $h_r f, f\omega^{-n}(h_r) \in V = (HR, \omega)U$ since V is an ideal of R. Thus, $h_r(0)f(n), f(n)\omega^n\omega^{-n}(h_r(0)) \in C_V$. It follows that $rf(n), f(n)r \in C_V$. Now, consider $a, b \in C_V$, then there exist $f, g \in V$ such that $f(n) = a$ and $g(n) = b$ for some $n \in \mathbb{N} \cup \{0\}$. Since V is an ideal, $f + g \in V$. Thus, $a + b = f(n) + g(n) = (f + g)(n) \in C_V$. Therefore, C_V is an ideal of R.

Since R satisfies the right Beachy-Blair condition so there exists a nonempty finite subset $Y = \{y_1, y_2, y_3, \ldots, y_s\}$ of C_V such that $r_R(Y) = 0$. Thus, for each $y_i \in Y_0$ there exists $f_i \in V$ such that $f_i(n) = y_i$ for some $n \in \mathbb{N}$, where $1 \leq i \leq s$. It follows that $V_1 = \{f_1, f_2, f_3, \ldots, f_s\}$ be a subset of V. Then $Y \subseteq C_{V_1}$ which implies that $r_R(C_{V_1}) = 0$. Now, we show $r_{(HR,\omega)}(V_1) = 0$. Let $g \in r_{(HR,\omega)}(V_1)$. Then $f_i g = 0$ for all $f_i \in V_1$. Since R is skew Hurwitz series-wise Armendariz, ω-compatible and torsion-free as a $\mathbb{Z}$-module so $f_i(n)g(m) = 0$ for every $n \in \text{supp}(f_i)$ and $m \in \text{supp}(g)$ from Theorem 3.1. Thus, $g(m) = 0$ which implies that $g = 0$. This proves that $r_{(HR,\omega)}(V_1) = 0$.

Since V_1 is a subset of V and V is an ideal of (HR, ω) generated by the right ideal U so

$$V_1 = \left\{ f_i = \sum_{j=1}^{m_s} g_i^j f_i^j \mid g_i^j \in (HR, \omega), f_i^j \in U, 1 \leq i \leq s, 1 \leq j \leq m_s \right\}.$$

Now consider $U_1 = \left\{ f_i^j \in U \mid 1 \leq i \leq s, 1 \leq j \leq m_s \right\}$ which is a finite subset of U. Thus, $r_{(HR,\omega)}(U_1) \subseteq r_{(HR,\omega)}(V_1) = 0$. Hence, (HR, ω) satisfies the right Beachy-Blair conditions. □

As a direct consequence of the above theorem, we obtain the following corollary.

Corollary 3.1. *Let R be a reduced ring and be torsion-free as a $\mathbb{Z}$-module. If R satisfies the right Beachy-Blair condition then the Hurwitz series ring HR satisfies the right Beachy-Blair condition.*

Proof. Let ω be an identity automorphism of R, so $(HR, \omega) \cong HR$. Thus, from Theorem 3.2, we obtain the result. □

4. Skew Hurwitz Series Modules With the right Beachy-Blair condition

In this section, we discuss the right R-module with the right Beachy-Blair condition to the skew Hurwitz series module $MH_{(HR,\omega)}$. In particular, we prove that a right R-module with the right Beachy-Blair condition passes to the skew Hurwitz series module $MH_{(HR,\omega)}$ under certain conditions.

Due to Annin [3], a right R-module M_R is called ω-compatible if for any $m \in M_R$ and $p \in R$, $mp = 0$ if and only if $m\omega(p) = 0$, where $\omega : R \to R$ is an endomorphism of R. It follows that, if M_R is ω-compatible, $mp = 0$ if and only if $m\omega^k(p) = 0$ for all k.

According to Lee and Zhou [24], a module M_R is called ω-reduced if for any $m \in M_R$ and $a \in R$, $ma = 0$ implies $mR \cap M_R a = 0$ and ω-compatible, where $\omega : R \to R$ is a ring of endomorphism of R with $\omega(1) = 1$. Henceforth, they also proved the following lemma.

Lemma 4.1. *The following are equivalent for a module M_R.*

(a) *M_R is ω-reduced.*
(b) *The following conditions holds: for any $m \in M_R$ and $a \in R$;*
 (i) *$ma = 0$ implies $mRa = mR\omega(a) = 0$;*
 (ii) *$ma\omega(a) = 0$ implies $ma = 0$;*
 (iii) *$ma^2 = 0$ implies $ma = 0$.*

In [24], Lee and Zhou introduced the concept of ω-Armendariz of power series type and defined as follows.

Definition 4.1. A right R-module M_R is said to be ω-Armendariz of power series type if the following conditions are satisfied.

(a) For any $m(x) = \sum_{i=0}^{\infty} m_i x^i \in M[[x; \omega]]$ and $f(x) = \sum_{i=0}^{\infty} a_j x^j \in R[[x; \omega]]$, $m(x)f(x) = 0$ implies $m_i\omega^i(a_j) = 0$ for each $i, j \geq 0$.
(b) For any $m \in M_R$ and $a \in R$, $ma = 0$ if and only if $m\omega(a) = 0$.

Motivated by the above definition, Sharma and Singh [36] introduced the concept of ω-Armendariz of skew Hurwitz series type and defined as follows.

Definition 4.2. Let M_R be a right R-module and $\omega : R \to R$ be an endomorphism of R. A right R-module M_R is said to be ω-Armendariz of skew Hurwitz series type if the following conditions are satisfied.

(a) For every skew Hurwitz series $\phi \in MH_{(HR,\omega)}$ and $g \in (HR, \omega)$, $\phi g = 0$ implies $\phi(p)\omega^p g(q) = 0$ for all p, q.
(b) For any $m \in M_R$ and $a \in R$, $ma = 0$ if and only if $m\omega(a) = 0$.

Next theorem shows that every ω-reduced module is ω-Armendariz of skew Hurwitz series type under some additional conditions.

Theorem 4.1. *Let M_R be a right R-module and ω be an automorphism of R. If M_R be ω-reduced and torsion-free as a $\mathbb{Z}$-module then M_R is ω-Armendariz of skew Hurwitz series type.*

Proof. Following the proof of Sharma and Singh [36, Theorem 4.5], we obtain the result. □

Recently, Ouyang et al. [28] generalized further the concept of rings with the right Beachy-Blair condition to a right R-module with the right Beachy-Blair condition. They proved the relationship between a right R-module M_R with the right Beachy-Blair condition and its skew polynomial, skew monoid and skew generalized power series extensions. Moreover, they extended the well known results of Beachy and Blair [4], and Desale and Varadarajan [9]. In the following theorem, we prove that the right Beachy-Blair condition passes to the skew Hurwitz series module $HM_{(HR,\omega)}$.

Theorem 4.2. *Let M_R be a right R-module which is ω-Armendariz of skew Hurwitz series type and torsion-free as a $\mathbb{Z}$-module, and ω be an automorphism of R. If M_R satisfies the right Beachy-Blair condition then the skew Hurwitz series module $HM_{(HR,\omega)}$ satisfies the right Beachy-Blair condition.*

Proof. Suppose M_R satisfies the right Beachy-Blair condition and let I be a submodule of $HM_{(HR,\omega)}$ with $r_{(HR,\omega)}(I) = 0$. Then $N = (HR, \omega)I$ is a submodule of $HM_{(HR,\omega)}$ which is generated by I. Put $C_N = \cup_{\phi \in N} C_\phi$, where $C_\phi = \{\phi(n) | \phi \in N, n \in \text{supp}(\phi)\}$ which is a nonempty subset of M_R. Now, we show $r_R(C_N) = 0$. Let $a \in r_R(C_N)$, $\phi(n)a = 0$ for all $n \in \text{supp}(\phi)$ and all $\phi \in N$. Which gives $0 = \phi(n)a = \phi(n)h_a(0) = \phi(n)\omega^n(h_a(0))$ since M_R is ω-Armendariz of skew Hurwitz series type and torsion-free as a $\mathbb{Z}$-module. It follows that $\phi h_a = 0$. Thus, $h_a = 0$ which implies $a = 0$. Therefore, $r_R(C_N) = 0$.

Now, we show C_N is a submodule of M_R. Since ω is an automorphism of R so for any $r \in R$, $h_r, \omega^{-n}(h_r) \in (HR, \omega)$. Then $h_r\phi, \phi\omega^{-n}(h_r) \in N$ since N is a submodule of $HM_{(HR,\omega)}$. Thus, $h_r(0)\phi(n), \phi(n)\omega^n\omega^{-n}(h_r(0)) \in C_N$. Therefore, $r\phi(n), \phi(n)r \in C_N$ for all $n \in \text{supp}(\phi)$. Let $a, b \in C_N$, then there exist $\phi_1, \phi_2 \in C_N$ such that $\phi_1(n) = a$ and $\phi_2(n) = b$ for some $n \in \mathbb{N} \cup \{0\}$. Since N is a submodule, $\phi_1 + \phi_2 \in N$. Thus, $a + b = \phi_1(n) + \phi_2(n) = (\phi_1 + \phi_2)(n) \in C_N$. Hence, C_N is a submodule of M_R.

Since M_R satisfies the right Beachy-Blair condition so there exists a nonempty finite subset $V = \{v_1, v_2, v_3, \ldots, v_s\}$ of C_N such that $r_R(V) = 0$. Thus, for each v_i there exists $\phi_i \in N$ such that $\phi_i(n) = v_i$ for some $n \in \mathbb{N}$, where $1 \le i \le s$. It follows that $N_1 = \{\phi_1, \phi_2, \phi_3, \ldots, \phi_s\}$ be a subset of N. Then $V \subseteq C_{N_1}$ which implies that

$r_R(C_{N_1}) = 0$. Now, we show $r_{(HR,\omega)}(N_1) = 0$. Let $g \in r_{(HR,\omega)}(N_1)$. Then $\phi_i g = 0$ for all $\phi_i \in N_1$. Since R is ω-Armendariz of skew Hurwitz series type and torsion-free as a $\mathbb{Z}$-module so $\phi_i(n)g(m) = 0$ for every $n \in \operatorname{supp}(\phi_i)$ and $m \in \operatorname{supp}(g)$ from Theorem 4.1. Thus, $g = 0$. This proves that $r_{(HR,\omega)}(N_1) = 0$.

Since N_1 is a subset of N and N is a submodule of $HM_{(HR,\omega)}$ generated by the submodule I so

$$N_1 = \left\{ \phi_i = \sum_{j=1}^{m_s} g_i^j \phi_i^j \mid g_i^j \in (HR,\omega), \phi_i^j \in I, 1 \le i \le s, 1 \le j \le m_s \right\}.$$

Now, consider $I_1 = \left\{ \phi_i^j \in I \mid 1 \le i \le s, 1 \le j \le m_s \right\}$ which is a finite subset of I. Thus, $r_{(HR,\omega)}(I_1) \subseteq r_{(HR,\omega)}(N_1) = 0$. Hence, the skew Hurwitz series module $HM_{(HR,\omega)}$ satisfies the right Beachy-Blair condition. □

Here, we obtain the following result as a special case of Theorem 4.2.

Corollary 4.1. *Let M_R be a right R-module, reduced and torsion-free as $\mathbb{Z}$-module. If M_R satisfies the right Beachy-Blair condition then the skew Hurwitz series module $HM_{(HR)}$ satisfies the right Beachy-Blair condition.*

Proof. Let ω be an identity automorphism of R, so $HM_{(HR,\omega)} \cong HM_{HR}$. Thus, from Theorem 4.2, we can get the proof. □

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[1]Department of Mathematics,
Indian Institute of Technology,
New Delhi-110016, India
Email address: rksharmaiitd@gmail.com

[2]Department of Computer Science and Engineering,
Jamia Hamdard (Deemed to be University),
New Delhi-110062, India
Email address: amit.bhooshan84@gmail.com