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SOME RESULTS CONCERNED WITH HANKEL DETERMINANT

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Abstract. In this paper, we discuss different versions of the boundary Schwarz lemma and Hankel determinant for $\mathcal{K}(\alpha)$ class. Also, for the function $f(z) = z + c_2z^2 + c_3z^3 + \cdots$ defined in the unit disc such that $f \in \mathcal{K}(\alpha)$, we estimate a modulus of the angular derivative of $f(z)$ function at the boundary point z_0 with $f(z_0) = \frac{z_0}{1+\alpha}$ and $f'(z_0) = \frac{1}{1+\alpha}$. That is, we shall give an estimate below $|f''(z_0)|$ according to the first nonzero Taylor coefficient of about two zeros, namely $z = 0$ and $z_1 \neq 0$. The sharpness of this inequality is also proved.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z) = z + c_2z^2 + c_3z^3 + \cdots$ which are analytic in $E = \{z : |z| < 1\}$. Also, $\mathcal{K}(\alpha)$ be the subclass of $\mathcal{A}$ consisting of all functions f which satisfy

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z) - \alpha\right| < 1, \tag{1.1}$$

where $\alpha \in \mathbb{C}$. There are a lot of interesting studies regarding inequality (1.1) [16,17,24].

The certain analytic functions which is in the class of $\mathcal{K}(\alpha)$ on the unit disc E are considered in this paper. The subject of the present paper is to discuss some properties of the function $f(z)$ which belongs to the class of $\mathcal{K}(\alpha)$ by applying Schwarz lemma. Schwarz lemma is a highly popular topic in electrical engineering. As exemplary applications, the use of positive real functions and boundary analysis of these functions for circuit synthesis can be given. Moreover, it is also possible to

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utilize Schwarz lemma for the analysis of transfer functions in control engineering and to design multi-notch filter structures in signal processing [14, 15].

Let $f \in \mathcal{A}$. The q^{th} Hankel determinant of f for $n \geq 0$ and $q \geq 1$ is stated by Noonan and Thomas [23] as

$$H_q(n) = \begin{vmatrix} c_n & c_{n+1} & \cdots & c_{n+q-1} \\ c_{n+1} & c_{n+2} & \cdots & c_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n+q-1} & c_{n+q} & \cdots & c_{n+2q-2} \end{vmatrix}, \quad c_1 = 1.$$

From the Hankel determinant for $n = 1$ and $q = 2$, we have

$$H_2(1) = \begin{vmatrix} c_1 & c_2 \\ c_2 & c_3 \end{vmatrix} = c_3 - c_2^2.$$

Similarly, for $u = z - z_1$ and $f \in \mathcal{A}$, we have

$$D_s(m) = \begin{vmatrix} a_m & a_{m+1} & \cdots & a_{m+s-1} \\ a_{m+1} & a_{m+2} & \cdots & a_{m+s} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m+s-1} & a_{m+s} & \cdots & a_{m+2s-2} \end{vmatrix}, \quad a_1 = 1.$$

From the Hankel determinant for $m = 1$ and $s = 2$, we have

$$D_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2.$$

Here, the Hankel determinant $H_2(1) = c_3 - c_2^2$ and $D_2(1) = a_3 - a_2^2$ are well-known as Fekete-Szegö functional [22]. In [23], authors have obtained the upper bounds of the Hankel determinant $|c_2c_4 - c_3^2|$. Also, in [20], author have obtained the upper bounds the Hankel determinant $A_n^{(k)}$. Moreover, in [21], authors have given bounds for the Second Hankel determinant for class $\mathcal{M}_\alpha$. In [1], Schwarz lemma at the boundary has been examined for a class $\mathcal{K}$ of analytic functions, and the modulus of the second derivative has been estimated from below in terms of Hankel determinants $H_2(1)$.

We will obtain consideration for $f''(z)$ from below by using $H_2(1)$ and $D_2(1)$ determinants. In this consideration, the coefficients in Taylor expansion of $f(z)$ at $z = 0$ and $z = z_1$ points are used. The functions we use for our main results are as follows. The relationship between the Fekete-Szegö function, that is the Hankel determinant $H_2(1)$, and the second derivative of the function will be considered. In this consideration, the Taylor coefficients that form the analytic function $f(z)$ and the coefficients that form the Hankel determination will be correlated. In this correlation, Schwarz lemma and its results will be used.

Let $f \in \mathcal{K}(\alpha)$ and consider the following function

$$t(z) = \left(\frac{z}{f(z)}\right)^2 f'(z) - \alpha = 1 - \alpha + \left(c_3 - c_2^2\right) z^2 + \left(2c_4 - 4c_2c_3 + 2c_2^3\right) z^3 + \cdots .$$

It is an analytic function in E and $t(0) = 1 - \alpha$. Consider the function

$$T(z) = \frac{R(z)}{\frac{z-z_1}{1-\overline{z_1}z}}, \quad R(z) = \frac{t(z) - t(0)}{1 - \overline{t(0)}t(z)}.$$

Here, $T(z)$ is an analytic function in E, $T(0) = 0$ and $|T(z)| < 1$ for $z \in E$.

Several studies on Schwarz lemma exist in literature as it has a wide applicability area. Some examples are about being estimated from below the modulus of the derivative of the function at some boundary point of the unit disc which is also called as boundary version of Schwarz lemma. The classical Schwarz lemma implies the inequality

$$|f'(z_0)| \geq 1 \tag{1.2}$$

which is known as the Schwarz lemma on the boundary, and also as a part of the Lindelöf principle. The inequality (1.2) and its generalizations have important aplications in geometric theory of functions [2–7, 13–15, 15, 18]. Mercer [10] proves a version of the Schwarz lemma where the images of two points are known. Also, he considers some Schwarz and Carathéodory inequalities at the boundary, as consequences of a lemma due to Rogosinski [11]. In addition, he obtains an new boundary Schwarz lemma, for analytic functions mapping the unit disk to itself [12]. In [9], authors have given simple proofs of various versions of the Schwarz lemma for real-valued harmonic functions and for holomorphic (more generally harmonic quasiregular, shortly HQR) mappings with the strip codomain. In [8], the authors have given different applications of the Schwarz lemma and the Jack lemma.

The following lemma, known as the Julia-Wolff lemma, is needed in the sequel (see [19]). In addition, the second derivative of the function $f(z)$ will be considered from below. Therefore, here the existence of the second derivative gives the result of the Julia-Wolff lemma.

Lemma 1.1 (Julia-Wolff lemma)**.** *Let f be an analytic function in E, $f(0) = 0$ and $f(E) \subset E$. If, in addition, the function f has an angular limit $f(z_0)$ at $z_0 \in \partial E$, $|f(z_0)| = 1$, then the angular derivative $f'(z_0)$ exists and $1 \leq |f'(z_0)| \leq \infty$.*

Corollary 1.1. *The analytic function f has a finite angular derivative $f'(z_0)$ if and only if f' has the finite angular limit $f'(z_0)$ at $z_0 \in \partial E$.*

2. Main Results

In this section, we discuss different versions of the boundary Schwarz lemma and Hankel determinant for $\mathcal{K}(\alpha)$ class. Also, for the function $f(z) = z + c_2z^2 + c_3z^3 + \cdots$ defined in the unit disc such that $f \in \mathcal{K}(\alpha)$, we estimate a modulus of the angular derivative of $f(z)$ function at the boundary point z_0 with $f(z_0) = \frac{z_0}{1+\alpha}$ and $f'(z_0) = \frac{1}{1+\alpha}$. That is, we shall give an estimate below $|f''(z_0)|$ according to the first nonzero Taylor coefficient of about two zeros, namely $z = 0$ and $z_1 \neq 0$. The sharpness of this inequality is also proved. Motivated by the results of the work presented in [2], the following result has been obtained.

Theorem 2.1. *Let $f \in \mathcal{K}(\alpha)$ and $\left(\frac{z_1}{f(z_1)}\right)^2 f'(z_1) = 1$ for $0 < |z_1| < 1$. Suppose that, for some $z_0 \in \partial E$, f has an angular limit $f(z_0)$ at z_0, $f(z_0) = \frac{z_0}{1+\alpha}$ and $f'(z_0) = \frac{1}{1+\alpha}$. Then we have the inequality*

(2.1)
$$|f''(z_0)| \geq \frac{|\alpha|^2}{\left(1-|1-\alpha|^2\right)|1+\alpha|^2}\left(2+2\frac{1-|z_1|^2}{|1-z_1|^2}+\frac{\left(1-|1-\alpha|^2\right)|z_1|^2-|H_2(1)|}{\left(1-|1-\alpha|^2\right)|z_1|^2+|H_2(1)|}\right.$$
$$\left.\times\left[1+\frac{A}{B}\cdot\frac{1-|z_1|^2}{|1-z_1|^2}\right]\right),$$

where
$$A = \left(1-|1-\alpha|^2\right)^2|z_1|^4+|D_2(1)|\left(1-|z_1|^2\right)^2|H_2(1)|-\left(1-|1-\alpha|^2\right)^2|D_2(1)|\,|z_1|$$
$$-\left(1-|1-\alpha|^2\right)|H_2(1)|\,|z_1|,$$
$$B = \left(1-|1-\alpha|^2\right)^2|z_1|^4+|D_2(1)|\left(1-|z_1|^2\right)^2|H_2(1)|+\left(1-|1-\alpha|^2\right)^2|D_2(1)|\,|z_1|$$
$$+\left(1-|1-\alpha|^2\right)|H_2(1)|\,|z_1|.$$

This result is sharp for $\alpha \in \mathbb{R}$, with equality for each possible value of $|H_2(1)|$ and $|D_2(1)|$.

Proof. Let
$$q(z) = \frac{z-z_1}{1-\overline{z_1}z}.$$
In addition, let $h : E \to E$ be an analytic and a point $z_1 \in E$ in order to satisfy
$$\left|\frac{h(z)-h(z_1)}{1-\overline{h(z_1)}h(z)}\right| \leq \left|\frac{z-z_1}{1-\overline{z_1}z}\right| = |q(z)|$$
and
$$|h(z)| \leq \frac{|h(z_1)|+|q(z)|}{1+|h(z_1)|\,|q(z)|}, \tag{2.2}$$
by Schwarz-pick lemma [6]. If $p : E \to E$ is analytic function and $0 < |z_1| < 1$, letting
$$h(z) = \frac{p(z)-p(0)}{z\left(1-\overline{p(0)}p(z)\right)}$$
in (2.2), we obtain
$$\left|\frac{p(z)-p(0)}{z\left(1-\overline{p(0)}p(z)\right)}\right| \leq \frac{\left|\frac{p(z_1)-p(0)}{z_1\left(1-\overline{p(0)}p(z_1)\right)}\right|+|q(z)|}{1+\left|\frac{p(z_1)-p(0)}{z_1\left(1-\overline{p(0)}p(z_1)\right)}\right||q(z)|}$$

and

$$|p(z)| \le \frac{|p(0)| + |z| \frac{|C|+|q(z)|}{1+|C||q(z)|}}{1 + |p(0)|\,|z| \frac{|C|+|q(z)|}{1+|C||q(z)|}}, \tag{2.3}$$

where

$$C = \frac{p(z_1) - p(0)}{z_1 \left(1 - \overline{p(0)} p(z_1)\right)}.$$

Without loss of generality, we will assume that $z_0 = 1$. If we take

$$p(z) = \frac{R(z)}{z^2 \left(\frac{z - z_1}{1 - \overline{z_1} z}\right)^2},$$

then

$$p(0) = \frac{H_2(1)}{\left(1 - |1-\alpha|^2\right) z_1^2}, \quad p(z_1) = \frac{D_2(1)\left(1 - |z_1|^2\right)^2}{\left(1 - |1-\alpha|^2\right) z_1^2}$$

and

$$C = \frac{\frac{D_2(1)\left(1-|z_1|^2\right)^2}{\left(1-|1-\alpha|^2\right)z_1^2} + \frac{H_2(1)}{\left(1-|1-\alpha|^2\right)z_1^2}}{z_1\left(1 + \frac{D_2(1)\left(1-|z_1|^2\right)^2}{\left(1-|1-\alpha|^2\right)z_1^2} \frac{H_2(1)}{\left(1-|1-\alpha|^2\right)z_1^2}\right)},$$

where $|C| \le 1$. Let $|p(0)| = \beta$ and

$$\mathrm{T} = \frac{\left|\frac{D_2(1)\left(1-|z_1|^2\right)^2}{\left(1-|1-\alpha|^2\right)z_1^2}\right| + \left|\frac{H_2(1)}{\left(1-|1-\alpha|^2\right)z_1^2}\right|}{|z_1|\left(1 + \left|\frac{D_2(1)\left(1-|z_1|^2\right)^2}{\left(1-|1-\alpha|^2\right)z_1^2}\right| \left|\frac{H_2(1)}{\left(1-|1-\alpha|^2\right)z_1^2}\right|\right)}.$$

From (2.3), we get

$$|R(z)| \le |z|^2 |q(z)|^2 \frac{\beta + |z| \frac{\mathrm{T}+|q(z)|}{1+\mathrm{T}|q(z)|}}{1 + \beta\,|z| \frac{\mathrm{T}+|q(z)|}{1+\mathrm{T}|q(z)|}}$$

and

$$\frac{1 - |R(z)|}{1 - |z|} \ge \frac{1 + \beta\,|z| \frac{\mathrm{T}+|q(z)|}{1+\mathrm{T}|q(z)|} - \beta\,|z|^2 |q(z)|^2 - |q(z)|^2 |z|^3 \frac{\mathrm{T}+|q(z)|}{1+\mathrm{T}|q(z)|}}{(1 - |z|)\left(1 + \beta\,|z| \frac{\mathrm{T}+|q(z)|}{1+\mathrm{T}|q(z)|}\right)}.$$

Let $\kappa(z) = 1 + \beta\,|z| \frac{\mathrm{T}+|q(z)|}{1+\mathrm{T}|q(z)|}$ and $\tau(z) = 1 + \mathrm{T}\,|q(z)|$. Therefore, we obtain

$$\begin{aligned} \frac{1 - |R(z)|}{1 - |z|} \ge & \frac{1}{\kappa(z)\tau(z)} \left\{ \frac{1 - |z|^3 |q(z)|^3}{1 - |z|} + \mathrm{T}\,|q(z)| \frac{1 - |z|^3 |q(z)|}{1 - |z|} \right. \\ & \left. + \beta\,|z|\,|q(z)| \frac{1 - |z|\,|q(z)|}{1 - |z|} + \beta\,|z|\,\mathrm{T} \frac{1 - |z|\,|q(z)|}{1 - |z|} \right\}. \end{aligned} \tag{2.4}$$

Since

$$\lim_{z\to 1}\kappa(z)=\lim_{z\to 1}\left(1+\beta\left|z\right|\frac{\mathrm{T}+\left|q(z)\right|}{1+\mathrm{T}\left|q(z)\right|}\right)=1+\beta,$$
$$\lim_{z\to 1}\tau(z)=\lim_{z\to 1}\left(1+\mathrm{T}\left|q(z)\right|\right)=1+\mathrm{T},$$
$$\lim_{z\to 1}\frac{1-\left|z\right|^{i}\left|\frac{z-z_1}{1-\overline{z_1}z}\right|^{j}}{1-\left|z\right|}=i+j\frac{1-\left|z_1\right|^2}{\left|1-z_1\right|^2},$$

for nonnegative integers i and j and

$$1-\left|q(z)\right|^2=1-\left|\frac{z-z_1}{1-\overline{z_1}z}\right|^2=\frac{\left(1-\left|z_1\right|^2\right)\left(1-\left|z\right|^2\right)}{\left|1-\overline{z_1}z\right|^2},$$

passing to the angular limit in (2.4) gives

$$\begin{aligned}\left|R'(1)\right|\geq&\frac{2}{(1+\beta)(1+\mathrm{T})}\left(3+3\frac{1-\left|z_1\right|^2}{\left|1-z_1\right|^2}+\mathrm{T}\left[3+3\frac{1-\left|z_1\right|^2}{\left|1-z_1\right|^2}\right]\right.\\&\left.+\beta\left[1+\frac{1-\left|z_1\right|^2}{\left|1-z_1\right|^2}\right]+\beta\mathrm{T}\left[1+3\frac{1-\left|z_1\right|^2}{\left|1-z_1\right|^2}\right]\right)\\=&2+2\frac{1-\left|z_1\right|^2}{\left|1-z_1\right|^2}+\frac{1-\beta}{1+\beta}\left[1+\frac{1-\mathrm{T}}{1+\mathrm{T}}\frac{1-\left|z_1\right|^2}{\left|1-z_1\right|^2}\right].\end{aligned}$$

Moreover, since

$$\frac{1-\beta}{1+\beta}=\frac{1-\left|p(0)\right|}{1+\left|p(0)\right|}=\frac{1-\frac{\left|H_2(1)\right|}{\left(1-\left|1-\alpha\right|^2\right)\left|z_1\right|^2}}{1+\frac{\left|H_2(1)\right|}{\left(1-\left|1-\alpha\right|^2\right)\left|z_1\right|^2}}=\frac{\left(1-\left|1-\alpha\right|^2\right)\left|z_1\right|^2-\left|H_2(1)\right|}{\left(1-\left|1-\alpha\right|^2\right)\left|z_1\right|^2+\left|H_2(1)\right|},$$

$$\frac{1-\mathrm{T}}{1+\mathrm{T}}=\frac{1-\dfrac{\left|\frac{D_2(1)\left(1-\left|z_1\right|^2\right)^2}{\left(1-\left|1-\alpha\right|^2\right)z_1^2}\right|+\left|\frac{H_2(1)}{\left(1-\left|1-\alpha\right|^2\right)z_1^2}\right|}{\left|z_1\right|\left(1+\left|\frac{D_2(1)\left(1-\left|z_1\right|^2\right)^2}{\left(1-\left|1-\alpha\right|^2\right)z_1^2}\right|\left|\frac{H_2(1)}{\left(1-\left|1-\alpha\right|^2\right)z_1^2}\right|\right)}}{1+\dfrac{\left|\frac{D_2(1)\left(1-\left|z_1\right|^2\right)^2}{\left(1-\left|1-\alpha\right|^2\right)z_1^2}\right|+\left|\frac{H_2(1)}{\left(1-\left|1-\alpha\right|^2\right)z_1^2}\right|}{\left|z_1\right|\left(1+\left|\frac{D_2(1)\left(1-\left|z_1\right|^2\right)^2}{\left(1-\left|1-\alpha\right|^2\right)z_1^2}\right|\left|\frac{H_2(1)}{\left(1-\left|1-\alpha\right|^2\right)z_1^2}\right|\right)}}$$

and

$$\frac{1-\mathrm{T}}{1+\mathrm{T}}=\frac{\left|z_1\right|\left(1+\left|\frac{D_2(1)\left(1-\left|z_1\right|^2\right)^2}{\left(1-\left|1-\alpha\right|^2\right)z_1^2}\right|\left|\frac{H_2(1)}{\left(1-\left|1-\alpha\right|^2\right)z_1^2}\right|\right)-\left|\frac{D_2(1)\left(1-\left|z_1\right|^2\right)^2}{\left(1-\left|1-\alpha\right|^2\right)z_1^2}\right|-\left|\frac{H_2(1)}{\left(1-\left|1-\alpha\right|^2\right)z_1^2}\right|}{\left|z_1\right|\left(1+\left|\frac{D_2(1)\left(1-\left|z_1\right|^2\right)^2}{\left(1-\left|1-\alpha\right|^2\right)z_1^2}\right|\left|\frac{H_2(1)}{\left(1-\left|1-\alpha\right|^2\right)z_1^2}\right|\right)+\left|\frac{D_2(1)\left(1-\left|z_1\right|^2\right)^2}{\left(1-\left|1-\alpha\right|^2\right)z_1^2}\right|+\left|\frac{H_2(1)}{\left(1-\left|1-\alpha\right|^2\right)z_1^2}\right|}\cdot\frac{A_1}{B_1},$$

where

$$A_1=\left(1-\left|1-\alpha\right|^2\right)^2\left|z_1\right|^4+\left|D_2(1)\right|\left(1-\left|z_1\right|^2\right)^2\left|H_2(1)\right|-\left(1-\left|1-\alpha\right|^2\right)^2\left|D_2(1)\right|\left|z_1\right|$$

$$- \left(1 - |1-\alpha|^2\right) |H_2(1)| \, |z_1| \, ,$$
$$B_1 = \left(1 - |1-\alpha|^2\right)^2 |z_1|^4 + |D_2(1)| \left(1 - |z_1|^2\right)^2 |H_2(1)| + \left(1 - |1-\alpha|^2\right)^2 |D_2(1)| \, |z_1|$$
$$+ \left(1 - |1-\alpha|^2\right) |H_2(1)| \, |z_1| \, ,$$

we obtain

$$|R'(1)| \geq 2 + 2\frac{1-|z_1|^2}{|1-z_1|^2} + \frac{\left(1 - |1-\alpha|^2\right)|z_1|^2 - |H_2(1)|}{\left(1 - |1-\alpha|^2\right)|z_1|^2 + |H_2(1)|}\left[1 + \frac{A_2}{B_2} \cdot \frac{1-|z_1|^2}{|1-z_1|^2}\right],$$

where

$$A_2 = \left(1 - |1-\alpha|^2\right)^2 |z_1|^4 + |D_2(1)| \left(1 - |z_1|^2\right)^2 |H_2(1)| - \left(1 - |1-\alpha|^2\right)^2 |D_2(1)| \, |z_1|$$
$$- \left(1 - |1-\alpha|^2\right) |H_2(1)| \, |z_1| \, ,$$
$$B_2 = \left(1 - |1-\alpha|^2\right)^2 |z_1|^4 + |D_2(1)| \left(1 - |z_1|^2\right)^2 |H_2(1)| + \left(1 - |1-\alpha|^2\right)^2 |D_2(1)| \, |z_1|$$
$$+ \left(1 - |1-\alpha|^2\right) |H_2(1)| \, |z_1| \, .$$

From definition of $R(z)$, we have

$$R'(z) = \frac{1-|t(0)|^2}{\left(1-\overline{t(0)}t(z)\right)^2} t'(z)$$

and

$$|R'(1)| = \frac{1-|1-\alpha|^2}{|\alpha|^2} \, |f''(1)| \, |1+\alpha|^2 \, .$$

Thus, we obtain the inequality (2.1).

In order to show that the inequality (2.1) is sharp, choose arbitrary real numbers z_1, x and y such that $0 < x < \left(1 - |1-\alpha|^2\right)|z_1|^2$, $0 < y < \frac{\left(1-|1-\alpha|^2\right)|z_1|^2}{\left(1-|z_1|^2\right)^2}$.

Let

$$\mathtt{K} = \frac{\frac{y}{z_1^2}\left(1-|z_1|^2\right)^2 - \frac{x}{z_1^2}}{z_1\left(1 - \left(1-|z_1|^2\right)^2 \frac{y}{z_1^2}\frac{x}{z_1^2}\right)}.$$

Let

$$R(z) = z^2 \left(\frac{z-z_1}{1-\overline{z_1}z}\right)^2 \frac{\frac{x}{z_1^2} + z\frac{\mathtt{K}+\frac{z-z_1}{1-\overline{z_1}z}}{1+\mathtt{K}\frac{z-z_1}{1-\overline{z_1}z}}}{1 + \frac{x}{z_1^2} z \frac{\mathtt{K}+\frac{z-z_1}{1-\overline{z_1}z}}{1+\mathtt{K}\frac{z-z_1}{1-\overline{z_1}z}}}. \tag{2.5}$$

From (2.5), with the simple calculations, we obtain

$$\frac{R''(0)}{2!} = x, \quad \frac{R''(z_1)}{2!} = y$$

and

$$R'(1) = 2 + 2\frac{1-|z_1|^2}{(1-z_1)^2} + \frac{z_1^2 - x}{z_1^2 + x}\left[1 + \frac{z_1^4 - y\left(1-|z_1|^2\right)^2 x - y\left(1-|z_1|^2\right)^2 z_1 + xz_1}{z_1^4 - y\left(1-|z_1|^2\right)^2 x + y\left(1-|z_1|^2\right)^2 z_1 - xz_1}\right].$$

Choosing suitable signs of the numbers z_1, x and y, we conslude from the last equality that the inequality (2.1) is sharp. □

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