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# A NEW EXTENSION OF BANACH-CARISTI THEOREM AND ITS APPLICATION TO NONLINEAR FUNCTIONAL EQUATIONS

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Abstract. In this paper, we present a new extension of Banach-Caristi type theorem for multivalued mappings. We show that our result is not a consequence of multivalued version of Banach contraction principle due to Nadler. We provide an application of our result to the solution of functional equations.

## 1. Preliminaries

Caristi [4] introduced an important generalization of the Banach contraction principle as follows.

**Theorem 1.1** ([4]). *Let* $(\Lambda, \eta)$ *be a complete metric space* (*MS, in short*) *and* $\Im : \Lambda \to \Lambda$ *be a self-map satisfying*

$$\eta(\varsigma, \Im(\varsigma)) \leq \phi(\varsigma) - \phi(\Im(\varsigma)),$$

*for all* $\varsigma \in \Lambda$, *where* $\phi : \Lambda \to [0, \infty)$ *is a lower semicontinous mapping. Then* $\Im$ *admits a fixed point.*

Caristi's theorem has a close connection with Ekeland's variational principle [7, 8]. Weston [20] established a characterization for the metric completeness in terms of Caristi's theorem. Agarwal and Khamsi [1] extended Caristi's result to vector valued metric spaces.

In 1969, Nadler [17] established a number of very significant fixed point results for multivalued maps using the Hausdorff concept, i.e., by considering the distance

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between two arbitrary sets. Khan [12] studied some interesting common fixed points for multivalued maps.

Let $(\Lambda, \eta)$ be a complete MS and let $CB(\Lambda)$ denote the class of all nonempty closed and bounded subsets of $\Lambda$. Then for $\mathcal{A}, \mathcal{B} \in CB(\Lambda)$, define the map $\mathcal{H} : CB(\Lambda) \times CB(\Lambda) \to [0, \infty)$ by

$$\mathcal{H}(\mathcal{A}, \mathcal{B}) = \max \left\{ \sup_{\xi \in \mathcal{B}} \Delta(\xi, \mathcal{A}), \sup_{\delta \in \mathcal{A}} \Delta(\delta, \mathcal{B}) \right\},$$

where $\Delta(\delta, \mathcal{B}) = \inf_{\xi \in \mathcal{B}} \eta(\delta, \xi)$. $\mathcal{H}$ is called the Pompeiu-Hausdorff metric induced by $\eta$.

**Definition 1.1** ([17])**.** $\varsigma \in \Lambda$ is said to be a fixed point of the multivalued map $\Im : \Lambda \to CB(\Lambda)$ if $\varsigma \in \Im(\varsigma)$. The set of all fixed points of $\Im$ is denoted by $Fix(\Im)$.

*Remark* 1.1. In the MS $(CB(\Lambda), \mathcal{H})$, $\varsigma \in \Lambda$, is a fixed point of $\Im$ if and only if $\Delta(\varsigma, \Im(\varsigma)) = 0$.

The following results are important in the present context.

**Lemma 1.1** ([3,5])**.** *Let $(\Lambda, \eta)$ be a MS and $U, V, W \in CB(\Lambda)$. Then*

(a) $\Delta(\mu, V) \leq \eta(\mu, \gamma)$ *for any* $\gamma \in V$ *and* $\mu \in \Lambda$*;*
(b) $\Delta(\mu, V) \leq \mathcal{H}(U, V)$ *for any* $\mu \in U$*;*
(c) $\Delta(\mu, U) \leq [\eta(\mu, \nu) + \Delta(\nu, U)]$ *for all* $\mu, \nu \in \Lambda$*.*

**Lemma 1.2** ([17])**.** *Let $U, V \in CB(\Lambda)$ and let $\varsigma \in U$. Then for any $p > 0$ there exists $\xi \in V$ such that*

$$\eta(\varsigma, \xi) \leq \mathcal{H}(U, V) + p.$$

*However, there may not be a point $\xi \in V$ such that*

$$\eta(\varsigma, \xi) \leq \mathcal{H}(U, V).$$

*If $V$ is compact, then such a point $\xi$ exists, i.e., $\eta(\varsigma, \xi) \leq \mathcal{H}(U, V)$.*

**Lemma 1.3** ([17])**.** *Let $\{U_n\}$ be a sequence in $CB(\Lambda)$ and $\lim_{n\to\infty} \mathcal{H}(U_n, U) = 0$ for some $U \in CB(\Lambda)$. If $\upsilon_n \in U_n$ and $\lim_{n\to\infty} \eta(\upsilon_n, \upsilon) = 0$ for some $\upsilon \in \Lambda$, then $\upsilon \in U$.*

**Lemma 1.4** ([16])**.** *If $\{\varsigma_n\}$ is a sequence in a MS $(\Lambda, \eta)$ such that there exists a constant $\lambda \in [0, 1)$ satisfying*

$$\eta(\varsigma_{n+1}, \varsigma_n) \leq \lambda \eta(\varsigma_n, \varsigma_{n-1}), \quad \text{for all } n \geq 1,$$

*then the sequence $\{\varsigma_n\}$ is Cauchy.*

Caristi type conditions have been applied to multivalued mappings by Jachymski [10], Feng and Liu [9], Latif and Kutbi [15] and many more. Also, generalized Caristi's fixed point theorems have been studied by Latif [14], Suzuki [19], and several others.

Recently, Khojateh et al. [13] gave some applications of Caristi's theorem in MS, whereas Karapinar et al. [11] extended the Banach and Caristi type theorems to $b$-metric spaces. In the present paper, we introduce a new extension of Banach and

Caristi type theorem to a complete MS for multivalued mappings. In Section 2, we present the main result and in Section 3 we provide an application of our result to the solution of a particular type of nonlinear functional equations. For some recent work on the application of multivalued fixed point results to the solution of functional/integral equations, we refer to [2, 6, 18].

## 2. Main Results

In this section, we present our main result which is a new extension of Banach-Caristi theorem.

**Theorem 2.1.** *Let* $(\Lambda, \eta)$ *be a complete MS and* $\Im : \Lambda \to CB(\Lambda)$ *be a multivalued map such that* $\Im(\varsigma)$ *is compact for each* $\varsigma \in \Lambda$. *Suppose that the function* $\phi : \Lambda \to \mathbb{R}$ *satisfies the following conditions:*

(a) $\phi$ *is bounded below* (*i.e.,* $\inf \phi(\varsigma) > -\infty$)*;*

(b) $\Delta(\varsigma, \Im(\varsigma)) > 0$ *implies* $\mathcal{H}(\Im(\varsigma), \Im(\xi)) \leq (\phi(\varsigma) - \phi(\xi))\eta(\varsigma, \xi)$ *for all* $\xi \in \Lambda$.

*Then* $\Im$ *has a fixed point.*

*Proof.* Consider $\varsigma_0 \in \Lambda$ and choose $\varsigma_1 \in \Im(\varsigma_0)$. Since $\Im(\varsigma_1)$ is compact, by Lemma 1.2, we can select $\varsigma_2 \in \Im(\varsigma_1)$ satisfying $\eta(\varsigma_1, \varsigma_2) \leq \mathcal{H}(\Im(\varsigma_0), \Im(\varsigma_1))$. Similarly, we can choose $\varsigma_3 \in \Im(\varsigma_2)$ satisfying $\eta(\varsigma_2, \varsigma_3) \leq \mathcal{H}(\Im(\varsigma_1), \Im(\varsigma_2))$ and so on.

Continuing in this way, we construct a sequence $\{\varsigma_n\}_{n=0}^{\infty}$ satisfying

$$\eta(\varsigma_n, \varsigma_{n+1}) \leq \mathcal{H}(\Im(\varsigma_{n-1}), \Im(\varsigma_n)).$$

We assume that $\varsigma_n \notin \Im(\varsigma_n)$ (i.e., $\Delta(\varsigma_n, \Im(\varsigma_n)) > 0$) for all $n \geq 0$, since otherwise we obtain a fixed point and the proof is completed.

Let $\alpha_n = \eta(\varsigma_{n-1}, \varsigma_n)$. Using condition (b), we have

$$\begin{aligned} \alpha_{n+1} = \eta(\varsigma_n, \varsigma_{n+1}) &\leq \mathcal{H}(\Im(\varsigma_{n-1}), \Im(\varsigma_n)) \\ &\leq (\phi(\varsigma_{n-1}) - \phi(\varsigma_n))\eta(\varsigma_{n-1}, \varsigma_n) \\ &= (\phi(\varsigma_{n-1}) - \phi(\varsigma_n))\alpha_n. \end{aligned} \tag{2.1}$$

So, $0 < \frac{\alpha_{n+1}}{\alpha_n} \leq \phi(\varsigma_{n-1}) - \phi(\varsigma_n)$ for each $n \in \mathbb{N}$.

Thus, the sequence $\{\phi(\varsigma_n)\}$ is positive and non-increasing (i.e., bounded and monotone). Hence, it converges to some $r \geq 0$.

Further, for each $n \in \mathbb{N}$,

$$\begin{aligned} \sum_{k=1}^{n} \frac{\alpha_{k+1}}{\alpha_k} &\leq \sum_{k=1}^{n} (\phi(\varsigma_{k-1}) - \phi(\varsigma_k)) \\ &= (\phi(\varsigma_0) - \phi(\varsigma_1)) + (\phi(\varsigma_1) - \phi(\varsigma_2)) + \cdots + (\phi(\varsigma_{n-1}) - \phi(\varsigma_n)) \\ &= (\phi(\varsigma_0) - \phi(\varsigma_n)) \\ &\to \phi(\varsigma_0) - r \quad \text{as} \quad n \to \infty. \end{aligned}$$

Therefore, $\sum_{n=1}^{\infty} \frac{\alpha_{n+1}}{\alpha_n} < \infty$, which implies that $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 0$. Thus, for $\lambda \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $\frac{\alpha_{n+1}}{\alpha_n} \leq \lambda$ for all $n \geq n_0$. This implies that $\eta(\varsigma_{n+1}, \varsigma_n) \leq$

$\lambda\eta(\varsigma_n, \varsigma_{n-1})$ for all $n \geq n_0$. Using Lemma 1.4, we see that the sequence $\{\varsigma_n\}$ is Cauchy and since $(\Lambda, \eta)$ is complete, $\varsigma_n \to \varsigma$ as $n \to \infty$ for some $\varsigma \in \Lambda$.

We claim that $\varsigma$ is a fixed point of $\Im$. We have

$$\begin{aligned}
\Delta(\varsigma, \Im(\varsigma)) &\leq [\eta(\varsigma, \varsigma_{n+1}) + \Delta(\varsigma_{n+1}, \Im(\varsigma))] \quad \text{(using (c) of Lemma 1.1)}\\
&\leq [\eta(\varsigma, \varsigma_{n+1}) + \mathcal{H}(\Im(\varsigma_n), \Im(\varsigma))] \quad \text{(using (b) of Lemma 1.1)}\\
&\leq [\eta(\varsigma, \varsigma_{n+1}) + (\phi(\varsigma_n) - \phi(\varsigma))\eta(\varsigma_n, \varsigma)] \quad \text{(using (b) of the hypothesis)}\\
&\to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Therefore, $\Delta(\varsigma, \Im(\varsigma)) = 0$, i.e., $\varsigma \in \Im(\varsigma)$. □

*Remark* 2.1. It should be noted that the right hand side of Caristi's condition from Theorem 1.1 does not depend on the distance function, whereas in our condition from Theorem 2.1, the right hand side depends on the distance function. As such, Theorem 2.1 should better be treated as a variant of a Caristi type result instead of a generalization of the same. This is more so, because one can observe that when $\Im$ is a single-valued mapping, our result does not reduce to original Caristi's result. For these reasons our result does not bear a direct connection with the multivalued results studied in [9, 10, 15].

Next, we provide an example to validate Theorem 2.1.

*Example* 2.1. Consider $\Lambda = \{0, 1, 2\}$ and $\eta : \Lambda \times \Lambda \to [0, \infty)$ be defined as $\eta(0, 1) = 1$, $\eta(0, 2) = 2$, $\eta(1, 2) = 1$, $\eta(\varsigma, \varsigma) = 0$ and $\eta(\varsigma, \xi) = \eta(\xi, \varsigma)$ for all $\varsigma, \xi \in \Lambda$. Then $(\Lambda, \eta)$ is a complete MS. Define the multivalued map $\Im : \Lambda \to CB(\Lambda)$ by

$$\Im(\varsigma) = \begin{cases} \{0\}, & \text{if } \varsigma \neq 2,\\ \{0, 2\}, & \text{if } \varsigma = 2. \end{cases}$$

Also, define $\phi : \Lambda \to \mathbb{R}$ by $\phi(0) = 0$, $\phi(1) = 5$ and $\phi(2) = 3$. Clearly, $(\Lambda, \eta)$ is a complete MS and $\Im(\varsigma)$ is compact for each $\varsigma \in \Lambda$. Further, we observe that $\Delta(\varsigma, \Im(\varsigma)) > 0$ for $\varsigma = 1$. Indeed, we have

$$\begin{aligned}
\Delta(0, \Im 0) =&\Delta(0, \{0\}) = 0,\\
\Delta(1, \Im 1) =&\Delta(1, \{0\}) = 1,\\
\Delta(2, \Im 2) =&\Delta(2, \{0, 2\}) = 0.
\end{aligned}$$

Now,

$$\begin{aligned}
\mathcal{H}(\Im 1, \Im 0) =&\mathcal{H}(\{0\}, \{0\}) = 0,\\
\mathcal{H}(\Im 1, \Im 1) =&0,\\
\mathcal{H}(\Im 1, \Im 2) =&\mathcal{H}(\{0\}, \{0, 2\}) = 2.
\end{aligned}$$

Hence, it is easy to see that

$$\begin{aligned}
\mathcal{H}(\Im 1, \Im 0) &\leq (\phi(1) - \phi(0))\eta(1, 0),\\
\mathcal{H}(\Im 1, \Im 1) &\leq (\phi(1) - \phi(0))\eta(1, 1),\\
\mathcal{H}(\Im 1, \Im 2) &\leq (\phi(1) - \phi(0))\eta(1, 2).
\end{aligned}$$

Therefore, all conditions of Theorem 2.1 are satisfied and we see that $\Im$ has a fixed point. Here, $Fix(\Im) = \{0, 2\}$.

*Remark* 2.2. Note that $\mathcal{H}(\Im 1, \Im 2) = 2 > \eta(1, 2) = 1$. Hence, Theorem 2.1 is not a consequence of the multivalued version of the Banach contraction principle due to Nadler [17].

## 3. Application to Functional Equations

Mathematical optimization problems often make use of dynamic programming to obtain the optimal solution. In many such optimal problems, the corresponding dynamical program gets boiled down to solve a functional equation of the form:

$$u(t) = \sup_{s\in V}\{g(t, s) + G(t, s, u(h(t, s)))\}, \quad t \in W, \tag{3.1}$$

where $h : W \times V \to W$, $g : W \times V \to \mathbb{R}$ and $G : W \times V \times \mathbb{R} \to \mathbb{R}$. Let $M$ and $N$ be Banach spaces. $W \subset M$ is called a state space and $V \subset N$ is called a decision space. The process under study is a multistage process. We define the following:

- $B(W) :=$ the collection of all bounded and closed real functions on $W$;
- $\|f\| := \sup_{t\in V} |f(t)|$, $f \in B(W)$.

The metric induced by $\|\cdot\|$ is given by

$$\eta(f_1, f_2) = \sup_{t\in W} |f_1(t) - f_2(t)|, \quad f_1, f_2 \in B(W). \tag{3.2}$$

Then $(B(W), \|\cdot\|)$ is a Banach space. Further, define the operator $\Im : B(W) \to B(W)$ by

$$\Im(f)(t) = \sup_{s\in V} g(t, s) + G(t, s, f(h(t, s))), \tag{3.3}$$

for all $f \in B(W)$ and $t \in W$. To prove an existence result, we need the following theorem.

**Theorem 3.1.** *Let* $\Im : B(W) \to B(W)$ *be defined by* (3.3). *Let* $\Im$ *be upper semi-continuous satisfying:*

(a) *$g$ and $G$ are bounded and continuous;*

(b) *for all $f_1, f_2 \in B(W)$ we have*

$$0 < \eta(f_1, f_2) < 1 \Rightarrow |G(t, s, f_1(t)) - G(t, s, f_2(t))| \le \frac{1}{2}\eta^2(f_1, f_2),$$

$$\eta(f_1, f_2) \ge 1 \Rightarrow |G(t, s, f_1(t)) - G(t, s, f_2(t))| \le \frac{2}{3}\eta^2(f_1, f_2), \tag{3.4}$$

*where $t \in W$ and $s \in V$.*

*Then* (3.1) *has a bounded solution.*

*Proof.* Let $\epsilon > 0$ and $t \in W$. Since $(B(W), \eta)$ is complete for $f_1, f_2 \in B(W)$ and $\epsilon > 0$ there exist $s_1, s_2 \in V$ such that

$$\Im(f_1)(t) \ge g(t, s_1) + G(t, s_1, f_1(h(t, s_1))), \tag{3.5}$$

but

$$\Im(f_1)(t) < g(t, s_1) + G(t, s_1, f_1(h(t, s_1))) + \epsilon \tag{3.6}$$

and

$$\Im(f_2)(t) \geq g(t, s_2) + G(t, s_2, f_2(h(t, s_2))). \tag{3.7}$$

But,

$$\Im(f_2)(t) < g(t, s_2) + G(t, s_2, f_2(h(t, s_2))) + \epsilon. \tag{3.8}$$

Consider the function $\delta : (0, \infty) \to (0, \infty)$ defined by

$$\delta(\varsigma) = \begin{cases} \dfrac{\varsigma^2}{2}, & \text{if } 0 < \varsigma < 1, \\ \dfrac{2}{3}\varsigma, & \text{if } \varsigma \geq 1. \end{cases}$$

Then (3.4) reduces to

$$|G(t, s, f_1(t)) - G(t, s, f_2(t))| \leq \delta(\eta(f_1, f_2)). \tag{3.9}$$

We observe that $\delta(\varsigma) < \varsigma$ for all $\varsigma \in (0, \infty)$. From (3.6), (3.7) and (3.8), we have that

$$\begin{aligned} \Im(f_1)(t) - \Im(f_2)(t) &< |G(t, s_1, f_1(h(t, s_1))) - G(t, s_2, f_2(h(t, s_2)))| + \epsilon \\ &\leq \delta(\eta(f_1, f_2)) + \epsilon. \end{aligned} \tag{3.10}$$

Similarly, we have

$$\Im(f_2)(t) - \Im(f_1)(t) < \delta(\eta(f_1, f_2)) + \epsilon. \tag{3.11}$$

From (3.10) and (3.11)

$$|\Im(f_2)(t) - \Im(f_1)(t)| < \delta(\eta(f_1, f_2)) + \epsilon. \tag{3.12}$$

Thus,

$$\eta(\Im(f_1), \Im(f_2)) < \delta(\eta(f_1, f_2)) + \epsilon. \tag{3.13}$$

Since $\epsilon > 0$ is arbitrary, we obtain

$$\eta(\Im(f_1), \Im(f_2)) \leq \delta(\eta(f_1, f_2)) < \eta(f_1, f_2), \tag{3.14}$$

(since $\delta(\varsigma) < \varsigma$, for each $\varsigma \in (0, \infty)$). Now, define $\phi : B(W) \to \mathbb{R}$ such that $\phi(f) = [\|f\|]^2$, where $f \in B(W)$ and $[\cdot]$ denotes the greatest integer function. Now, all such functions $f_i \in B(W)$ which satisfy $\eta(f_i, (f_j)) > 0$, $i \neq j$, we observe that

$$\eta(\Im(f_1), \Im(f_2)) \leq \delta(\eta(f_1, f_2)) < \eta(f_1, f_2) \leq |\phi(f_1) - \phi(f_2)|\eta(f_1, f_2),$$

(since in this case $|\phi(f_1) - \phi(f_2)| \geq 1$). Thus, we observe that Theorem 2.1 is applicable to the operator $\Im$, so $\Im$ has a fixed point $f^* \in B(W)$, which in turn is a bounded solution of the functional equation (3.1). □

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