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ESTIMATES FOR INITIAL COEFFICIENTS OF CERTAIN SUBCLASSES OF BI-CLOSE-TO-CONVEX ANALYTIC FUNCTIONS

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Abstract. In this paper we find bounds on the modulii of the second, third and fourth Taylor-Maclaurin's coefficients for functions in a subclass of *bi-close-to-convex* analytic functions, which includes the class studied by Srivastava et al. as particular case. Our estimates on the second and third coefficients improve upon earlier bounds. The result on the fourth coefficient is new. Our bounds are obtained by refining well known estimates for the initial coefficients of the Carthéodory functions.

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions $f(z)$ represented by the following *normalized* Taylor-Maclaurin's series:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the *open* unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| \ < 1\}$. A function $f \in \mathcal{A}$ is said to be *univalent* in $\mathbb{U}$ if $f(z)$ is one-to-one in $\mathbb{U}$. As usual, we denote by $\mathcal{S}$ the subclass of functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$. The function $f \in \mathcal{S}$ has a *compositional inverse* f^{-1}, defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \quad (w \in \text{range of } f).$$

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It is well known that for every function $f \in \mathcal{S}$ the compositional inverse function $f^{-1}(w)$ is analytic in some disc $|w| < r_0(f)$, $r_0(f) \geq \frac{1}{4}$. Moreover, $f^{-1}(w)$ has the Taylor-Maclaurin series expansion of the form:

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} b_n w^n \quad (|w| < r_0(f)),$$

where

$$b_n = \frac{(-1)^{n+1}}{n!}|A_{ij}|$$

and $|A_{ij}|$ is the $(n-1)^{\text{th}}$ order determinant whose entries are defined, in terms of the coefficients of $f(z)$, by the following:

$$|A_{ij}| = \begin{cases} [(i-j+1)n + j - 1]a_{i-j+2}, & \text{if } i+1 \geq j, \\ 0, & \text{if } i+1 < j. \end{cases}$$

For initial values of n we, therefore, have:

$$b_2 = -a_2, \quad b_3 = 2a_2^2 - a_3, \quad b_4 = 5a_2a_3 - 5a_2^3 - a_4, \tag{1.2}$$

and so on.

The function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if $f \in \mathcal{S}$ and $f^{-1}(w)$ has univalent *analytic continuation* to the unit disk $\mathbb{U}$. For example, the function

$$f(z) = ze^{-Az}$$

is bi-univalent in $\mathbb{U}$ if $|A| \leq \frac{1}{e}$ [10]. For some more examples see [12, 15, 19]. We denote by σ, the class of analytic bi-univalent functions in $\mathbb{U}$ given by (1.1). Investigation on the class σ was initiated by Lewin [14]. He showed that $|a_2| \leq 1.51$ for every $f \in \sigma$. Subsequently, Brannan and Clunie [3] surmised that $|a_2| \leq \sqrt{2}$. Netanyahu [16] finally proved that $|a_2| \leq \frac{4}{3}$ $(f \in \sigma)$. Later Brannan and Taha [4] introduced and studied new sub-classes of bi-univalent functions (also see Taha [20]). For a detailed history of the developments on the class of functions σ see [2, 13].

In this paper we shall also investigate bi-univalent functions defined on

$$\Delta = \{z \in \mathbb{C} : 1 < |z| < \infty\}.$$

Let Σ denote the class of analytic functions of the form:

$$h(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n} \quad (z \in \Delta), \tag{1.3}$$

which are univalent in Δ. The inverse of a function in Σ is represented by

$$h^{-1}(w) = w + \sum_{n=0}^{\infty} \frac{B_n}{w^n} \quad (M < |w| < \infty, M > 1). \tag{1.4}$$

We say that the function $h \in \Sigma$ is bi-univalent in Δ if $h^{-1}(w)$ has analytic continuation to Δ.

In order to describe certain sub classes of $\mathcal{S}$ and Σ we shall also need the class $\mathcal{P}$ consisting of functions $P(z)$ which are analytic in $\mathbb{U}$, satisfy $|\arg(P(z))| \leq \frac{\pi}{2}$ $(z \in \mathbb{U})$ and $P(0) = 1$. The functions $P(z) \in \mathcal{P}$ are named after Carthéodory.

It is well known that if f in $\mathcal{A}$ is such that $f' \in \mathcal{P}$, then $f \in \mathcal{S}$. In fact, f is *close-to-convex* [7, 10]. We denote the class of these functions by Q. Chichra [6] studied the class of functions Q_λ $(\lambda \geq 0)$ consisting of functions $f \in \mathcal{A}$ and satisfying $(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) \in \mathcal{P}$. We observe that in the particular case $\lambda = 1$, we have $Q_1 := Q$. Chichra [6] further more, proved that

$$Q_{\lambda_1} \subseteq Q_{\lambda_2} \quad (0 \leq \lambda_2 \leq \lambda_1) \quad \text{(also see [8])}.$$

Therefore,

$$Q_\lambda \subseteq Q_1 := Q \subset \mathcal{S} \quad (\lambda \geq 1).$$

Frasin and Aouf [9] introduced the following subclass of *bi-close-to-convex* analytic functions analogous to the subclass Q_λ studied by Chichra [6].

Definition 1.1 (See [9])**.** The function $f(z)$ given by (1.1) is said to be in the class σQ_λ^α $(0 < \alpha \leq 1, \lambda \geq 1)$ if the following conditions are satisfied:

$$f \in \sigma \quad \text{and} \quad \left|\arg\left((1-\lambda)\frac{f(z)}{z} + \lambda f'(z)\right)\right| < \frac{\alpha\pi}{2} \quad (z \in \mathbb{U}) \tag{1.5}$$

and

$$\left|\arg\left((1-\lambda)\frac{g(w)}{w} + \lambda g'(w)\right)\right| < \frac{\alpha\pi}{2} \quad (w \in \mathbb{U}), \tag{1.6}$$

where g is the analytic continuation of f^{-1} to $\mathbb{U}$.

We observe that in the particular case $\lambda = 1$, the class $\sigma Q_1^\alpha := \sigma Q^\alpha$ was earlier studied by Srivastava et al. [19]. More recently, Çağlar et al. [5] introduced a more general class of bi-univalent analytic functions than the class σQ_λ^α (also see Srivastava et al. [1, 18, 21]). However, in this paper we shall restrict our attention to the class σQ_λ^α.

In addition to the class σQ_λ^α, in this paper we shall also study the following subclass of Σ.

Definition 1.2. The function $h(z)$ given by (1.3) is said to be in the class $\Sigma\Theta_\lambda^\alpha$ $(0 < \alpha \leq 1, \lambda \geq 1)$ if $h \in \Sigma$ and the following conditions are satisfied:

$$\left|\arg\left((1-\lambda)\frac{h(z)}{z} + \lambda h'(z)\right)\right| < \frac{\alpha\pi}{2} \quad (z \in \Delta) \tag{1.7}$$

and

$$\left|\arg\left((1-\lambda)\frac{H(w)}{w} + \lambda H'(w)\right)\right| < \frac{\alpha\pi}{2} \quad (w \in \Delta), \tag{1.8}$$

where H is the analytic continuation of h^{-1} to Δ.

In the present paper we develop an elementary method to find new estimates for $|a_2|$ and $|a_3|$ for $f \in \sigma Q_\lambda^\alpha$, which improve upon bounds of Frasin and Aouf [9] and afortiori, the bounds obtained earlier by Srivastava et al. [19]. We also extend a result of Hayami and Owa [12] and find estimate on $|a_4|$ for $f \in \sigma Q_\lambda^\alpha$. Further more, we find estimates on the initial coefficients $|b_0|$, $|b_1|$ and $|b_2|$ for functions in the class $\Sigma\Theta_\lambda^\alpha$. We note that very recently Hamidi et al. [11] studied coefficient estimate problem for a class of functions similar to our class $\Sigma\Theta_\lambda^\alpha$ under the additional restriction that the initial coefficients of the functions are missing. Thus our Theorem 2.2, proved below, on bounds of initial coefficients attempts to bridge this gap and supplements the work in [11]. The methods adopted and developed in this paper are applicable for finding improved coefficient estimates for the several sub-classes of bi-univalent functions studied in [5, 17] and [18].

2. Coefficient Bounds for the Function Classes σQ_λ^α and $\Sigma\Theta_\lambda^\alpha$

In this section we denote by $g(w)$ the analytic continuation of the function $f^{-1}(w)$ to the unit disc $\mathbb{U}$. We state and prove the following.

Theorem 2.1. *Let the function $f(z)$ given by* (1.1)*, be in the class $\sigma Q_\lambda^\alpha\,(0 < \alpha \leq 1$ and $\lambda \geq 1)$. Then*

$$|a_2| \leq \begin{cases} \frac{2\alpha}{\sqrt{2\alpha(1+2\lambda)+(1-\alpha)(1+\lambda)^2}}, & 1 \leq \lambda \leq 1+\sqrt{2}, \\ \frac{2\alpha}{(1+\lambda)}, & \lambda > 1+\sqrt{2}, \end{cases} \tag{2.1}$$

$$|a_3| \leq \frac{2\alpha}{1+2\lambda} \tag{2.2}$$

and

$$|a_4| \leq \frac{2\alpha}{1+3\lambda} \begin{cases} 1 + \frac{2(1-\alpha)(1+\lambda)\left\{6\alpha(1+2\lambda)+(1-2\alpha)(1+\lambda)^2\right\}}{3[2\alpha(1+2\lambda)+(1-\alpha)(1+\lambda)^2]^{\frac{3}{2}}}, & 1 \leq \lambda \leq 1+\sqrt{2},\ 0<\alpha\leq 1, \\ 1 + \frac{2(1-\alpha)\left\{6\alpha(1+2\lambda)+(5-4\alpha)(1+\lambda)^2\right\}}{3(1+\lambda)^2}, & 1+\sqrt{2} < \lambda \leq \lambda_0, 0<\alpha\leq 1 \\ & \text{or } \lambda > \lambda_0,\ 0<\alpha\leq \frac{1}{2}, \\ 1 + \frac{2(1-\alpha)\left\{6\alpha(1+2\lambda)+4(2-\alpha)(1+\lambda)^2\right\}}{3(1+\lambda)^2}, & \lambda > \lambda_0,\ \frac{1}{2} < \alpha \leq 1, \end{cases} \tag{2.3}$$

where λ_0 is the positive root of the quadratic equation

$$2(1-2\alpha)\lambda^2 + 3(1+3\alpha)\lambda + (1+3\alpha) = 0.$$

Proof. Let the function $f(z)$ be a member of the class $\sigma Q_\lambda^\alpha\,(\lambda \geq 1,\ 0<\alpha\leq 1)$. Then by Definition 1.1, we have the following:

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) = [P(z)]^\alpha \tag{2.4}$$

and

$$(1-\lambda)\frac{g(w)}{w}+\lambda f'(w)=[Q(w)]^{\alpha}, \tag{2.5}$$

respectively, where $P(z)$ and $Q(w)$ are members of the Carthéodory class $\mathcal{P}$ and have the forms:

$$P(z)=1+c_1z+c_2z^2+c_3z^3+\cdots \quad (z\in\mathbb{U}) \tag{2.6}$$

and

$$Q(w)=1+l_1w+l_2w^2+l_3w^3+\cdots \quad (w\in\mathbb{U}), \tag{2.7}$$

respectively. Now, equating the coefficients of $(1-\lambda)\frac{f(z)}{z}+\lambda f'(z)$ with the coefficients of $[P(z)]^{\alpha}$, we get

$$(1+\lambda)a_2=\alpha c_1 \quad \text{or} \quad a_2=\frac{\alpha}{1+\lambda}c_1, \tag{2.8}$$

$$(1+2\lambda)a_3=\alpha c_2+\frac{\alpha(\alpha-1)}{2}c_1^2, \tag{2.9}$$

$$(1+3\lambda)a_4=\alpha c_3+\alpha(\alpha-1)c_1c_2+\frac{\alpha(\alpha-1)(\alpha-2)}{6}c_1^3. \tag{2.10}$$

Similarly, a comparison of coefficients of both sides of (2.5) yields:

$$(1+\lambda)a_2=-\alpha l_1, \tag{2.11}$$

$$(1+2\lambda)(2a_2^2-a_3)=\alpha l_2+\frac{\alpha(\alpha-1)}{2}l_1^2 \tag{2.12}$$

and

$$-(1+3\lambda)(5a_2^3-5a_2a_3+a_4)=\alpha l_3+\alpha(\alpha-1)l_1l_2+\frac{\alpha(\alpha-1)(\alpha-2)}{6}l_1^3. \tag{2.13}$$

In order to find improved estimates on $|a_2|$ and $|a_3|$, we first establish certain relations involving c_1, l_1, c_2 and l_2. To this end we observe that (2.8) and (2.11), together give

$$c_1=-l_1. \tag{2.14}$$

We add (2.9) with (2.12), then use the relation $c_1=-l_1$ and get the following:

$$2(1+2\lambda)a_2^2=\alpha\,(c_2+l_2)+\alpha(\alpha-1)c_1^2.$$

Putting $a_2=\frac{\alpha}{(1+\lambda)}c_1$ from (2.8) we have after simplification:

$$c_1^2=\frac{(1+\lambda)^2}{2\alpha(1+2\lambda)+(1-\alpha)(1+\lambda)^2}(c_2+l_2). \tag{2.15}$$

The relation (2.15) also gives the following refined estimates:

$$|c_1|\le\frac{2\,(1+\lambda)}{\sqrt{2\alpha(1+2\lambda)+(1-\alpha)(1+\lambda)^2}} \quad (1\le\lambda\le1+\sqrt{2}) \tag{2.16}$$

and

$$|c_2 + l_2| \leq \frac{4\ [2\alpha(1+2\lambda) + (1-\alpha)(1+\lambda)^2]}{(1+\lambda)^2} \quad (\lambda \geq 1+\sqrt{2}). \tag{2.17}$$

Now using the estimates (2.16) for the range $1 \leq \lambda \leq 1+\sqrt{2}$ and $|c_1| \leq 2$ for the range $\lambda > 1+\sqrt{2}$ in the expression (2.8) we get:

$$|a_2| \leq \begin{cases} \frac{2\alpha}{\sqrt{2\alpha(1+2\lambda)+(1-\alpha)(1+\lambda)^2}}, & 1 \leq \lambda \leq 1+\sqrt{2}, \\ \frac{2\alpha}{(1+\lambda)}, & \lambda > 1+\sqrt{2}. \end{cases}$$

We thus get the claimed bound of (2.1).

We now express a_3 in terms of the coefficients of the functions $P(z)$ and $Q(w)$. For this, we subtract (2.12) from (2.9) and get

$$2(1+2\lambda)(a_3 - a_2^2) = \alpha(c_2 - l_2) + \frac{\alpha(\alpha-1)}{2}(c_1^2 - l_1^2).$$

The relation $c_1^2 = l_1^2$ from (2.14), reduces the above expression to

$$a_3 = a_2^2 + \frac{\alpha}{2(1+2\lambda)}(c_2 - l_2). \tag{2.18}$$

Next putting $a_2 = \frac{\alpha}{1+\lambda}c_1$ and then using (2.15) for c_1^2, we obtain

$$\begin{aligned} a_3 =& \frac{\alpha^2}{(1+\lambda)^2}c_1^2 + \frac{\alpha}{2(1+2\lambda)}(c_2 - l_2), \\ =& \frac{\alpha^2}{2\alpha(1+2\lambda) + (1-\alpha)(1+\lambda)^2}(c_2 + l_2) + \frac{\alpha}{2(1+2\lambda)}(c_2 - l_2), \\ =& \alpha\left(\frac{\alpha}{2\alpha(1+2\lambda) + (1-\alpha)(1+\lambda)^2} + \frac{1}{2(1+2\lambda)}\right)c_2 \\ &+ \alpha\left(\frac{\alpha}{2\alpha(1+2\lambda) + (1-\alpha)(1+\lambda)^2} - \frac{1}{2(1+2\lambda)}\right)l_2. \end{aligned}$$

Since

$$\begin{aligned} &\frac{\alpha}{2\alpha(1+2\lambda) + (1-\alpha)(1+\lambda)^2} - \frac{1}{2(1+2\lambda)} \\ =& \frac{-(1-\alpha)(1+\lambda)^2}{2(1+2\lambda)(2\alpha(1+2\lambda) + (1-\alpha)(1+\lambda)^2)} < 0, \end{aligned}$$

an application of triangle inequality gives the following

$$\begin{aligned} |a_3| \leq& \alpha\left(\frac{\alpha}{2\alpha(1+2\lambda) + (1-\alpha)(1+\lambda)^2} + \frac{1}{2(1+2\lambda)}\right)|c_2| \\ &+ \alpha\left(\frac{1}{2(1+2\lambda)} - \frac{\alpha}{2\alpha(1+2\lambda) + (1-\alpha)(1+\lambda)^2}\right)|l_2|. \end{aligned}$$

Therefore, the well known estimates $|c_2| \le 2$ and $|l_2| \le 2$ (cf. [4]), give the following:

$$|a_3| \le \frac{2\alpha}{(1+2\lambda)}. \tag{2.19}$$

This is precisely our assertion at (2.2).

We next derive a relation between $c_1(c_2 - l_2)$ and $c_3 + l_3$ for our future use. For this purpose we add (2.13) and (2.10). After simplification we get the following:

$$-(1+3\lambda)(5a_2^3 - 5a_2a_3) = \alpha(c_3 + l_3) + \alpha(\alpha-1)c_1(c_2 - l_2). \tag{2.20}$$

By substituting $a_3 = a_2^2 + \frac{\alpha}{2(1+2\lambda)}(c_2 - l_2)$ from (2.18) and $a_2 = \frac{\alpha}{1+\lambda}c_1$ in the above equation (2.20) we have

$$c_1(c_2 - l_2) = \mu_0(c_3 + l_3), \tag{2.21}$$

where

$$\mu_0 = \frac{2(1+\lambda)(1+2\lambda)}{5\alpha(1+3\lambda) + 2(1-\alpha)(1+\lambda)(1+2\lambda)}.$$

We observe that $0 < \mu_0 \le 2$ for every $\lambda \ge 1$ if $0 \le \alpha \le \frac{1}{2}$. However, if $\frac{1}{2} \le \alpha < 1$, then $0 < \mu_0 \le 2$ for $1 < \lambda \le \lambda_0$, where λ_0 is the positive root of the quadratic equation

$$2(1-2\alpha)\lambda^2 + 3(1+3\alpha)\lambda + (1+3\alpha) = 0.$$

Moreover, $\lambda_0 > \frac{97}{16}$.

We are now ready to find a bound for $|a_4|$. As in our estimate for $|a_3|$ in this case also we shall express a_4 in terms of the first three coefficients of $P(z)$ and $Q(w)$. For this purpose we subtract (2.13) from (2.10) and get

$$\begin{aligned}2(1+3\lambda)a_4 = &-(1+3\lambda)(5a_2^3 - 5a_2a_3) + \alpha(c_3 - l_3) + \alpha(\alpha-1)(c_1c_2 - l_1l_2)\\ &+ \frac{\alpha(\alpha-1)(\alpha-2)}{6}(c_1^3 - l_1^3).\end{aligned}$$

The relation $c_1 = -l_1$ reduces the above expression to

$$\begin{aligned}2(1+3\lambda)a_4 = &-(1+3\lambda)(5a_2^3 - 5a_2a_3)\\ &+ \alpha(c_3 - l_3) + \alpha(\alpha-1)c_1(c_2 + l_2) + \frac{\alpha(\alpha-1)(\alpha-2)}{3}c_1^3.\end{aligned} \tag{2.22}$$

In (2.22) we replace $-(1+3\lambda)(5a_2^3 - 5a_2a_3)$ by the expression on the right hand side of the equality of (2.20) and use the relation $c_1 = -l_1$. This gives on simplification the following:

$$2(1+3\lambda)a_4 = 2\alpha c_3 + \alpha(\alpha-1)c_1(c_2 - l_2) + \alpha(\alpha-1)c_1(c_2 + l_2) + \frac{\alpha(\alpha-1)(\alpha-2)}{3}c_1^3. \tag{2.23}$$

First suppose that λ and α are constrained by the requirement $1 \le \lambda \le \lambda_0$ and $0 < \alpha \le 1$ or $\lambda > \lambda_0$ and $0 < \alpha \le \frac{1}{2}$. Then in the equation (2.23) replacing $c_1(c_2 - l_2)$ by

$\mu_0(c_3+l_3)$ from (2.21) we get:

$$\begin{aligned}2(1+3\lambda)a_4 =& 2\alpha c_3+\alpha(\alpha-1)\frac{2(1+\lambda)(1+2\lambda)}{5\alpha(1+3\lambda)+2(1-\alpha)(1+\lambda)(1+2\lambda)}(c_3+l_3)\\ &+\alpha(\alpha-1)c_1\ (c_2+l_2)+\frac{\alpha(\alpha-1)(\alpha-2)}{3}c_1^3\\ =&\frac{10\alpha^2(1+3\lambda)+2\alpha(1-\alpha)(1+\lambda)(1+2\lambda)}{5\alpha(1+3\lambda)+2(1-\alpha)(1+\lambda)(1+2\lambda)}c_3\\ &-\frac{2\alpha(1-\alpha)(1+\lambda)(1+2\lambda)}{5\alpha(1+3\lambda)+2(1-\alpha)(1+\lambda)(1+2\lambda)}l_3\\ &-\alpha(1-\alpha)c_1(c_2+l_2)+\frac{\alpha(\alpha-1)(\alpha-2)}{3}c_1c_1^2.\end{aligned}\tag{2.24}$$

Suppose that we furthermore restrict λ in the range $1\leq\lambda\leq 1+\sqrt{2}<\lambda_0,\ 0<\alpha\leq 1$. Then in (2.24) we substitute the expression in the right hand side of the equality of (2.15) in place of c_1^2 and get

$$\begin{aligned}2(1+3\lambda)a_4 =&\frac{10\alpha^2(1+3\lambda)+2\alpha(1-\alpha)(1+\lambda)(1+2\lambda)}{5\alpha(1+3\lambda)+2(1-\alpha)(1+\lambda)(1+2\lambda)}c_3\\ &-\frac{2\alpha(1-\alpha)(1+\lambda)(1+2\lambda)}{5\alpha(1+3\lambda)+2(1-\alpha)(1+\lambda)(1+2\lambda)}l_3-\alpha(1-\alpha)c_1(c_2+l_2)\\ &+\frac{\alpha(1-\alpha)(2-\alpha)}{3}\frac{(1+\lambda)^2}{2\alpha(1+2\lambda)+(1-\alpha)(1+\lambda)^2}c_1(c_2+l_2)\\ =&\frac{10\alpha^2(1+3\lambda)+2\alpha(1-\alpha)(1+\lambda)(1+2\lambda)}{5\alpha(1+3\lambda)+2(1-\alpha)(1+\lambda)(1+2\lambda)}c_3\\ &-\frac{2\alpha(1-\alpha)(1+\lambda)(1+2\lambda)}{5\alpha(1+3\lambda)+2(1-\alpha)(1+\lambda)(1+2\lambda)}l_3\\ &-\alpha(1-\alpha)\left[\frac{\{6\alpha(1+2\lambda)+(1-2\alpha)(1+\lambda)^2\}}{3[2\alpha(1+2\lambda)+(1-\alpha)(1+\lambda)^2]}\right]c_1(c_2+l_2).\end{aligned}\tag{2.25}$$

Now, we apply the triangle inequality in (2.25) and get the following:

$$\begin{aligned}2(1+3\lambda)|a_4| \leq&\frac{10\alpha^2(1+3\lambda)+2\alpha(1-\alpha)(1+\lambda)(1+2\lambda)}{5\alpha(1+3\lambda)+2(1-\alpha)(1+\lambda)(1+2\lambda)}|c_3|\\ &+\frac{2\alpha(1-\alpha)(1+\lambda)(1+2\lambda)}{5\alpha(1+3\lambda)+2(1-\alpha)(1+\lambda)(1+2\lambda)}|l_3|\\ &+\alpha(1-\alpha)\left[\frac{\{6\alpha(1+2\lambda)+(1-2\alpha)(1+\lambda)^2\}}{3[2\alpha(1+2\lambda)+(1-\alpha)(1+\lambda)^2]}\right]|c_1(c_2+l_2)|.\end{aligned}$$

Note that we made use of the fact that if $1\leq\lambda\leq 1+\sqrt{2}$ and $0<\alpha\leq 1$ then

$$6\alpha(1+2\lambda)+(1-2\alpha)(1+\lambda)^2>0.$$

The well known estimates $|c_n| \leq 2$, $|l_n| \leq 2\,(n = 2,3)$, and the refined bound (2.16) for $|c_1|$ yields the following:

$$\begin{aligned}2(1+3\lambda)|a_4| \leq & \frac{10\alpha^2(1+3\lambda)+2\alpha(1-\alpha)(1+\lambda)(1+2\lambda)}{5\alpha(1+3\lambda)+2(1-\alpha)(1+\lambda)(1+2\lambda)}2\\ & + \frac{2\alpha(1-\alpha)(1+\lambda)(1+2\lambda)}{5\alpha(1+3\lambda)+2(1-\alpha)(1+\lambda)(1+2\lambda)}2\\ & + \alpha(1-\alpha)\left[\frac{\{6\alpha(1+2\lambda)+(1-2\alpha)(1+\lambda)^2\}}{3[2\alpha(1+2\lambda)+(1-\alpha)(1+\lambda)^2]}\right]\\ & \times \frac{2(1+\lambda)}{\sqrt{[2\alpha(1+2\lambda)+(1-\alpha)(1+\lambda)^2]}}4\end{aligned}$$

or

$$\begin{aligned}|a_4| \leq & \frac{2\alpha}{1+3\lambda}\left(1+\frac{2(1-\alpha)(1+\lambda)\{6\alpha(1+2\lambda)+(1-2\alpha)(1+\lambda)^2\}}{3[2\alpha(1+2\lambda)+(1-\alpha)(1+\lambda)^2]^{\frac{3}{2}}}\right)\\ & (1 \leq \lambda \leq 1+\sqrt{2}, 0 < \alpha \leq 1).\end{aligned}$$

We get the first bound of (2.3).

Next, suppose that $1+\sqrt{2} < \lambda \leq \lambda_0$ and $0 < \alpha \leq 1$ or $\lambda > \lambda_0$ and $0 < \alpha \leq \frac{1}{2}$. We apply the triangle inequality in (2.24) and get

$$\begin{aligned}2(1+3\lambda)|a_4| \leq & \frac{10\alpha^2(1+3\lambda)+2\alpha(1-\alpha)(1+\lambda)(1+2\lambda)}{5\alpha(1+3\lambda)+2(1-\alpha)(1+\lambda)(1+2\lambda)}|c_3|\\ & + \frac{2\alpha(1-\alpha)(1+\lambda)(1+2\lambda)}{5\alpha(1+3\lambda)+2(1-\alpha)(1+\lambda)(1+2\lambda)}|l_3|\\ & + \alpha(1-\alpha)|c_1(c_2+l_2)| + \frac{\alpha(\alpha-1)(\alpha-2)}{3}|c_1^3|.\end{aligned}$$

The estimates $|c_n| \leq 2\,(n = 1,3)$, $|l_3| \leq 2$ together with the estimate (2.17) for $|c_2+l_2|$ yields the following:

$$\begin{aligned}|a_4| \leq & \frac{2\alpha}{(1+3\lambda)}\left(1+\frac{2(1-\alpha)\{6\alpha(1+2\lambda)+(5-4\alpha)(1+\lambda)^2\}}{3(1+\lambda)^2}\right)\\ & \left(1+\sqrt{2} \leq \lambda \leq \lambda_0,\ 0 \leq \alpha < 1 \,\text{or}\, \lambda > \lambda_0,\ 0 \leq \alpha \leq \frac{1}{2}\right).\end{aligned}$$

We get the second estimate in (2.3).

Lastly, if $\lambda > \lambda_0$ and $\frac{1}{2} < \alpha < 1$, then we apply the triangle inequality in (2.23) and get

$$2(1+3\lambda)|a_4| \leq 2\alpha|c_3| + \alpha(1-\alpha)|c_1||(c_2-l_2)| + \alpha(1-\alpha)|c_1||(c_2+l_2)| + \frac{\alpha(1-\alpha)(2-\alpha)}{3}|c_1^3|.$$

By using the well bounds $|c_n| \leq 2\,(n = 1, 2, 3)$, $|l_2| \leq 2$ and the refined estimate (2.17) for $|c_2 + l_2|$ we have

$$2(1+3\lambda)|a_4| \leq 4\alpha + 8\alpha(1-\alpha) + \frac{8\alpha(1-\alpha)[2\alpha(1+2\lambda)+(1-\alpha)(1+\lambda)^2]}{(1+\lambda^2} + \frac{8\alpha(1-\alpha)(2-\alpha)}{3}$$

or

$$|a_4| \leq \frac{2\alpha}{(1+3\lambda)}\left(1 + \frac{2(1-\alpha)\left\{6\alpha(1+2\lambda)+4(2-\alpha)(1+\lambda)^2\right\}}{3(1+\lambda)^2}\right) \left(\lambda > \lambda_0,\, \frac{1}{2} < \alpha < 1\right).$$

This is precisely the third estimate in (2.3). Thus, the proof of Theorem 2.1 is completed. □

Theorem 2.2. *Let the function $h(z)$, given by (1.3), be in the class $\Sigma\Theta_\lambda^\alpha\,(\lambda \geq 1,\, 0 < \alpha \leq 1)$. Then*

$$|b_1| \leq \frac{2\alpha}{2\lambda - 1} \tag{2.26}$$

and

(2.27)

$$|b_2| \leq \begin{cases} \frac{2\alpha}{3\lambda-1}\left(1 + \frac{2(1-\alpha)|1-2\alpha|}{3}\right) & 1 \leq \lambda \leq \lambda_1,\, 0 < \alpha \leq 1 \,or\, \lambda > \lambda_1,\, 0 < \alpha \leq \frac{1}{2}, \\ \frac{2\alpha}{3\lambda-1}\left(1 + \frac{4(1-\alpha)(1+\alpha)}{3}\right) & \lambda > \lambda_1,\, \frac{1}{2} < \alpha \leq 1, \end{cases}$$

where λ_1 is the largest positive root of the quadratic equation

$$2(1-2\alpha)\lambda^2 + 3(3\alpha-1)\lambda + 1 - 3\alpha = 0.$$

Proof. We adopt and suitably modify the lines of proof of Theorem 2.1 here. Therefore, we choose to omit the details and sketch only the main steps. In this case we have the following:

$$(1-\lambda)\frac{h(z)}{z} + \lambda h'(z) = [P(z)]^\alpha \tag{2.28}$$

and

$$(1-\lambda)\frac{H(w)}{w} + \lambda H'(w) = [Q(w)]^\alpha, \tag{2.29}$$

respectively, where

$$P(z) = 1 + \frac{c_1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \cdots \quad (z \in \Delta) \tag{2.30}$$

and

$$Q(w) = 1 + \frac{l_1}{w} + \frac{l_2}{w^2} + \frac{l_3}{w^3} + \cdots \quad (w \in \Delta) \tag{2.31}$$

are functions with positive real part in Δ. By comparing coefficients we get:

$$(1-\lambda)b_0 = \alpha c_1, \tag{2.32}$$

$$(1-2\lambda)b_1 = \alpha c_2 + \frac{1}{2}\alpha(\alpha-1)c_1^2, \tag{2.33}$$

$$(1-3\lambda)b_2 = \alpha c_3 + \alpha(\alpha-1)c_1c_2 + \frac{1}{6}\alpha(\alpha-1)(\alpha-2)c_1^3, \tag{2.34}$$

$$(1-\lambda)b_0 = -\alpha l_1, \tag{2.35}$$

$$(1-2\lambda)b_1 = -\alpha l_2 - \frac{1}{2}\alpha(\alpha-1)l_1^2 \tag{2.36}$$

and

$$-(1-3\lambda)(b_2+b_0b_1) = \alpha l_3 + \alpha(\alpha-1)l_1l_2 + \frac{1}{6}\alpha(\alpha-1)(\alpha-2)l_1^3. \tag{2.37}$$

The equations (2.32) and (2.35) give the following relation between c_1 and l_1:

$$c_1 = -l_1. \tag{2.38}$$

Similarly, the equations (2.33) and (2.36) provide the following relation among c_1, c_2 and l_2

$$c_2 + l_2 = (1-\alpha)c_1^2. \tag{2.39}$$

We add (2.36) and (2.33) which yields, after simplification, the following:

$$2(1-2\lambda)b_1 = \alpha(c_2 - l_2). \tag{2.40}$$

By applying the triangle inequality and using the well known estimates $|c_2| \leq 2$ and $|l_2| \leq 2$ we obtain

$$|b_1| \leq \frac{2\alpha}{2\lambda - 1}. \tag{2.41}$$

This is precisely our assertion at (2.26).

In order to find a bound for $|b_2|$ we subtract (2.37) from (2.34) and after simplification get

$$2(1-3\lambda)b_2 = -(1-3\lambda)b_0b_1 + \alpha(c_3-l_3) + \alpha(\alpha-1)c_1(c_2+l_2) + \frac{1}{3}\alpha(\alpha-1)(\alpha-2)c_1^3. \tag{2.42}$$

Similarly addition of (2.37) and (2.34) yields:

$$-(1-3\lambda)b_0b_1 = \alpha(c_3+l_3) + \alpha(\alpha-1)c_1(c_2-l_2). \tag{2.43}$$

By substituting $b_1 = \frac{\alpha(c_2-l_2)}{2(1-2\lambda)}$ from (2.40) and $b_0 = \frac{\alpha c_1}{1-\lambda}$ in the above equation (2.43) we have

$$c_1(c_2 - l_2) = \mu_1(c_3 + l_3), \tag{2.44}$$

where

$$\mu_1 = \frac{2(\lambda-1)(2\lambda-1)}{(3\lambda-1)\alpha + 2(1-\alpha)(\lambda-1)(2\lambda-1)}.$$

We notice that $0 < \mu_1 \leq 2$ for every $\lambda \geq 1$ if $0 < \alpha \leq \frac{1}{2}$. However, if $\frac{1}{2} < \alpha \leq 1$, then $0 < \mu_1 \leq 2$ for $1 \leq \lambda \leq \lambda_1$, where λ_1 is the largest positive root of the quadratic equation

$$2(1-2\alpha)\lambda^2 + 3(3\alpha-1)\lambda + 1 - 3\alpha = 0.$$

In (2.42) we replace $-(1-3\lambda)b_0b_1$ by the expression on the right hand side of the equality (2.43) and use the relation $c_1 = -l_1$. This gives on simplification the following:

$$2(1-3\lambda)b_2 = 2\alpha c_3 + \alpha(\alpha-1)c_1(c_2-l_2) + \alpha(\alpha-1)c_1(c_2+l_2) + \frac{1}{3}\alpha(\alpha-1)(\alpha-2)c_1^3.$$

By replacing $c_2 + l_2$ by $(1-\alpha)c_1^2$ from the relation (2.39) we obtain

$$2(1-3\lambda)b_2 = 2\alpha c_3 + \alpha(\alpha-1)c_1(c_2-l_2) + \frac{1}{3}\alpha(\alpha-1)(1-2\alpha)c_1^3. \tag{2.45}$$

We first suppose that λ and α are constrained by the requirement that $1 \leq \lambda \leq \lambda_1$ and $0 < \alpha \leq 1$ or $\lambda > \lambda_1$ and $0 < \alpha \leq \frac{1}{2}$. Now, we replace $c_1(c_2 - l_2)$ by $\mu_1(c_3 + l_3)$ from (2.44) and get:

$$\begin{aligned}
2(1-3\lambda)b_2 =& 2\alpha c_3 - \frac{2\alpha(1-\alpha)(\lambda-1)(2\lambda-1)}{(3\lambda-1)\alpha + 2(1-\alpha)(\lambda-1)(2\lambda-1)}(c_3+l_3) \\
&+ \frac{1}{3}\alpha(\alpha-1)(1-2\alpha)c_1^3 \\
=& \frac{2\alpha^2(3\lambda-1) + 2\alpha(1-\alpha)(\lambda-1)(2\lambda-1)}{(3\lambda-1)\alpha + 2(1-\alpha)(\lambda-1)(2\lambda-1)}c_3 \\
&- \frac{2\alpha(1-\alpha)(\lambda-1)(2\lambda-1)}{(3\lambda-1)\alpha + 2(1-\alpha)(\lambda-1)(2\lambda-1)}l_3 + \frac{\alpha(1-\alpha)(2\alpha-1)}{3}c_1^3.
\end{aligned}$$

By applying the triangle inequality together with the estimates $|c_n| \leq 1\,(n = 1, 3)$, $|l_3| \leq 2$ we have after simplification the following:

$$\begin{aligned}
|b_2| \leq& \frac{2\alpha}{3\lambda-1}\left(1 + \frac{2(1-\alpha)|1-2\alpha|}{3}\right) \\
&\left(1 \leq \lambda \leq \lambda_1 \text{ and } 0 < \alpha \leq 1 \text{ or } \lambda > \lambda_1 \text{ and } 0 < \alpha \leq \frac{1}{2}\right).
\end{aligned}$$

We get the first estimate in (2.27). Lastly, suppose that $\lambda > \lambda_1$ and $\frac{1}{2} < \alpha \leq 1$. We apply the triangle inequality and the familiar estimates $|c_n| \leq 2\,(n = 1, 2, 3)$ in (2.45) and get

$$|b_2| \leq \frac{2\alpha}{3\lambda-1}\left(1 + \frac{4(1-\alpha)(1+\alpha)}{3}\right) \quad \left(\lambda > \lambda_1,\, \frac{1}{2} < \alpha \leq 1\right).$$

This is precisely the second estimate in (2.27). The proof Theorem 2.2 is thus completed. □

3. Concluding Remarks

By Definition 1.1, to each function $f \in \sigma Q_\lambda^\alpha$ we associate a function of the Carthéodory class which is of the the form:

$$P(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \quad (z \in \mathbb{U}).$$

Similarly, its compositional inverse function g is also associated with a function $Q(z)$ of the Carthéodory class which is given by:

$$Q(z) = 1 + l_1 z + l_2 z^2 + l_3 z^3 + \cdots \quad (z \in \mathbb{U}).$$

The correspondence in both cases is one-to-one. In the present paper we derived suitable relationships between c_1 and l_1, and also among c_1, c_2, c_3 and l_3. These relations yielded refined bounds on $|c_1|$, $|c_1(c_2 - l_2)|$, $|c_2 + l_2|$ and $|c_3 + l_3|$, in suitable ranges of α and λ. Using the refined bounds we found estimates on $|a_3|$ and $|a_4|$ for functions in the class σQ_λ^α. We suitably adopted and amended the lines of proof of our Theorem 2.1 and found estimates on $|b_1|$ and $|b_2|$ for functions in the class $\Sigma\Theta_\lambda^\alpha$.

Recently Hayami and Owa [12] found bounds on $|a_4|$ and improved upon the bounds of Srivastava et al. [19] for $|a_3|$ for the class σQ^α. Thus, we have

(3.1)
$$|a_3| \leq \frac{2\alpha}{3}, \quad |a_4| \leq \frac{\alpha}{2}\left(1 + \frac{2(1-\alpha)(2+5\alpha)}{3(2+\alpha)}\sqrt{\frac{2}{2+\alpha}}\right) \quad (f \in \sigma Q^\alpha,\ 0 < \alpha \leq 1).$$

Also Frasin and Aouf [9] extended the work of Srivastava et al. [19] as follows:

(3.2)
$$|a_2| \leq \frac{2\alpha}{\sqrt{2\alpha(1+2\lambda) + (1-\alpha)(1+\lambda)^2}}, \quad |a_3| \leq \frac{2\alpha}{1+2\lambda} + \frac{4\alpha^2}{(1+\lambda)^2} \quad (f \in \sigma Q_\lambda^\alpha).$$

A comparison of (2.1) and (2.2) with (3.2) shows that our estimates on $|a_2|$ and $|a_3|$, for the class σQ_λ^α found in Theorem 2.1, improve upon the earlier bound obtained by Frasin and Aouf [9]. Also taking $\lambda = 1$ in Theorem 2.1 we get the estimates of Hayami and Owa [12] mentioned at (3.1).

In a recent paper Hamidi et al. [11] found bounds for functions in a class closely related to the function class $\Sigma\Theta_\lambda^\alpha$ studied in this paper, but under the restriction that initial coefficients are missing. Our work in Theorem 2.2 on coefficient bounds for initial coefficients supplements the results in [11].

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