Kragujevac Journal of Mathematics
Volume 47(4) (2023), Pages 499–509.

THREE SOLUTIONS FOR p-HAMILTONIAN SYSTEMS WITH IMPULSIVE EFFECTS

HADI HAGHSHENAS[1] AND GHASEM A. AFROUZI[2]

Abstract. In this paper, we give some new criteria that guarantee the existence of at least three weak solutions to a p-Hamiltonian boundary value problem generated by impulsive effects. To ensure the existence of these solutions, we use variational methods and critical point theory as our main tools.

1. Introduction.

In this research, we prove the existence of at least three weak solutions to the following second-order impulsive p-Hamiltonian system

(1.1)
$$\begin{cases} -\left(|u'|^{p-2}u'\right)' + A(t)|u|^{p-2}u = \lambda \nabla F(t,u) + \nabla G(t,u) + \nabla H(u), & \text{a.e. } t \in J, \\ \triangle(u_i'(t_j)) = I_{ij}(u_i(t_j)), \quad i = 1,2,\ldots,N, j = 1,2,\ldots,m, \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases}$$

Here, we assume that

- $N \geq 1$, $m \geq 2$, $p > 1$, $T > 0$ and $\lambda \in \mathbb{R}$;
- the function $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is measurable in $[0,T]$ and C^1 in $\mathbb{R}^N$;
- $G : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is a function such that $G(\cdot, x)$ is continuous on $[0,T]$ for all $x \in \mathbb{R}^N$ and $G(t,\cdot)$ is C^1 on $\mathbb{R}^N$ for almost every $t \in [0,T]$;
- $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $J = [0,T] \setminus \{t_1, t_2, \ldots, t_m\}$, $u(t) = (u_1(t), \ldots, u_N(t))$ and $\triangle(u_i'(t_j)) = u_i'(t_j^+) - u_i'(t_j^-)$ such that $u_i'(t_j^{\pm}) = \lim_{t \to t_j^{\pm}} u_i'(t)$;

Key words and phrases. Weak solution, p-Hamiltonian boundary value problem, impulsive effect, critical point theory, variational methods.

2010 *Mathematics Subject Classification.* Primary: 34B15, 34B37, 58E30.
DOI 10.46793/KgJMat2304.499H
Received: May 06, 2020.
Accepted: September 19, 2020.

- the functions $I_{ij} : \mathbb{R} \to \mathbb{R}$, $i = 1, 2, \ldots, N$, and $j = 1, 2, \ldots, m$, are continuous;
- $A(t) = (a_{ij}(t))_{N\times N}$ is an $N \times N$ continuous symmetric matrix and there is a positive constant $\underline{\lambda}$ such that $(A(t)|x|^{p-2}x, x) \geq \underline{\lambda}|x|^p$ for all $x \in \mathbb{R}^N$ and $t \in [0, T]$;
- $H : \mathbb{R}^N \to \mathbb{R}$ is a continuously differentiable function for which there is a constant $0 < L < \frac{\min\{1,\underline{\lambda}\}}{2p}$ such that $|H(x)| \leq L|x|^p$ for every $x \in \mathbb{R}^N$.

The study of the multiplicity of the solutions of Hamiltonian systems, as particular cases of dynamical systems, is mathematically important and interesting from a practical point of view. This is because these systems constitute a natural framework for the mathematical models of many natural phenomena in fluid mechanics, gas dynamics, nuclear physics, relativistic mechanics, etc. Inspired by the monographs [16] and [21], the existence and multiplicity of weak solutions for Hamiltonian systems have been investigated by many authors using variational methods. See [6, 7, 9, 11, 12, 17–19, 27–31, 33] and the references therein for example.

On the other hand, impulsive effects describe some discontinuous processes and occur in many research fields such as SIR epidemic models, controllability and optimization, etc. (see [8, 20]). In the past few decades, a series of nonlinear functional methods were applied for dealing with the existence of solutions to boundary value problems for impulsive differential equations. These include the coincidence degree theory, the comparison principles and fixed point theorems.

In particular, in the recent years, the variational method has been used successfully in the investigation of the existence and multiplicity of solutions to boundary value problems for differential equations with impulsive effects. See [1, 2] and the references therein. For the background, theory and applications of impulsive differential equations, we refer the interested readers to [4, 13, 23]. Recently, a great deal of work has been done on the existence of multiple solutions for second-order impulsive p-Hamiltonian systems. We refer the interested reader to [10, 15, 25, 26, 32], in which second-order Hamiltonian systems with impulsive effects have been examined.

Our results are motivated by the recent papers [14] and [24]. In [14], Li et al. have studied the three periodic solutions for p-Hamiltonian systems

$$\begin{cases} -(|u'|^{p-2}u')' + A(t)|u|^{p-2}u = \lambda\nabla F(t, u) + \mu\nabla G(t, u), \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \tag{1.2}$$

Their technical approach was based on the two general three critical points theorems of Averna and Bonanno [3] and Ricceri [22]. In [24], Shang and Zhang obtained three solutions to the perturbed Dirichlet boundary value problem

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = \lambda f(x, u) + g(x, u), \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

by using Theorem 2.1 below.

2. Preliminaries

In this article, we use the following theorem of Bonanno to prove the existence of three solutions for problem (1.1).

Theorem 2.1 ([5]). *Let X be a separable and reflexive real Banach space, and let $\phi, \psi : X \to \mathbb{R}$ be two continuously Gateaux differentiable functionals. Assume that ϕ is sequentially weakly lower semicontinuous and even, that ψ is sequentially weakly continuous and odd, and that, for some $a > 0$ and for each $\lambda \in [-a, a]$, the functional $\phi + \lambda\psi$ satisfies the Palais-Smale condition and*

$$\lim_{\|x\|\to+\infty} [\phi(x) + \lambda\psi(x)] = +\infty.$$

If there exists $k > 0$ such that

$$\inf_{x\in X} \phi(x) < \inf_{|\psi(x)|<k} \phi(x),$$

then, for every $b > 0$ there exists an open interval $\Lambda \subseteq [-b, b]$ and a positive real number σ such that for every $\lambda \in \Lambda$, the equation

$$\phi'(x) + \lambda\psi'(x) = 0,$$

admits at least three solutions in X whose norms are less than σ.

Here, we recall some basic concepts that will be used in what follows. Let

$$\begin{aligned} W_T^{1,p} =&\{u : [0, T] \to \mathbb{R}^N : u \text{ is absolutely continuous}, u(0) = u(T),\\ &u' \in L^p([0, T], \mathbb{R}^N)\}, \end{aligned}$$

which is endowed with the norm

$$\|u\| = \left(\int_0^T |u'(t)|^p + \Big(A(t)|u(t)|^{p-2}u(t), u(t)\Big)\, dt\right)^{\frac{1}{p}}.$$

Observe that

$$\begin{aligned} (A(t)|x|^{p-2}x, x) &= |x|^{p-2} \sum_{i,j=1}^{N} a_{ij}(t)x_i x_j \\ &\le |x|^{p-2} \sum_{i,j=1}^{N} |a_{ij}(t)||x_i||x_j| \\ &\le \left(\sum_{i,j=1}^{N} \|a_{ij}(t)\|_\infty\right) |x|^p. \end{aligned}$$

Then, there exists a constant $\overline{\lambda} \le \sum_{i,j=1}^{N} \|a_{ij}(t)\|_\infty$ such that $(A(t)|x|^{p-2}x, x) \le \overline{\lambda}|x|^p$ for all $x \in \mathbb{R}^N$. So,

$$\min\{1, \underline{\lambda}\}|||u|||^p \le \|u\|^p \le \max\{1, \overline{\lambda}\}|||u|||^p, \tag{2.1}$$

where

$$|||u||| = \left(\int_0^T |u(t)|^p dt + \int_0^T |u'(t)|^p dt\right)^{\frac{1}{p}},$$

is the usual norm of $W_T^{1,p}$. Let

$$k_0 = \sup_{u \in W_T^{1,p}\setminus\{0\}} \frac{\|u\|_\infty}{\|u\|}, \quad \|u\|_\infty = \sup_{t\in[0,T]} |u(t)|, \tag{2.2}$$

where $|\cdot|$ is the usual norm of $\mathbb{R}^N$. Since $W_T^{1,p} \hookrightarrow C^0$ is compact, one has $k_0 < +\infty$ and for each $u \in W_T^{1,p}$ there exists $\xi \in [0,T]$ such that $|u(\xi)| = \min_{t\in[0,T]} |u(t)|$. Hence, by Hölder's inequality, one has

$$\begin{aligned}
|u(t)| &= \left|\int_\xi^t u'(s)ds + u(\xi)\right| \\
&\leq \int_0^T |u'(s)|ds + \frac{1}{T}\int_0^T |u(\xi)|ds \\
&\leq \int_0^T |u'(s)|ds + \frac{1}{T}\int_0^T |u(s)|ds \\
&\leq T^{\frac{1}{q}}\left(\int_0^T |u'(s)|^p ds\right)^{\frac{1}{p}} + T^{-\frac{1}{p}}\left(\int_0^T |u(s)|^p ds\right)^{\frac{1}{p}} \\
&\leq \max\{T^{\frac{1}{q}}, T^{-\frac{1}{p}}\}\left(\left(\int_0^T |u'(s)|^p ds\right)^{\frac{1}{p}} + \left(\int_0^T |u(s)|^p ds\right)^{\frac{1}{p}}\right) \\
&\leq \sqrt[q]{2}\max\{T^{\frac{1}{q}}, T^{-\frac{1}{p}}\}\left(\int_0^T |u'(s)|^p ds + \int_0^T |u(s)|^p ds\right)^{\frac{1}{p}} \\
&= \sqrt[q]{2}\max\{T^{\frac{1}{q}}, T^{-\frac{1}{p}}\}|||u|||,
\end{aligned}$$

for each $t \in [0,T]$ and $q = \frac{p}{p-1}$. So, by (2.1) and the above expression, we obtain

$$\|u\|_\infty \leq \sqrt[q]{2}\max\{T^{\frac{1}{q}}, T^{-\frac{1}{p}}\}|||u||| \leq \sqrt[q]{2}\max\{T^{\frac{1}{q}}, T^{-\frac{1}{p}}\}\left(\min\{1,\underline{\lambda}\}\right)^{-\frac{1}{p}}\|u\|.$$

Then, from this and (2.2) it follows that

$$k_0 \leq k = \sqrt[q]{2}\max\{T^{\frac{1}{q}}, T^{-\frac{1}{p}}\}\left(\min\{1,\underline{\lambda}\}\right)^{-\frac{1}{p}}.$$

As usual, a weak solution to problem (1.1) is any $u \in W_T^{1,p}$ such that

$$\int_0^T \left[\left(|u'(t)|^{p-2}u'(t), v'(t)\right) + \left(A(t)|u(t)|^{p-2}u(t), v(t)\right)\right] dt - \int_0^T (\nabla G(t,u(t)), v(t))\, dt$$

$$- \int_0^T (\nabla H(u(t)), v(t))\, dt + \sum_{j=1}^{p}\sum_{i=1}^{N} I_{ij}(u_i(t_j))v_i(t_j) - \lambda\int_0^T (\nabla F(t,u(t)), v(t))\, dt = 0,$$

for all $v \in W_T^{1,p}$.

3. The Main Results

Now, we present our main results.

Theorem 3.1. *Suppose that F, G, H and I_{ij} satisfy the following conditions.*

(H1) $H(\cdot)$ *is even.*

(H2) $G(t,\cdot)$ *is even and* $F(t,\cdot)$ *is odd for almost every* $t \in [0,T]$.

(H3) *The functions* I_{ij}, $i = 1,2,\dots,N,$ and $j = 1,2,\dots,m$, *are odd.*

(H4) $\lim_{|x|\to 0} \frac{|\nabla G(t,x)|}{|x|^{p-1}} = 0$ *uniformly for almost every* $t \in [0,T]$.

(H5) $\lim_{|x|\to +\infty} \frac{|\nabla G(t,x)|}{|x|^{p-1}} = 0$ *uniformly for almost every* $t \in [0,T]$.

(H6) *There exist constants* $c > 0$ *and* $1 \le q < p$ *such that*

$$|\nabla F(t,x)| \le c(1 + |x|^{q-1}),$$

for all $x \in \mathbb{R}^N$ *and almost every* $t \in [0,T]$.

(H7) *There is a constant* $B \ge 0$ *such that* $G(t,x) \ge 2r\frac{|x|^p}{p} - B$ *for all* $x \in \mathbb{R}^N$ *and almost every* $t \in [0,T]$. *Here,* $r = \sup\left\{\frac{1}{\int_0^T |u(t)|^p dt} : \ \|u\| = 1\right\}$.

(H8) *For any* $i \in \{1,2,\dots,N\}$ *and* $j \in \{1,2,\dots,m\}$ *there exist constants* $a_{ij} > 0$, $b_{ij} > 0$ *and* $\gamma_{ij} \in [0,1]$ *such that*

$$I_{ij}(y) \ge -a_{ij} - b_{ij} y^{\gamma_{ij}} \quad (y \ge 0) \quad \text{and} \quad I_{ij}(y) \le a_{ij} + b_{ij}(-y)^{\gamma_{ij}} \quad (y \le 0).$$

Then, for every $b > 0$ *there exist an open interval* $\Lambda \subseteq [-b,b]$ *and a positive real number* σ *such that for every* $\lambda \in \Lambda$, *problem* (1.1) *admits at least three solutions in* $W_T^{1,p}$ *whose norms are less than* σ.

Proof. Let $X = W_T^{1,p}$ be endowed with $\|\cdot\|$, and for each u in X let

$$\phi(u) = \frac{1}{p}\|u\|^p - \int_0^T G(t,u(t))dt - \int_0^T H(u(t))dt,$$

$$\psi(u) = \frac{1}{\lambda}\sum_{j=1}^{m}\sum_{i=1}^{N}\int_0^{u_i(t_j)} I_{ij}(t)dt - \int_0^T F(t,u(t))dt.$$

Then, for every $u, v \in X$,

$$\begin{aligned}\phi'(u)(v) =& \int_0^T \Big[\Big(|u'(t)|^{p-2}u'(t), v'(t)\Big) + \Big(A(t)|u(t)|^{p-2}u(t), v(t)\Big)\Big]\, dt \\ &- \int_0^T (\nabla G(t,u(t)), v(t))\, dt - \int_0^T (\nabla H(u(t)), v(t))\, dt,\\ \psi'(u)(v) =& \frac{1}{\lambda}\sum_{j=1}^{m}\sum_{i=1}^{N} I_{ij}(u_i(t_j))v_i(t_j) - \int_0^T (\nabla F(t,u(t)), v(t))\, dt.\end{aligned}$$

Since the critical points of the functional $\phi + \lambda\psi$ on X are exactly the weak solutions of problem (1.1), our aim is to apply Theorem 2.1 to ϕ and ψ. It is well-known that ϕ is a continuously Gateaux differentiable and sequentially weakly lower semicontinuous functional. Moreover, ψ is continuously Gateaux differentiable and sequentially weakly continuous. Also, by (H1), (H2) and (H3), ϕ is even and ψ is odd. Owning to the assumption (H8), we have that

$$\begin{aligned}\int_0^z I_{ij}(t)dt &\geq -a_{ij}z - \frac{b_{ij}}{\gamma_{ij}+1} z^{\gamma_{ij}+1} \\ &= -a_{ij}|z| - \frac{b_{ij}}{\gamma_{ij}+1}|z|^{\gamma_{ij}+1} \quad (z \geq 0),\end{aligned}$$

and

$$\begin{aligned}\int_z^0 I_{ij}(t)dt &\leq -a_{ij}z - \frac{b_{ij}(-1)^{\gamma_{ij}}}{\gamma_{ij}+1} z^{\gamma_{ij}+1} \\ &= a_{ij}|z| + \frac{b_{ij}}{\gamma_{ij}+1}|z|^{\gamma_{ij}+1} \quad (z < 0).\end{aligned}$$

Therefore, for every $i \in \{1, 2, \ldots, N\}$, $j \in \{1, 2, \ldots, m\}$ and $z \in \mathbb{R}$,

$$\int_0^z I_{ij}(t)dt \geq -a_{ij}|z| - \frac{b_{ij}}{\gamma_{ij}+1}|z|^{\gamma_{ij}+1}. \tag{3.1}$$

Thanks to (H4), given $\varepsilon > 0$ small enough, we may find a constant $C_\varepsilon > 0$ such that

$$|G(t, x)| \leq C_\varepsilon + \frac{\varepsilon}{p}|x|^p, \tag{3.2}$$

for every $x \in \mathbb{R}^N$ and almost every $t \in [0, T]$. Also, taking (H6) into account, we get

$$|F(t, x)| \leq c|x| + \frac{c}{q}|x|^q, \tag{3.3}$$

for every $x \in \mathbb{R}^N$ and almost every $t \in [0, T]$. Now by (3.1), (3.2) and (3.3), for all $u \in X$ and $\lambda \in \mathbb{R}$, we obtain

$$\begin{aligned}\phi(u) + \lambda\psi(u) =& \frac{1}{p}\|u\|^p - \int_0^T G(t, u(t))dt - \int_0^T H(u(t))dt \\ &+ \sum_{j=1}^m \sum_{i=1}^N \int_0^{u_i(t_j)} I_{ij}(t)dt - \lambda \int_0^T F(t, u(t))dt \\ \geq& \frac{1}{p}\|u\|^p - \int_0^T \left(C_\varepsilon + \frac{\varepsilon}{p}|u(t)|^p\right) dt - L\int_0^T |u(t)|^p dt \\ &- \lambda \int_0^T \left(c|u(t)| + \frac{c}{q}|u(t)|^q\right) dt \\ &- \sum_{j=1}^m \sum_{i=1}^N a_{ij}|u(t_j)| - \sum_{j=1}^m \sum_{i=1}^N \frac{b_{ij}}{\gamma_{ij}+1}|u(t_j)|^{\gamma_{ij}+1} \\ \geq& \frac{1}{p}\left(1 - \frac{2^{p-1}\varepsilon + Lp}{\min\{1, \underline{\lambda}\}}\right)\|u\|^p - \frac{1}{q}\left(\min\{1, \underline{\lambda}\}\right)^{-\frac{q}{p}} 2^{\frac{q(p-1)}{p}} \lambda c\|u\|^q \\ &- \left(\min\{1, \underline{\lambda}\}\right)^{-\frac{1}{p}} 2^{\frac{p-1}{p}} \lambda c\|u\| - C_\varepsilon T\end{aligned}$$

$$-\sum_{j=1}^{m}\sum_{i=1}^{N} a_{ij}|u(t_j)| - \sum_{j=1}^{m}\sum_{i=1}^{N} \frac{b_{ij}}{\gamma_{ij}+1}|u(t_j)|^{\gamma_{ij}+1}.$$

Since $p > q$ and ε is small enough,

$$\lim_{\|u\|\to+\infty} [\phi(u) + \lambda\psi(u)] = +\infty. \tag{3.4}$$

Now, we prove that $\varphi_\lambda = \phi + \lambda\psi$ satisfies the (P-S) condition. Suppose that $\{u_n\}_{n=1}^{\infty}$ is a (P-S) sequence of φ_λ, that is, there exists $C > 0$ such that

$$\varphi_\lambda(u_n) \to C, \quad \varphi'_\lambda(u_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Assume that $\|u_n\| \to +\infty$. Then, (3.4) contradicts to the $\varphi_\lambda(u_n) \to C$, hence, $\{u_n\}_{n=1}^{\infty}$ is bounded in $W_T^{1,p}$. We may assume that there exists $u_0 \in W_T^{1,p}$ satisfying $u_n \to u_0$ weakly in $W_T^{1,p}$, $u_n \to u_0$ in $L^p[0,T]$, $u_n(t) \to u_0(t)$ for almost every $t \in [0,T]$. Observe that

$$\begin{aligned}\varphi'_\lambda(u_n)(u_n - u_0) = &\int_0^T \Big[\Big(|u'_n(t)|^{p-2}u'_n(t), u'_n(t) - u'_0(t)\Big) \\ &+ \Big(A(t)|u_n(t)|^{p-2}u_n(t), u_n(t) - u_0(t)\Big) \Big] dt \\ &- \int_0^T (\nabla G(t, u_n(t)), u_n(t) - u_0(t))\, dt \\ &- \int_0^T (\nabla H(u_n(t)), u_n(t) - u_0(t))\, dt \\ &- \int_0^T (\nabla F(t, u_n(t)), u_n(t) - u_0(t))\, dt \\ &+ \frac{1}{\lambda}\sum_{j=1}^{m}\sum_{i=1}^{N} I_{ij}\left((u_n)_i(t_j)\right)\left((u_n)_i(t_j) - (u_0)_i(t_j)\right).\end{aligned}$$

We already know that

$$\varphi'_\lambda(u_n)(u_n - u_0) \to 0 \quad \text{as} \quad n \to \infty.$$

Clearly,

$$\lim_{n\to\infty} \int_0^T (\nabla H(u_n(t)), u_n(t) - u_0(t))\, dt = 0.$$

By (H5), given $\varepsilon > 0$, we may find a constant $C_\varepsilon > 0$ such that

$$|\nabla G(t,x)| \le C_\varepsilon + \varepsilon |x|^{p-1},$$

for every $x \in \mathbb{R}^N$ and almost every $t \in [0,T]$. So,

$$\int_0^T (\nabla G(t, u_n(t)), u_n(t) - u_0(t))\, dt \to 0 \quad \text{as} \quad n \to \infty.$$

Moreover, by (H6)

$$\int_0^T (\nabla F(t, u_n(t)), u_n(t) - u_0(t))\, dt \to 0 \quad \text{as} \quad n \to \infty.$$

Also, $\sum_{j=1}^{m}\sum_{i=1}^{N} I_{ij}\left((u_n)_i(t_j)\right)\left((u_n)_i(t_j)-(u_0)_i(t_j)\right)\to 0$ as $n\to\infty$. Therefore,

$$\int_0^T \left[\left(|u_n'(t)|^{p-2}u_n'(t), u_n'(t)-u_0'(t)\right)+\left(A(t)|u_n(t)|^{p-2}u_n(t), u_n(t)-u_0(t)\right)\right]dt\to 0,$$

as $n\to\infty$. This, together with the weak convergence of $u_n\to u_0$ in $W_T^{1,p}$, implies that $u_n\to u_0$ in $W_T^{1,p}$ as $n\to\infty$. Hence, φ_λ satisfies the (P-S) condition. Finally, we prove that $\inf_{u\in X}\phi(u)<\inf_{|\psi(u)|<k}\phi(u)$ for some $k>0$. To this end, we choose a nonnegative function $v\in W_T^{1,p}$ with $\|v\|=1$. By condition (H7), a simple calculation shows that

$$\begin{aligned}\phi(sv)&=\frac{1}{p}\|sv\|^p-\int_0^T G(t,sv(t))dt-\int_0^T H(sv(t))dt\\ &\le\frac{s^p}{p}\|v\|^p-2\frac{s^p r}{p}\int_0^T|v(t)|^p dt+BT+\frac{Ls^p}{\min\{1,\underline{\lambda}\}}\|v\|^p\\ &\le\left(\frac{L}{\min\{1,\underline{\lambda}\}}-\frac{1}{p}\right)s^p+BT\to-\infty,\end{aligned}\tag{3.5}$$

as $s\to\infty$. Since $\frac{1}{2p}>\frac{L}{\min\{1,\underline{\lambda}\}}$, (3.5) implies that $\phi(sv)<0$ for $s>0$ large enough. So, we choose a large enough $s_0>0$, and let $u_1=s_0v$ such that $\phi(u_1)<0$. Thus, $\inf_{u\in X}\phi(u)<0$. From (H4), for every $\varepsilon>0$, there exists $\rho_0(\varepsilon)>0$ such that

$$|\nabla G(t,x)|\le\varepsilon|x|^{p-1},\quad \text{if } 0<|x|<\rho_0(\varepsilon).$$

Thus,

$$\int_0^T G(t,u(t))dt\le\int_0^T\frac{\varepsilon}{p}|u(t)|^p dt\le\frac{\varepsilon}{p\min\{1,\underline{\lambda}\}}\|u\|^p.$$

By choosing $\varepsilon=\frac{1}{2}\min\{1,\underline{\lambda}\}$, we get

$$\phi(u)\ge\left(\frac{1}{2p}-\frac{L}{\min\{1,\underline{\lambda}\}}\right)\|u\|^p>0.$$

Hence, there exists $k>0$ such that $\inf_{|\psi(u)|<k}\phi(u)=0$. So,

$$\inf_{u\in X}\phi(u)<\inf_{|\psi(u)|<k}\phi(u).$$

Now, all the assumptions of Theorem 2.1 are verified. Thus, for every $b>0$ there exists an open interval $\Lambda\subseteq[-b,b]$ and a positive real number σ such that for every $\lambda\in\Lambda$, problem (1.1) admits at least three solutions in $W_T^{1,p}$ whose norms are less than σ. $\square$

Theorem 3.2. *If F, G, H and I_{ij} satisfy the assumptions* (Hi) *for $i=1,2,3,4,5,6,8$ and* (H'7)*, which asserts that $\lim_{|x|\to 0}\frac{G(t,x)}{|x|^p}=+\infty$ for almost every $t\in[0,T]$, then for every $b>0$ there exist an open interval $\Lambda\subseteq[-b,b]$ and a positive real number σ such that for every $\lambda\in\Lambda$, problem* (1.1) *admits at least three solutions in $W_T^{1,p}$ whose norms are less than σ.*

Proof. The proof is similar to that of Theorem 3.1. So we only give a sketch of it. By the proof of Theorem 3.1, the functionals ϕ, ψ are sequentially weakly lower semicontinuous and continuously Gateaux differentiable in $W_T^{1,p}$, ϕ is even and ψ is odd. For every $\lambda \in \mathbb{R}$, the functional $\phi + \lambda\psi$ satisfies the (P-S) condition and

$$\lim_{\|u\|\to+\infty} [\phi(u) + \lambda\psi(u)] = +\infty.$$

Owning to the assumption (H'7), we can find $\delta > 0$ such that, for every $M > 0$ one has $|G(t,x)| > M|x|^p$ for $0 < |x| \leq \delta$ and almost every $t \in [0,T]$. We choose a nonzero nonnegative function $v \in C_0^\infty([0,T])$, put $M > \frac{3\|v\|^p}{2p\int_0^T |v(t)|^p dt}$ and take $\varepsilon > 0$ small enough. Then, we obtain

$$\begin{aligned}
\phi(\varepsilon v) &= \frac{1}{p}\|\varepsilon v\|^p - \int_0^T G(t,\varepsilon v(t))dt - \int_0^T H(\varepsilon v(t))dt \\
&\leq \frac{1}{p}\varepsilon^p\|v\|^p - M\varepsilon^p\int_0^T |v(t)|^p dt + \frac{L\varepsilon^p}{\min\{1,\underline{\lambda}\}}\|v\|^p \\
&< \frac{3}{2p}\varepsilon^p\|v\|^p - M\varepsilon^p\int_0^T |v(t)|^p dt < 0.
\end{aligned}$$

So, we get

$$\inf_{u\in W_T^{1,p}} \phi(u) < 0.$$

By the proof of Theorem 3.1, we know that there exists $k > 0$ such that

$$\inf_{u\in X} \phi(u) < \inf_{|\psi(u)|<k} \phi(u).$$

Hence, our conclusion follows from Theorem 2.1. □

When $I_{ij} = G = H \equiv 0$, the problem (1.1) reduces to the following ordinary problem which has been considered in [14] by Li et al.

$$\begin{cases} -(|u'|^{p-2}u')' + A(t)|u|^{p-2}u = \lambda\nabla F(t,u), & \text{a.e. } t \in [0,T], \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \tag{3.6}$$

By a reasoning just like that of Theorem 3.1, we obtain the following result.

Theorem 3.3. *If $F(t,\cdot)$ is odd for almost every $t \in [0,T]$ and there exist constants $c > 0$ and $1 \leq q < p$ such that*

$$|\nabla F(t,x)| \leq c(1 + |x|^{q-1}),$$

for all $x \in \mathbb{R}^N$ and almost every $t \in [0,T]$, then for every $b > 0$ there exist an open interval $\Lambda \subseteq [-b,b]$ and a positive real number σ such that for every $\lambda \in \Lambda$, problem (3.6) *admits at least three solutions in $W_T^{1,p}$ whose norms are less than σ.*

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[1]Department of Mathematics, Faculty of Mathematical Sciences,
University of Mazandaran,
Babolsar, Iran.
Email address: haghshenas60@gmail.com

[2]Department of Mathematics, Faculty of Mathematical Sciences,
University of Mazandaran,
Babolsar, Iran.
Email address: afrouzi@umz.ac.ir