Kragujevac Journal of Mathematics
Volume 47(2) (2023), Pages 309–329.

ON THE SIMPLICIAL COMPLEXES ASSOCIATED TO THE CYCLOTOMIC POLYNOMIAL

ALEKSANDRA KOSTIĆ[1]

Abstract. Musiker and Reiner in [9] studied coefficients of cyclotomic polynomial in terms of topology of associated simplicial complexes. They determined homotopy type of associated complexes for all cyclotomic polynomials, except for cyclotomic polynomials whose degree is a product of three prime numbers. Using discrete Morse theory for simplicial complexes we partially answer a question posed by the two authors regarding homotopy type of the associated complexes when degree of the cyclotomic polynomial is a product of three prime numbers.

1. Introduction

Cyclotomic polynomials are an important type of polynomials in algebraic number theory, Galois theory and geometry. If n is a positive integer, then the n^{th} cyclotomic polynomial is defined as the unique monic, irreducible polynomial having all n^{th} primitive roots of unity as its zeros. It has degree given by Euler phi function $\phi(n)$, with formula

$$\Phi_n(x) = \prod_{j \in (\mathbb{Z}/n\mathbb{Z})^\times} (x - \xi^j),$$

where ξ is the n^{th} root of unity in $\mathbb{C}$. Additionally, cyclotomic polynomial $\Phi_n(x)$ has integer coefficients which are well-studied. Musiker and Reiner in [9] interpreted these coefficients topologically, as the torsion in the homology of a certain simplicial complex associated with the degree of the cyclotomic polynomial. The idea for these simplicial complexes originally appeared in [3] and reappeared in [1, 7]. In what follows, we give a review of associated simplicial complexes. It is sufficient to interpret the coefficients

Key words and phrases. Cyclotomic polynomial, simplicial complexes, discrete Morse theory, homotopy type.
2010 *Mathematics Subject Classification.* Primary: 55U10. Secondary: 55P10, 55P15.
DOI 10.46793/KgJMat2302.309K
Received: July 16, 2020.
Accepted: August 25, 2020.

of the cyclotomic polynomial for squarefree n. Therefore, we fix such a squarefree $n = p_1 \cdots p_d$. Let

$$K_{p_1,\ldots,p_d} := K_{p_1} * \cdots * K_{p_d}$$

be the simplicial join of $K_{p_1}, \ldots, K_{p_d}$, where K_{p_i} is a 0-dimensional abstract simplicial complex with p_i vertices which are labeled by residues $\{0 \pmod{p_i}, 1 \pmod{p_i}, \ldots, (p_i - 1) \pmod{p_i}\}$. The facets of $K_{p_1,\ldots,p_d}$ are labeled by a sequence of residues $(j_1 \pmod{p_1}, \ldots, j_d \pmod{p_d})$ and by the Chinese Reminder Theorem, they can be denoted by residue $j \pmod{n}$ (denote this facet by $F_{j \pmod{n}}$). Let $A \subseteq \{0, 1, \ldots, \phi(n)\}$. We denote by K_A the subcomplex of $K_{p_1,\ldots,p_d}$ which is generated by the facets $\{F_{j \pmod{n}}\}$, where

$$j \in A \cup \{\phi(n) + 1, \phi(n) + 2, \ldots, n - 2, n - 1\}.$$

It turns out that subcomplexes $K_\emptyset$ and $K_{\{j\}}$, where $j \in \{0, \ldots, \phi(n)\}$ have a very nice feature, which Musiker and Reiner proved in the next two theorems. Let $[z_{j \pmod{n}}] := \partial[F_{j \pmod{n}}]$ denote the $(d-2)$-cycle which is its image under the simplical boundary map ∂.

Theorem 1.1. ([9, Theorem 7.1.]). *Let $n = p_1 \cdots p_d$ be squarefree.*

(i) *One has a homology isomorphism*

$$\tilde{H}_*(K_\emptyset) \cong \tilde{H}_*(\mathbb{S}^{d-2}),$$

with $\tilde{H}_{d-2}(K_\emptyset) \cong \mathbb{Z}$ generated by the cycle $[z_{\phi(n) \pmod{n}}]$.

(ii) *If $\Phi_n(x) = \sum_{j=0}^{\phi(n)} c_j x^j$, then for $j = 0, 1, \ldots, \phi(n)$, one has*

$$[z_{j \pmod{n}}] = c_j [z_{\phi(n) \pmod{n}}] \text{ in } \tilde{H}_{d-2}(K_\emptyset) \cong \mathbb{Z}$$

and a homology isomorphism

$$\tilde{H}_*(K_{\{j\}}) \cong \tilde{H}_*(\mathbb{B}^{d-1} \cup_{f_j} \mathbb{S}^{d-2}),$$

where $\deg(f_j) = c_j$.

Theorem 1.2. ([9, Theorem 7.5.]). *For $d \geq 4$ and every $A \subseteq \{0, 1, \ldots, \phi(n)\}$, the complex K_A is simply-connected. Consequently, for $d \neq 3$, one has the following.*

(i) *The complex $K_\emptyset$ is homotopy equivalent to $\mathbb{S}^{d-2}$ and contains $[z_{\phi(n) \pmod{n}}]$ as a fundamental $(d-2)$-cycle.*

(ii) *For $j = 0, 1, \ldots, \phi(n)$, the cyclotomic polynomial coefficient c_j gives the degree of the attaching map from the oriented boundary $[z_{j \pmod{n}}]$ of the facet $F_{j \pmod{n}}$ into the homotopy $(d-2)$-sphere $K_\emptyset$, with respect to the choice of $[z_{\phi(n) \pmod{n}}]$ as the fundamental cycle.*

(iii) *In particular, the complex $K_{\{j\}}$ is homotopy equivalent to $\mathbb{S}^{d-2} \cup_{f_j} \mathbb{B}^{d-1}$, where* $\deg(f_j) = c_j$.

For $d \geq 4$ the fundamental group of K_A is determined by its 2-skeleton, which is the same as 2-skeleton of $K_{p_1,\ldots,p_d}$ since the subcomplex $K_\emptyset$, and consequently every subcomplex K_A, contains the full $(d-2)$-skeleton of $K_{p_1,\ldots,p_d}$ [9, Proposition 5.5.].

This skeleton is shellable [9, Proposition 5.1.], hence homotopy equivalent to a wedge of $(d-2)$-spheres.

The homotopy types of $K_\emptyset$ and $K_{\{j\}}$ remain as opened question when $d = 3$. Namely, in Question 7.6, Musiker and Rainer ask the following.

1) Is $K_\emptyset$ homotopy equivalent to the circle $\mathbb{S}^1$?
2) Is $K_{\{j\}}$ homotopy equivalent to $\mathbb{B}^2 \cup_{f_j} \mathbb{S}^1$, where $\deg(f_j) = c_j$, $j = 0, 1, \ldots, \phi(n)$?

In [8], authors show, giving a counter-example, that Theorem 1.2 does not follow generally when $d = 3$. For $n = 3 \cdot 5 \cdot 7$, $K_\emptyset$ is not homotopy equivalent to the circle $\mathbb{S}^1$ and $K_{\{j\}}$ is not homotopy equivalent to $\mathbb{B}^2 \cup_{f_j} \mathbb{S}^1$ for $j = 7$.

In this paper, by using discrete Morse theory, we prove that for $n = 3 \cdot 5 \cdot p$, where $p \geq 7$ is an arbitrary prime number, Theorem 1.2 holds for certain classes of prime p modulo 15, while for the others we show it does not hold. This result is given in the following two theorems.

Theorem 1.3. *Let $p \equiv k \pmod{15}$, where $k \in \{1, 2, 13, 14\}$, and $n = 3 \cdot 5 \cdot p$.*

(1) *The complex $K_\emptyset$ is homotopy equivalent to $\mathbb{S}^1$.*
(2) *If $\Phi_n(x) = \sum_{j=0}^{\phi(n)} c_j x^j$, then for $j \in \{0, 1, \ldots, \phi(n)\}$, the complex $K_{\{j\}}$ is homotopy equivalent to $\mathbb{S} \cup_{f_j} \mathbb{B}^2$, where $\deg(f_j) = c_j$.*

Theorem 1.4. *Let $p \equiv k \pmod{15}$, where $k \in \{4, 7, 8, 11\}$, and $n = 3 \cdot 5 \cdot p$. The complex $K_\emptyset$ is not homotopy equivalent to $\mathbb{S}^1$.*

The paper is organized as follows. In Section 2, we briefly introduce notation of simplicial complexes, define an acyclic discrete vector field and its critical elements. In Section 3 we study the structure of the subcomplex $K_\emptyset$. Additionally, we construct an appropriate acyclic discrete vector field on $K_\emptyset$. In Section 4, we prove Theorem 1.3 by using results from Section 3. Finally, in Section 5 we prove Theorem 1.4.

2. Basic Concepts

2.1. **Simplicial complex.** Here, we present the basic notation and terminology concerning simplicial complexes which we will use intensively in this paper. For more details see [10].

An *abstract simplicial complex* K is a collection of finite non-empty sets such that, if $\sigma \in K$ and $\emptyset \neq \tau \subseteq \sigma$, then $\tau \in K$. If $\sigma \in K$, and σ has $n+1$ elements, we refer to σ as an n-simplex. If we want to emphasize that σ is n-dimensional simplex, i.e., n-simplex, we use notation $\sigma^{(n)}$.

A non-empty subset τ of σ is called a *face* of σ. Those simplices that are not faces of any other simplex in K are called *facets*.

Definition 2.1. Let K be any simplicial complex and let σ be any face of K. The *star* $\mathrm{St}(\sigma)$ of σ is the subcomplex of K consisting of all faces τ containing σ and of all faces of τ, i.e.,

$$\mathrm{St}(\sigma) = \{s \in K \mid (\exists \tau \in K)\ (\sigma \subseteq \tau \text{ and } s \subseteq \tau)\}.$$

Definition 2.2. The *link* of σ, denoted by $\mathrm{Lk}(\sigma)$, is the subcomplex of K consisting of all faces in $\mathrm{St}(\sigma)$ that do not intersect σ, i.e.,

$$\mathrm{Lk}(\sigma) = \{\tau \in K \mid \tau \in \mathrm{St}(\sigma) \text{ and } \sigma \cap \tau = \emptyset\}.$$

To simplify notation, if $\sigma = \{v\}$, where v is a vertex, we write $\mathrm{St}(v)$ for $\mathrm{St}(\{v\})$ and $\mathrm{Lk}(v)$ for $\mathrm{Lk}(\{v\})$. From the previous, it is clear that $\mathrm{St}(v) = v * \mathrm{Lk}(v)$. In order to simplify notation we denote the union $\bigcup_{i=1}^{k} \mathrm{St}(v_i)$ by $\mathrm{St}(v_1, \dots, v_k)$.

2.2. **Discrete Morse theory.** This subsection aims to give a brief introduction and some of the main results from Forman's discrete Morse theory. Discrete Morse theory (shorter DMT) is based on pairing faces of the complex, which actually represent forming sequences of collapses on the complex. We will use this theory in order to prove homotopical equivalence between certain simplicial complexes. For a more thorough background concerning DMT, we refer the reader to [4–6].

Definition 2.3. A function $F : K \to \mathbb{R}$ is discrete Morse function if, for every $\alpha^{(p)} \in K$,

(1) $f(\beta^{(p+1)}) \leq f(\alpha^{(p)})$ for at most one $\beta^{(p+1)} \supset \alpha^{(p)}$, and
(2) $f(\gamma^{(p-1)}) \geq f(\alpha^{(p)})$ for at most one $\gamma^{(p-1)} \subset \alpha^{(p)}$.

Definition 2.4. Simplex $\alpha^{(p)}$ is critical simplex if $f(\beta^{(p+1)}) > f(\alpha^{(p)})$ for all $\beta^{(p+1)} \supset \alpha^{(p)}$ and $f(\gamma^{(p-1)}) < f(\alpha^{(p)})$ for all $\gamma^{(p-1)} \subset \alpha^{(p)}$.

Forman proved that the topology of a simplicial complex is related to its critical simplex in a very strong way. This connection is given in the next theorem.

Theorem 2.1 ([5])**.** *Suppose K is a simplicial complex with a discrete Morse function. Then, K is homotopy equivalent to a* CW *complex with exactly one cell of dimension p for each critical simplex of dimension p.*

The number of critical simplices is not a topological invariant as it depends on the discrete Morse function. According to the previous theorem, the goal is to find Morse function with as small critical simplicies as possible. For this purpose, we introduce discrete vector field, which is (under some conditions) an equivalent concept.

Definition 2.5. Discrete vector field V on a finite simplicial complex K is the set of pairs $\{\alpha^{(p)}, \beta^{(p+1)}\}$, where $\alpha^{(p)} \subset \beta^{(p+1)}$, and each simplex is in at most one pair. We say that $\{\alpha^{(p)}, \beta^{(p+1)}\}$ is a matching in V. Simplex γ in K is critical or unmatched with respect to V if γ is not contained in any pair in V.

For a simplicial complex K and a discrete vector field V on K, let $\mathcal{C}_k(K, V)$ denote the set of all critical k-simplices in the simplicial complex K with respect to V and let

$$\mathcal{C}(K, V) = \bigcup_{k=0}^{\dim K} \mathcal{C}_k(K, V).$$

Definition 2.6. Given a discrete vector field V on a finite simplicial complex K, a V-path is a sequence of simplicies

$$\alpha_0^{(p)}, \beta_0^{(p+1)}, \alpha_1^{(p)}, \beta_1^{(p+1)}, \ldots, \alpha_{r+1}^{(p)}, \beta_{r+1}^{(p+1)},$$

such that, for each $i \in \{0, \ldots, r\}$, pair $\{\alpha_i, \beta_i\} \in V$ and $\beta_i \supset \alpha_{i+1} \neq \alpha_i$. This path is non-trivially closed if $r > 0$ and $\alpha_0 = \alpha_{r+1}$.

If a discrete vector field V does not contain a non-trivial closed V-path we say that V is acyclic.

Theorem 2.2 ([5])**.** *A discrete vector field V on a finite simplicial complex K is a discrete vector field of some Morse function if and only if V is acyclic.*

Namely, for a discrete Morse function f, we can easily define a discrete vector field in the following way: $\{\alpha^{(p)}, \beta^{(p+1)}\} \in V$ whenever $f(\beta^{(p+1)}) \leq f(\alpha^{(p)})$. Previous theorem give a condition when we can do the converse process.

On the other hand, matching $\{\alpha^{(p)}, \beta^{(p+1)}\}$ in a discrete vector field V on a finite simplicial complex K can be represent by an arrow from a simplex $\alpha^{(p)}$ to a simplex $\beta^{(p+1)}$ of K. According to this, a modified Hasse (directed) diagram of the complex K corresponds to V. Hasse diagram is modified in the following way: arrows are reversed each time when for $\beta^{(p+1)}$ and its face $\alpha^{(p)}$ one has $\{\alpha^{(p)}, \beta^{(p+1)}\} \in V$. We denote this diagram by $D(K, V)$. Directed path from α to β in $D(K, V)$ we denote by $\alpha \to \beta$. It turns out that if V is an acyclic discrete vector field then $D(K, V)$ is acyclic directed graph, that is, $\alpha \to \beta$ and $\beta \to \alpha$ implies $\alpha = \beta$. The symbol $\alpha \not\to \beta$ we use to denote that a directed path from α to β does not exist in $D(K, V)$. Generally, for families K_1 and K_2, we write $K_1 \to K_2$ if there are $\alpha \in K_1$ and $\beta \in K_2$ such that $\alpha \to \beta$. The symbol $K_1 \not\to K_2$ is used to denote the non-existence of such a directed path.

We will use the next theorem in further work in order to prove the existence of a homotopical equivalence between a certain simplicial complexes.

Theorem 2.3. ([6, Theorem 4.4]). *Suppose that K_0 is a subcomplex of K such that $K_0 \not\to K \setminus K_0$ and such that all critical faces belong to K_0. Then it is possible to collapse K to K_0. In particular, K and K_0 are homotopy equivalent. Hence, K has no homology in dimensions strictly greater than* $\dim K_0$.

3. Complex $K_\emptyset$

Let $n = 3 \cdot 5 \cdot p$, where p is a prime. As the case when $p = 7$ was investigated in [8], we can assume that $p > 7$. For $n = 3 \cdot 5 \cdot p$ simplicial complex $K_{3,5,p}$ has $p + 8$ vertices:

$$0 \pmod 3,\ 1 \pmod 3,\ 2 \pmod 3,\ 0 \pmod 5,\ 1 \pmod 5,\ 2 \pmod 5,$$
$$3 \pmod 5,\ 4 \pmod 5,\ 0 \pmod p,\ 1 \pmod p,\ \ldots,\ p-1 \pmod p.$$

In order to simplify the notation we label these vertices by numbers $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \ldots, p + 7$, respectively.

Subcomplex $K_\emptyset$ is a two-dimensional complex built of facets:

$$\begin{array}{lllllll}
F_{(8p-7) \pmod n}, & \dots, & F_{(8p-1) \pmod n}, & F_{8p \pmod n}, & \dots, & F_{(9p-8) \pmod n}, \\
F_{(9p-7) \pmod n}, & \dots, & F_{(9p-1) \pmod n}, & F_{9p \pmod n}, & \dots, & F_{(10p-8) \pmod n}, \\
\vdots & & \vdots & \vdots & & \vdots \\
F_{(14p-7) \pmod n}, & \dots, & F_{(14p-1) \pmod n}, & F_{14p \pmod n}, & \dots, & F_{(15p-8) \pmod n}, \\
F_{(15p-7) \pmod n}, & \dots, & F_{(15p-1) \pmod n}. & & &
\end{array}$$

Note that facet $F_{(dp-i) \pmod n}$ contains vertex $(p-i) \pmod p$ which is labeled by number $p-i+8$ for all $d \in \{8,\dots,15\}$, $i \in \{1,\dots,7\}$. Similarly, facet $F_{(dp+i) \pmod n}$ contains vertex $i \pmod p$ which is labeled by number $i+8$ for all $d \in \{8,\dots,14\}$, $i \in \{0,\dots,p-8\}$. Let

$$\left[a_j^i, b_j^i, i\right] = \begin{cases} F_{((8+j-1)p+i-8) \pmod n}, & \text{for } i \in \{8,\dots,p\}, \\ \\ F_{((8+j-1)p+i-8-p) \pmod n}, & \text{for } i \in \{p+1,\dots,p+7\}. \end{cases}.$$

Then, the above set of facets are:

$$\begin{array}{lllllll}
\left[a_1^{p+1}, b_1^{p+1}, p+1\right], & \dots, & \left[a_1^{p+7}, b_1^{p+7}, p+7\right], & \left[a_1^8, b_1^8, 8\right], & \dots, & \left[a_1^p, b_1^p, p\right], \\
\left[a_2^{p+1}, b_2^{p+1}, p+1\right], & \dots, & \left[a_2^{p+7}, b_2^{p+7}, p+7\right], & \left[a_2^8, b_2^8, 8\right], & \dots, & \left[a_2^p, b_2^p, p\right], \\
\vdots & & \vdots & \vdots & & \vdots \\
\left[a_7^{p+1}, b_7^{p+1}, p+1\right], & \dots, & \left[a_7^{p+7}, b_7^{p+7}, p+7\right], & \left[a_7^8, b_7^8, 8\right], & \dots, & \left[a_7^p, b_7^p, p\right], \\
\left[a_8^{p+1}, b_8^{p+1}, p+1\right], & \dots, & \left[a_8^{p+7}, b_8^{p+7}, p+7\right], & & &
\end{array}$$

respectively.

As every facet of $K_\emptyset$ contains exactly one vertex from the set of vertices $\{8, 9, \dots, p+7\}$ it is clear that

$$K_\emptyset = \bigcup_{i=8}^{p+7} \mathrm{St}(i).$$

Therefore, we begin our analysis of the complex $K_\emptyset$ with analysis of its subcomplexes $\mathrm{St}(8), \dots, \mathrm{St}(p+7)$.

As the number of 2-simplicies of $K_\emptyset$ is $7p+7$, we can notice that the subcomplex $\mathrm{St}(i)$ is built of facets $\left\{[a_j^i, b_j^i, i]\right\}_{j=1}^{7}$ when $i \in \{8,\dots,p\}$ and facets $\left\{[a_j^i, b_j^i, i]\right\}_{j=1}^{8}$ when $i \in \{p+1,\dots,p+7\}$. Furthermore, it follows that

$$a_1^i = a_4^i = a_7^i, \quad a_2^i = a_5^i, \quad a_3^i = a_6^i, \quad \text{for } i \in \{8,\dots,p\},$$

and

$$a_1^i = a_4^i = a_7^i, \quad a_2^i = a_5^i = a_8^i, \quad a_3^i = a_6^i, \quad \text{for } i \in \{p+1,\dots,p+7\},$$

because $8p \equiv 11p \equiv 14p$, $9p \equiv 12p \equiv 15p$ and $10p \equiv 13p$ modulo 3. Similarly, as $8p \equiv 13p$, $9p \equiv 14p$ and $10p \equiv 15p$ modulo 5, we can conclude that

$$b_1^i = b_6^i, \quad b_2^i = b_7^i, \quad \text{for } i \in \{8, \dots, p\},$$

and

$$b_1^i = b_6^i, \quad b_2^i = b_7^i, \quad b_3^i = b_8^i, \quad \text{for } i \in \{p+1, \dots, p+7\}.$$

Figure 1 shows subcomplex $\mathrm{St}(i)$ depending on the index $i \in \{8, \dots, p+7\}$.

(A) $i \in \{8, \dots, p\}$ (B) $i \in \{p+1, \dots, p+7\}$

FIGURE 1. Simplicial complex $\mathrm{St}(i)$

3.1. **Discrete vector field on $K_\emptyset$.** In order to examine the topology of the complex $K_\emptyset$, we will look for a discrete vector field such that the number of critical 2-simplices are as small as possible. We will see below that finding an appropriate discrete vector field on $K_\emptyset$ can be reduced to finding an appropriate discrete vector field on its subcomplex $\mathrm{St}(p+1, \dots, p+7)$.

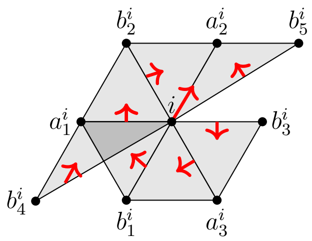
The simplicial subcomplex $\mathrm{St}(i)$, $i \in \{8, \dots, p\}$, is built of facets:

$$\begin{aligned}
&[a_1^i, b_1^i, i],\ [a_2^i, b_2^i, i],\ [a_3^i, b_3^i, i],\\
&[a_1^i, b_4^i, i],\ [a_2^i, b_5^i, i],\ [a_3^i, b_1^i, i],\\
&[a_1^i, b_2^i, i].
\end{aligned}$$

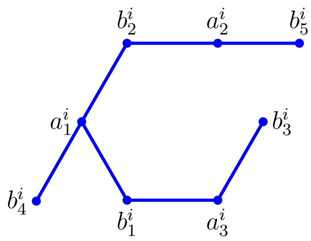
As a_1^i, a_2^i, a_3^i and $b_1^i, b_2^i, b_3^i, b_4^i, b_5^i$ are different vertices, we can define acyclic discrete vector field on $\mathrm{St}(i)$ as follows:

$$\begin{aligned}
S_i = \{&\{[b_1^i, i], [a_1^i, b_1^i, i]\},\ \{[b_2^i, i], [a_2^i, b_2^i, i]\},\ \{[b_3^i, i], [a_3^i, b_3^i, i]\},\\
&\{[b_4^i, i], [a_1^i, b_4^i, i]\},\ \{[b_5^i, i], [a_2^i, b_5^i, i]\},\ \{[a_3^i, i], [a_3^i, b_1^i, i]\},\\
&\{[a_1^i, i], [a_1^i, b_2^i, i]\},\ \{[i], [a_2^i, i]\}\}.
\end{aligned}$$

Discrete vector field S_i is an acyclic discrete vector field on $\mathrm{St}(i)$, as we can see on Figure 2 (A). Note that $\mathcal{C}(\mathrm{St}(i), S_i) = \mathrm{Lk}(i) \subset K_{3,5}$ and $\mathcal{C}_2(\mathrm{St}(i), S_i) = \emptyset$ (see Figure 2 (B)).



(A) S_i-paths



(B) $\mathcal{C}(\mathrm{St}(i), S_i)$

FIGURE 2. Discrete vector field S_i on complex $\mathrm{St}(i)$, $i \in \{8, \dots, p\}$

Lemma 3.1. *Let $n = 3 \cdot 5 \cdot p$, where $p > 7$ is a prime. If C is an arbitrary acyclic discrete vector field on $\mathrm{St}(p+1, \dots, p+7)$, then*

$$V = \left(\bigcup_{i=8}^{p} S_i \right) \cup C$$

is an acyclic discrete vector field on $K_\emptyset$.

Proof. It follows that

$$\left(\bigcup_{k \in \{8, \dots, \hat{i}, \dots, p+7\}} \mathrm{St}(k) \right) \cap \mathrm{St}(i) \subseteq \mathrm{Lk}(i).$$

As $\mathcal{C}(\mathrm{St}(i), S_i) = \mathrm{Lk}(i)$ for all $i \in \{8, \dots, p\}$, V is a well-defined discrete vector field on $K_\emptyset$ as each simplex is in at most one pair. Additionally, for all $i \in \{8, \dots, p\}$,

$$\bigcup_{k \in \{8, \dots, \hat{i}, \dots, p+7\}} \mathrm{St}(k) \not\longrightarrow \mathrm{St}(i) \setminus \mathrm{Lk}(i).$$

Hence, there are no non-trivial closed V-paths which contain simplices from the set $\bigcup_{i=8}^{p} (\mathrm{St}(i) \setminus \mathrm{Lk}(i))$. Note that

$$\bigcup_{i=8}^{p} \mathrm{Lk}(i) \setminus \mathrm{St}(p+1, \dots, p+7) \subseteq \mathcal{C}(K_\emptyset, V).$$

As

$$K_\emptyset = \left(\bigcup_{i=8}^{p} (\mathrm{St}(i) \setminus \mathrm{Lk}(i)) \right) \cup \left(\bigcup_{i=8}^{p} \mathrm{Lk}(i) \setminus \mathrm{St}(p+1, \dots, p+7) \right) \cup \mathrm{St}(p+1, \dots, p+7),$$

the discrete vector field V is acyclic on $K_\emptyset$. $\square$

According to the previous lemma, in what follows, we will focus on finding an appropriate discrete vector field on the subcomplex $\mathrm{St}(p+1, \dots, p+7)$.

3.2. **Discrete vector field on** $\mathrm{St}(p+1,\dots,p+7)$**.** As $\mathrm{St}(p+1,\dots,p+7) = \bigcup_{i=p+1}^{p+7}\mathrm{St}(i)$, we will find acyclic discrete vector fields without unpaired 2-simplices for the subcomplexes $\mathrm{St}(i)$, $i \in \{p+1,\dots,p+7\}$. We know that 2-simplices in $\mathrm{St}(i)$ are

$$[a_1^i, b_1^i, i],\ [a_2^i, b_2^i, i],\ [a_3^i, b_3^i, i],$$
$$[a_1^i, b_4^i, i],\ [a_2^i, b_5^i, i],\ [a_3^i, b_1^i, i],$$
$$[a_1^i, b_2^i, i],\ [a_2^i, b_3^i, i].$$

First, we consider the following discrete vector field on $\mathrm{St}(i)$:

$$\begin{aligned} V_i = \{&\{[b_1^i, i],\ [a_1^i, b_1^i, i]\},\ \{[b_2^i, i],\ [a_2^i, b_2^i, i]\},\ \{[b_3^i, i], [a_3^i, b_3^i, i]\},\\ &\{[b_4^i, i],\ [a_1^i, b_4^i, i]\},\ \{[b_5^i, i],\ [a_2^i, b_5^i, i]\},\ \{[a_3^i, i],\ [a_3^i, b_1^i, i]\},\\ &\{[a_1^i, i],\ [a_1^i, b_2^i, i]\},\ \{[a_2^i, i],\ [a_2^i, b_3^i, i]\}\}. \end{aligned}$$

Discrete vector field V_i is well-defined and pairs all facets of $\mathrm{St}(i)$, but it is not acyclic (see Figure 3).

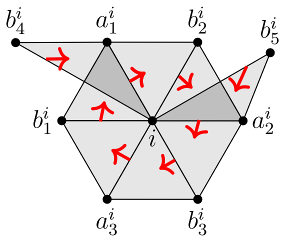


FIGURE 3. V_i-paths on complex $\mathrm{St}(i)$, $i \in \{p+1,\dots,p+7\}$

Namely, there is exactly one non-trivial closed V_i-path. This path contains facets $[a_1^i, b_1^i, i]$, $[a_2^i, b_2^i, i]$, $[a_3^i, b_3^i, i]$, $[a_3^i, b_1^i, i]$, $[a_1^i, b_2^i, i]$ and $[a_2^i, b_3^i, i]$ (see Figure 4).

Note that if $\alpha \supset [b_4^i, i]$, then $\alpha = [a_1^i, b_4^i, i]$. Similarly, if $\beta \supset [b_5^i, i]$, then $\beta = [a_2^i, b_5^i, i]$. Hence, there are no facets $\alpha, \beta \in \mathrm{St}(i)$ such that $\alpha \to [b_4^i, i]$ and $\beta \to [b_5^i, i]$ in V_i. Consequently, simplices $[b_4^i, i]$, $[b_5^i, i]$, $[a_1^i, b_4^i, i]$ and $[a_2^i, b_5^i, i]$ cannot be a part of a non-trivial closed V_i-path.

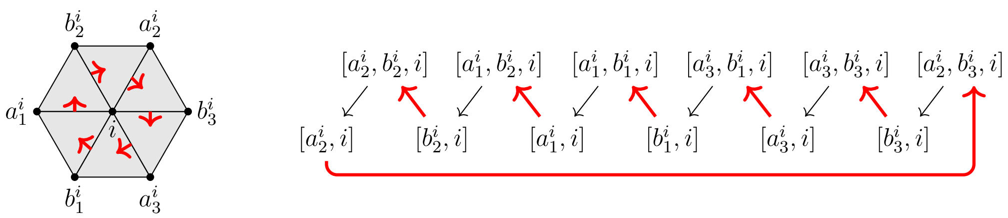


FIGURE 4. Non-trivial closed V_i-path

However, if we perform certain changes, we can make V_i acyclic. Namely, for some $(l,k) \in \{(1,1),(1,2),(2,2),(2,3),(3,1),(3,3)\}$, if we modify V_i in a way that we pair $[a_l^i, b_k^i]$ with $[a_l^i, b_k^i, i]$, instead paring $[b_k^i, i]$ or $[a_l^i, i]$, it becomes acyclic (see Figure 5).

If $\{[b^i_k,\, i],\, [a^i_l,\, b^i_k,\, i]\} \in V_i$, we replace the matching $\{[b^i_k,\, i],\, [a^i_l,\, b^i_k,\, i]\}$ by the matching $\{[a^i_l,\, b^i_k],\, [a^i_l,\, b^i_k,\, i]\}$. Thus, 1-simplex $[b^i_k,\, i]$ becomes unmatched, so we can add matching $\{[i],\, [b^i_k,\, i]\}$. Analogously, if $\{[a^i_l,\, i],\, [a^i_l,\, b^i_k,\, i]\} \in V_i$ we replace the matching $\{[a^i_l,\, i],\, [a^i_l,\, b^i_k,\, i]\}$ by the matching $\{[a^i_l,\, b^i_k],\, [a^i_l,\, b^i_k,\, i]\}$ and 1-simplex $[a^i_l,\, i]$ becomes unmatched. Then, we can add matching $\{[i],\, [a^i_l, i]\}$.

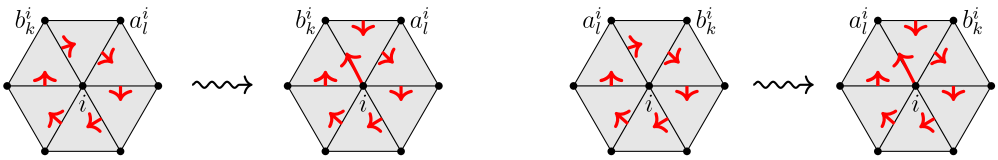


FIGURE 5. The two type of modification in discrete vector field V_i

Note that 0-simplex $[i]$ is not part of any non-trivial closed path in the mentioned modification of V_i. Namely, if $\alpha \to [i]$ for some 1-simplex $\alpha \supset [i]$, then α is not paired with any 0-simplex. Actually, α is pared with a 2-simplex.

Let

$$V_i([a^i_l, b^i_k]) := \Big(V_i \setminus \Big\{\{[a^i_l,\, i],\, [a^i_l,\, b^i_k,\, i]\}\Big\}\Big) \cup \Big\{\{[a^i_l,\, b^i_k],\, [a^i_l,\, b^i_k,\, i]\},\ \{[i],\, [a^i_l,\, i]\}\Big\}$$

if $\{[a^i_l,\, i],\, [a^i_l,\, b^i_k,\, i]\} \in V_i$ and

$$V_i([a^i_l, b^i_k]) := \Big(V_i \setminus \Big\{\{[b^i_k,\, i],\, [a^i_l,\, b^i_k,\, i]\}\Big\}\Big) \cup \Big\{\{[a^i_l,\, b^i_k],\, [a^i_l,\, b^i_k,\, i]\},\ \{[i],\, [b^i_k,\, i]\}\Big\}$$

if $\{[b^i_k, i],\, [a^i_l, b^i_k, i]\} \in V_i$.

According to the previous considerations, $V_i([a^i_l, b^i_k])$ is an acyclic discrete vector field on $\mathrm{St}(i)$, without critical 2-simplices, for all $i \in \{p+1, \dots, p+7\}$. Actually, $\mathcal{C}(\mathrm{St}(i), V_i([a^i_l, b^i_k]) = \mathrm{Lk}(i) \setminus [a^i_l, b^i_k] \subset K_{3,5}$.

The choice of (l, k) will depend on the rest of the complex $\mathrm{St}(p+1, \dots, p+7)$. Let $V_i([a^i_{l_i}, b^i_{k_i}])$ be the corresponding acyclic discrete vector field on $\mathrm{St}(i)$ for $i \in \{p+1, \dots, p+7\}$. If $[a^{p+1}_{l_{p+1}}, b^{p+1}_{k_{p+1}}]$, $[a^{p+2}_{l_{p+2}}, b^{p+2}_{k_{p+2}}], \dots, [a^{p+7}_{l_{p+7}}, b^{p+7}_{k_{p+7}}]$ are distinct 1-simplices then

$$C := \bigcup_{i=p+1}^{p+7} V_i([a^i_{l_i}, b^i_{k_i}])$$

is a well-defined discrete vector field on $\mathrm{St}(p+1, \dots, p+7)$. Generally, C does not have to be acyclic. Namely, there are no non-trivial closed C-paths which contain simplices from the only one subcomplex $\mathrm{St}(i)$, but there may be non-trivial closed paths containing simplices from the various subcomplexes $\mathrm{St}(i)$, $i \in \{p+1, \dots, p+7\}$.

For $i \in \{p+1, \dots, p+7\}$, note that the only "entrance" in $\mathrm{St}(i)$ with respect to C from $\mathrm{St}(p+1, \dots, \widehat{i}, \dots p+7)$ is through the 1-simplex $[a^i_{l_i}, b^i_{k_i}]$, whereas the set of "exits" are

$$\Big(\mathrm{Lk}(i) \setminus \{[a^i_{l_i}, b^i_{k_i}]\}\Big) \cap \{[a^j_{l_j}, b^j_{k_j}]\}_{j \in \{p+1, \dots, \widehat{i}, \dots, p+7\}}.$$

The only way to reach 1-simplex $[a_1^i, b_4^i]$ from the rest of the complex $\mathrm{St}(i)$ is through the 1-simplex $[b_4^i, i]$, i.e., $[b_4^i, i] \longrightarrow [a_1^i, b_4^i, i] \rightarrow [a_1^i, b_4^i]$. Similarly, 1-simplex $[a_2^i, b_5^i]$ can be reached from the rest of the complex $\mathrm{St}(i)$ through the 1-simplex $[b_5^i, i]$ only, i.e., $[b_5^i, i] \longrightarrow [a_2^i, b_5^i, i] \rightarrow [a_2^i, b_5^i]$. As $[b_4^i, i]$ and $[b_5^i, i]$ do not have entrance arrows in C, we can ignore 1-simplicies $[a_1^i, b_4^i]$ and $[a_2^i, b_5^i]$ as exits from $\mathrm{St}(i)$.

We form a directed graph $\mathrm{Flow}(C)$ of the "entrances/exits" through the subcomplexes $\mathrm{St}(i)$, $i \in \{p+1, \dots, p+7\}$, with respect to C. The graph $\mathrm{Flow}(C) = (A \sqcup B, E)$ is bipartite, where:

$$\begin{aligned} A =& \{\mathrm{St}(i) \mid i \in \{p+1, \dots, p+7\}\}, \\ B =& \{[a_{l_i}^i,\, b_{k_i}^i] \mid i \in \{p+1, \dots, p+7\}, \end{aligned}$$

and

$$\begin{aligned} E = \bigcup_{i=p+1}^{p+7} & \Big\{(\mathrm{St}(i), \alpha) \mid \alpha \in B \cap \mathrm{Lk}(i) \setminus \{[a_{l_i}^i, b_{k_i}^i], [a_1^i, b_4^i], [a_2^i, b_5^i]\}\Big\} \\ & \cup \Big\{([a_{l_i}^i,\, b_{k_i}^i], \mathrm{St}(i)) \mid i \in \{p+1, \dots, p+7\}\Big\}. \end{aligned}$$

Therefore, if $[a_{l_j}, b_{k_j}] \in \mathrm{St}(i) \setminus \{[a_1^i, b_4^i], [a_2^i, b_5^i]\}$ it follows that

$$[a_{l_i}, b_{k_i}] \rightarrow \mathrm{St}(i) \rightarrow [a_{l_j}, b_{k_j}] \rightarrow \mathrm{St}(j)$$

is path in $\mathrm{Flow}(C)$, for all distinct $i, j \in \{p+1, \dots, p+7\}$ (see Figure 6).

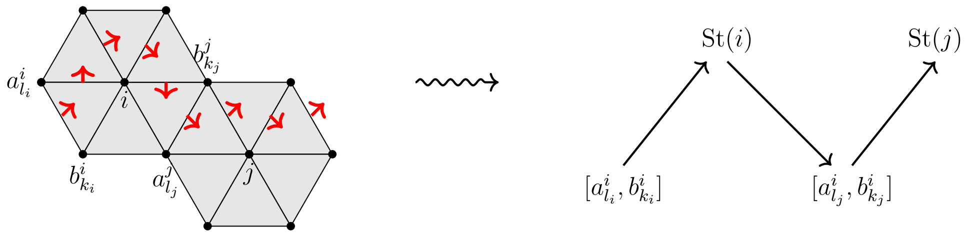


FIGURE 6. Forming a directed graph $\mathrm{Flow}(C)$

It is clear that if digraph $\mathrm{Flow}(C)$ is acyclic then C is an acyclic discrete vector field. In order to make C acyclic, we will choose appropriate 1-simplices $\{[a_{l_i}^i, b_{k_i}^i]\}_{i=p+1}^{p+7}$.

4. Proof of Theorem 1.3

Let $p_1, p_2 \geq 7$ be two distinct primes and $n_1 = 3 \cdot 5 \cdot p_1$ and $n_2 = 3 \cdot 5 \cdot p_2$. We consider subcomplexes $\mathrm{St}(p_1+1, \dots, p_1+7)$ and $\mathrm{St}(p_2+1, \dots, p_2+7)$ of $K_\emptyset$ for n_1 and n_2. The subcomplex $\mathrm{St}(p_1+1, \dots, p_1+7)$ consists of facets

$$\big\{F_{(dp_1 - i) \pmod{n_1}}\big\}_{d=\overline{8,15},\, i=\overline{1,7}},$$

while $\mathrm{St}(p_2+1, \dots, p_2+7)$ consists of facets

$$\big\{F_{(dp_2 - i) \pmod{n_2}}\big\}_{d=\overline{8,15},\, i=\overline{1,7}}.$$

It turns out that for certain primes p_1 and p_2, complexes $\mathrm{St}(p_1+1,\dots,p_1+7)$ and $\mathrm{St}(p_2+1,\dots,p_2+7)$ are isomorphic. Namely, when $p_1 \equiv p_2 \pmod{15}$, we can define the map $\pi : \mathrm{St}(p_1+1,\dots,p_1+7) \to \mathrm{St}(p_2+1,\dots,p_2+7)$ such that

$$p_1 + k \overset{\pi}{\mapsto} p_2 + k, \quad \text{for } k \in \{1,\dots,7\},$$

and π fixes every other vertex. Note that $dp_1 - i \equiv_{15} dp_2 - i$ when $p_1 \equiv_{15} p_2$ for all $i \in \{1,\dots,7\}$, $d \in \{8,\dots,15\}$. Additionally, $dp_1 - i \equiv_{p_1} p_1 - i$ and $dp_2 - i \equiv_{p_2} p_2 - i$. As $\pi(p_1 - i + 8) = p_2 - i + 8$, we can conclude that

$$\pi(F_{(dp_1 - i) \pmod{n_1}}) = F_{(dp_2 - i) \pmod{n_2}}.$$

Therefore, π is an isomorphism of the complexes.

If p is a prime number, then potential reminders modulo 15 are 1, 2, 4, 7, 8, 11, 13 and 14. According to the above, for a fixed reminder r modulo 15, we do not have to examine complexes $\mathrm{St}(p+1,\dots,p+7)$ for all primes $p \equiv r \pmod{15}$, it is enough to examine just for one of them.

Proof of Theorem 1.3. In order to show that $K_\emptyset \simeq \mathbb{S}^1$, we will construct an acyclic discrete vector field on $K_\emptyset$ without critical 2-simplices, with one critical 1-simplex and one critical 0-simplex. If such discrete vector field exists, by Theorem 2.1, $K_\emptyset$ is homotopy equivalent to a CW complex with exactly one 1-cell and one 0-cell. According to Theorem 1.1, $H_1(K_\emptyset) = \mathbb{Z}$. Therefore, $K_\emptyset$ is homotopy equivalent to $\mathbb{S}^1$.

Similarly, to show that $K_{\{j\}} \simeq \mathbb{S} \cup_{f_j} \mathbb{B}^2$ we will construct an acyclic discrete vector field on $K_{\{j\}}$ with one critical 2-simplex, one critical 1-simplex and one critical 0-simplex. Then, by Theorem 2.1, $K_{\{j\}}$ is homotopy equivalent to a CW complex with exactly one 2-cell, one 1-cell and one 0-cell. Consequently, as $\pi_1(K_{\{j\}})$ has a presentation where the generators are the 1-cells and the relations come from the 2-cells,

$$\pi_1(K_{\{j\}}) = \langle g \mid g^d = 1 \rangle,$$

where d is the degree of the attaching map from the boundary of 2-cell into the 1-cell. By Theorem 1.1, $H_1(K_{\{j\}}) = \mathbb{Z}/c_j\mathbb{Z}$. As $H_1(K_{\{j\}})$ is the abelianization of $\pi_1(K_{\{j\}})$, it follows that $d = c_j$. Finally, the complex $K_{\{j\}}$ is homotopy equivalent to $\mathbb{S} \cup_{f_j} \mathbb{B}^2$, where $\deg(f_j) = c_j$.

Note that if V is an acyclic discrete vector field on $K_\emptyset$ with one critical 1-simplex, one critical 0-simplex and without critical 2-simplices, then V is an acyclic discrete vector field on $K_{\{j\}}$ with one critical 2-simplex, one critical 1-simplex and one critical 0-simplex. The complex $K_{\{j\}}$ is obtained by adding the facets $F_{j \pmod{n}}$ to the complex $K_\emptyset$, so the critical 2-simplex with respect to V is facet $F_{j \pmod{n}}$.

Now, we divide analysis in several cases, depending on the remainder of the prime p modulo 15. We will focus on finding an acyclic discrete vector field on $K_\emptyset$ without critical 2-simplices, one critical 1-simplex and one critical 0-simplex for each case. For each case we will find an acyclic vector field C on $\mathrm{St}(p+1,\dots,p+7)$, hence, by

Lemma 3.1,

$$V = \left(\bigcup_{i=8}^{p} S_i\right) \cup C,$$

is an acyclic discrete vector field on $K_\emptyset$. For such defined discrete vector field V on $K_\emptyset$ it follows that

$$\mathcal{C}(K_\emptyset, V) = (\mathrm{Lk}(8,\ldots,p) \setminus \mathrm{St}(p+1,\ldots,p+7)) \;\cup\; \mathcal{C}(\mathrm{St}(p+1,\ldots,p+7), C).$$

Case 1: $p \equiv 1 \pmod{15}$.

The subcomplex $\mathrm{St}(p+1,\ldots,p+7)$ consists of the following 2-simplices:

$$\begin{array}{ccccccc}
[1,4,p+1]\,, & [2,5,p+2]\,, & [0,6,p+3]\,, & [1,7,p+4]\,, & [2,3,p+5]\,, & [0,4,p+6]\,, & [1,5,p+7]\,, \\
[2,5,p+1]\,, & [0,6,p+2]\,, & [1,7,p+3]\,, & [2,3,p+4]\,, & [0,4,p+5]\,, & [1,5,p+6]\,, & [2,6,p+7]\,, \\
[0,6,p+1]\,, & [1,7,p+2]\,, & [2,3,p+3]\,, & [0,4,p+4]\,, & [1,5,p+5]\,, & [2,6,p+6]\,, & [0,7,p+7]\,, \\
[1,7,p+1]\,, & [2,3,p+2]\,, & [0,4,p+3]\,, & [1,5,p+4]\,, & [2,6,p+5]\,, & [0,7,p+6]\,, & [1,3,p+7]\,, \\
[2,3,p+1]\,, & [0,4,p+2]\,, & [1,5,p+3]\,, & [2,6,p+4]\,, & [0,7,p+5]\,, & [1,3,p+6]\,, & [2,4,p+7]\,, \\
[0,4,p+1]\,, & [1,5,p+2]\,, & [2,6,p+3]\,, & [0,7,p+4]\,, & [1,3,p+5]\,, & [2,4,p+6]\,, & [0,5,p+7]\,, \\
[1,5,p+1]\,, & [2,6,p+2]\,, & [0,7,p+3]\,, & [1,3,p+4]\,, & [2,4,p+5]\,, & [0,5,p+6]\,, & [1,6,p+7]\,, \\
[2,6,p+1]\,, & [0,7,p+2]\,, & [1,3,p+3]\,, & [2,4,p+4]\,, & [0,5,p+5]\,, & [1,6,p+6]\,, & [2,7,p+7]\,.
\end{array}$$

We define discrete vector field C on $\mathrm{St}(p+1,\ldots,p+7)$ in the following way:

$$\begin{aligned}
C =& V_{p+1}([1,4]) \cup V_{p+2}([2,5]) \cup V_{p+3}([0,6]) \cup V_{p+4}([1,7]) \\
& \cup V_{p+5}([0,5]) \cup V_{p+6}([1,6]) \cup V_{p+7}([2,7]) \\
& \cup \{\{[1],[1,3]\},\{[2],[2,4]\},\{[3],[2,3]\},\{[4],[0,4]\},\{[5],[1,5]\},\{[6],[2,6]\}, \\
& \{[7],[0,7]\}\}.
\end{aligned}$$

Discrete vector field C is well-defined (see Figure 7). Additionally, Figure 7 shows that there are no non-trivial closed C-paths consisting of 0-simplices and 1-simplices. Graph $\mathrm{Flow}(C)$ is acyclic (see Figure 8), thus, there are no non-trivial closed C-paths which consist of 2-simplices and 1-simplices as well. Consequently, C is an acyclic discrete vector field on $\mathrm{St}(p+1,\ldots,p+7)$.

Note that the only critical simplex in $\mathrm{St}(p+1,\ldots,p+7)$ with respect to C is $[0]$, i.e., $\mathcal{C}(\mathrm{St}(p+1,\ldots,p+7), C) = \{[0]\}$. Additionally, it follows that

$$K_{3,5} \setminus \mathrm{St}(p+1,\ldots,p+7) = \{[0,3]\}.$$

On the other hand, $[0,3] \in K_{3,5}$ and

$$[0,3] \in F_{15p \pmod{n}} \subset \mathrm{St}(8).$$

Consequently,

$$[0,3] \in \mathrm{Lk}(8) \subset \mathrm{Lk}(8,\ldots,p).$$

Since $\mathrm{Lk}(8,\ldots,p) \subset K_{3,5}$, we conclude that

$$\mathrm{Lk}(8,\ldots,p) \setminus \mathrm{St}(p+1,\ldots,p+7) = \{[0,3]\}.$$

Finally,

$$\mathcal{C}(K_\emptyset, V) = \mathcal{C}(\mathrm{St}(p+1,\dots,p+7), C) \cup \{[0,3]\} = \{[0],[0,3]\},$$

so V is an acyclic discrete vector field on $K_\emptyset$ without critical 2-simplices, with one critical 1-simplex and one critical 0-simplex.

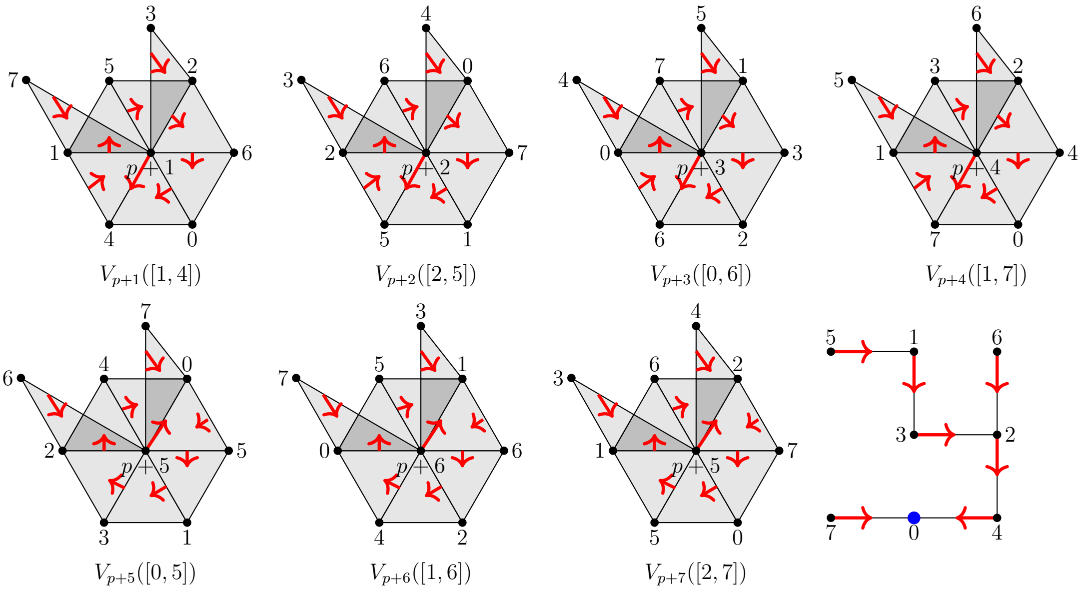


FIGURE 7. Gradient vector field C

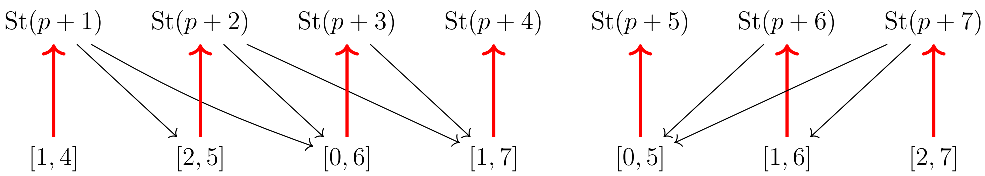


FIGURE 8. Digraph Flow(C)

Case 2: $p \equiv 2 \pmod{15}$.

Again, we look for an acyclic discrete vector field on $\mathrm{St}(p+1,\dots,p+7)$ such that the number of critical simplices are as small as possible. The subcomplex $\mathrm{St}(p+1,\dots,p+7)$ consists of facets:

$[0,7,p+1]$,	$[1,3,p+2]$,	$[2,4,p+3]$,	$[0,5,p+4]$,	$[1,6,p+5]$,	$[2,7,p+6]$,	$[0,3,p+7]$,
$[2,4,p+1]$,	$[0,5,p+2]$,	$[1,6,p+3]$,	$[2,7,p+4]$,	$[0,3,p+5]$,	$[1,4,p+6]$,	$[2,5,p+7]$,
$[1,6,p+1]$,	$[2,7,p+2]$,	$[0,3,p+3]$,	$[1,4,p+4]$,	$[2,5,p+5]$,	$[0,6,p+6]$,	$[1,7,p+7]$,
$[0,3,p+1]$,	$[1,4,p+2]$,	$[2,5,p+3]$,	$[0,6,p+4]$,	$[1,7,p+5]$,	$[2,3,p+6]$,	$[0,4,p+7]$,
$[2,5,p+1]$,	$[0,6,p+2]$,	$[1,7,p+3]$,	$[2,3,p+4]$,	$[0,4,p+5]$,	$[1,5,p+6]$,	$[2,6,p+7]$,
$[1,7,p+1]$,	$[2,3,p+2]$,	$[0,4,p+3]$,	$[1,5,p+4]$,	$[2,6,p+5]$,	$[0,7,p+6]$,	$[1,3,p+7]$,
$[0,4,p+1]$,	$[1,5,p+2]$,	$[2,6,p+3]$,	$[0,7,p+4]$,	$[1,3,p+5]$,	$[2,4,p+6]$,	$[0,5,p+7]$,
$[2,6,p+1]$,	$[0,7,p+2]$,	$[1,3,p+3]$,	$[2,4,p+4]$,	$[0,5,p+5]$,	$[1,6,p+6]$,	$[2,7,p+7]$.

Consider a discrete vector field

$$\begin{aligned} C =& V_{p+1}([0,7]) \cup V_{p+2}([2,3]) \cup V_{p+3}([0,4]) \cup V_{p+4}([1,5]) \\ & \cup V_{p+5}([2,5]) \cup V_{p+6}([0,6]) \cup V_{p+7}([1,7]) \\ & \cup \{\{[0],[0,3]\},\{[5],[0,5]\},\{[3],[1,3]\},\{[4],[1,4]\},\{[1],[1,6]\},\{[6],[2,6]\}, \\ & \{[7],[2,7]\}\} \end{aligned}$$

on $\mathrm{St}(p+1,\ldots,p+7)$. This discrete vector field is well-defined and acyclic. Namely, there are no non-trivial closed C-paths consisting of 0-simplices and 1-simplices (see Figure 9) and graph $\mathrm{Flow}(C)$ is acyclic (see Figure 10).

It follows that $\mathcal{C}(\mathrm{St}(p+1,\ldots,p+7),C) = \{[2],[2,4]\}$. Additionally,

$$K_{3,5} \subset \mathrm{St}(p+1,\ldots,p+7).$$

As $\mathrm{Lk}(8,\ldots,p) \subset K_{3,5}$, we can finally conclude that

$$\mathcal{C}(K_\emptyset, V) = \mathcal{C}(\mathrm{St}(p+1,\ldots,p+7),C) = \{[2],[2,4]\}.$$

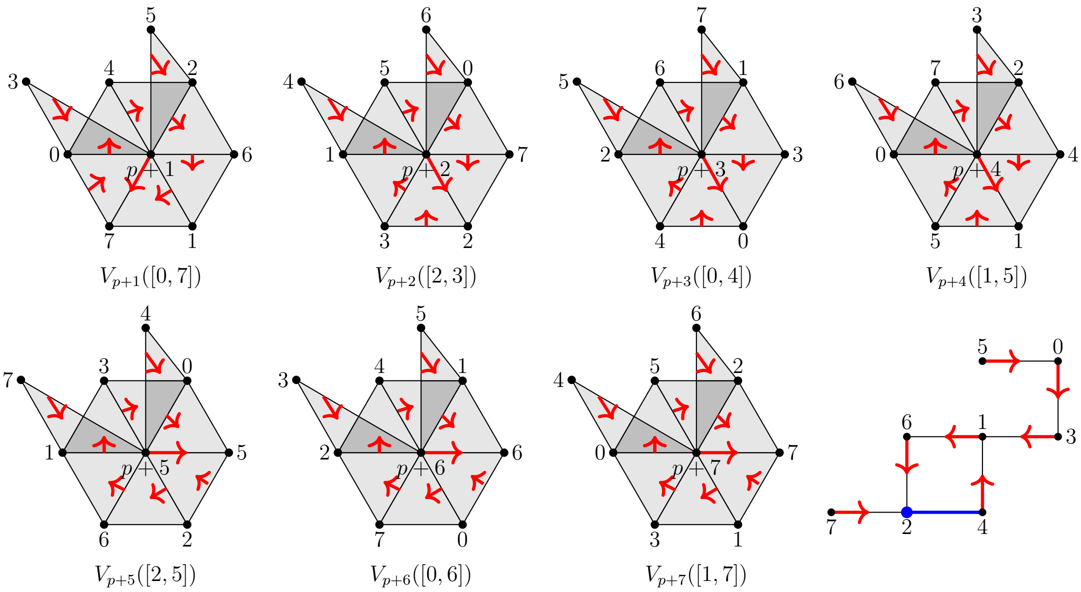


FIGURE 9. Gradient vector field C

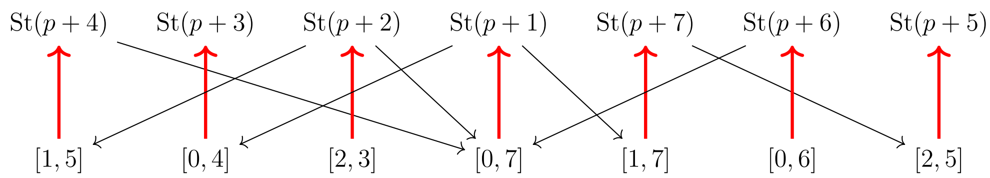


FIGURE 10. Digraph $\mathrm{Flow}(C)$

Case 3: $p \equiv 13 \pmod{15}$.

The subcomplex $\mathrm{St}(p+1,\dots,p+7)$ consists of the following 2-simplices:

$$\begin{array}{ccccccc}
[1,5,p+1]\,, & [2,6,p+2]\,, & [0,7,p+3]\,, & [1,3,p+4]\,, & [2,4,p+5]\,, & [0,5,p+6]\,, & [1,6,p+7]\,, \\
[2,3,p+1]\,, & [0,4,p+2]\,, & [1,5,p+3]\,, & [2,6,p+4]\,, & [0,7,p+5]\,, & [1,3,p+6]\,, & [2,4,p+7]\,, \\
[0,6,p+1]\,, & [1,7,p+2]\,, & [2,3,p+3]\,, & [0,4,p+4]\,, & [1,5,p+5]\,, & [2,6,p+6]\,, & [0,7,p+7]\,, \\
[1,4,p+1]\,, & [2,5,p+2]\,, & [0,6,p+3]\,, & [1,7,p+4]\,, & [2,3,p+5]\,, & [0,4,p+6]\,, & [1,5,p+7]\,, \\
[2,7,p+1]\,, & [0,3,p+2]\,, & [1,4,p+3]\,, & [2,5,p+4]\,, & [0,6,p+5]\,, & [1,7,p+6]\,, & [2,3,p+7]\,, \\
[0,5,p+1]\,, & [1,6,p+2]\,, & [2,7,p+3]\,, & [0,3,p+4]\,, & [1,4,p+5]\,, & [2,5,p+6]\,, & [0,6,p+7]\,, \\
[1,3,p+1]\,, & [2,4,p+2]\,, & [0,5,p+3]\,, & [1,6,p+4]\,, & [2,7,p+5]\,, & [0,3,p+6]\,, & [1,4,p+7]\,, \\
[2,6,p+1]\,, & [0,7,p+2]\,, & [1,3,p+3]\,, & [2,4,p+4]\,, & [0,5,p+5]\,, & [1,6,p+6]\,, & [2,7,p+7]\,.
\end{array}$$

Let

$$\begin{aligned}
C =&V_{p+1}([0,6]) \cup V_{p+2}([1,7]) \cup V_{p+3}([2,3]) \cup V_{p+4}([0,4]) \\
&\cup V_{p+5}([1,4]) \cup V_{p+6}([2,5]) \cup V_{p+7}([1,6]) \\
&\cup \{\{[5],[0,5]\},\{[7],[0,7]\},\{[3],[1,3]\},\{[1],[1,5]\},\{[4],[2,4]\},\{[6],[2,6]\}, \\
&\{[2],[2,7]\}\}
\end{aligned}$$

be a discrete vector field on $\mathrm{St}(p+1,\dots,p+7)$. As each simplex is in at most one pair, this discrete vector field is well-defined. Additionally, C is acyclic on $\mathrm{St}(p+1,\dots,p+7)$ and such that

$$\mathcal{C}(\mathrm{St}(p+1,\dots,p+7),C) = \{[0],[0,3]\}.$$

Namely, corresponding digraph $\mathrm{Flow}(C)$ is acyclic (see Figure 12), so there are no non-trivial closed C-paths which consist of 1-simplices and 2-simplices. Figure 11 shows that there are no non-trivial closed C-paths consisting of 0-simplices and 1-simplices as well.

Like in the previous case, $K_{3,5} \subset \mathrm{St}(p+1,\dots,p+7)$, and consequently,

$$\mathrm{Lk}(8,\dots,p) \subset \mathrm{St}(p+1,\dots,p+7),$$

because $\mathrm{Lk}(8,\dots,p) \subset K_{3,5}$. According to this, all critical simplices in $K_\emptyset$ with recpect to V are in $\mathrm{St}(p+1,\dots,p+7)$. Hence, we conclude

$$\mathcal{C}(K_\emptyset,V) = \mathcal{C}(\mathrm{St}(p+1,\dots,p+7),C) = \{[0],[0,3]\}.$$

Case 4: $p \equiv 14 \pmod{15}$.

The subcomplex $\mathrm{St}(p+1,\dots,p+7)$ is generated by facets:

$$\begin{array}{ccccccc}
[0,3,p+1]\,, & [1,4,p+2]\,, & [2,5,p+3]\,, & [0,6,p+4]\,, & [1,7,p+5]\,, & [2,3,p+6]\,, & [0,4,p+7]\,, \\
[2,7,p+1]\,, & [0,3,p+2]\,, & [1,4,p+3]\,, & [2,5,p+4]\,, & [0,6,p+5]\,, & [1,7,p+6]\,, & [2,3,p+7]\,, \\
[1,6,p+1]\,, & [2,7,p+2]\,, & [0,3,p+3]\,, & [1,4,p+4]\,, & [2,5,p+5]\,, & [0,6,p+6]\,, & [1,7,p+7]\,, \\
[0,5,p+1]\,, & [1,6,p+2]\,, & [2,7,p+3]\,, & [0,3,p+4]\,, & [1,4,p+5]\,, & [2,5,p+6]\,, & [0,6,p+7]\,, \\
[2,4,p+1]\,, & [0,5,p+2]\,, & [1,6,p+3]\,, & [2,7,p+4]\,, & [0,3,p+5]\,, & [1,4,p+6]\,, & [2,5,p+7]\,, \\
[1,3,p+1]\,, & [2,4,p+2]\,, & [0,5,p+3]\,, & [1,6,p+4]\,, & [2,7,p+5]\,, & [0,3,p+6]\,, & [1,4,p+7]\,, \\
[0,7,p+1]\,, & [1,3,p+2]\,, & [2,4,p+3]\,, & [0,5,p+4]\,, & [1,6,p+5]\,, & [2,7,p+6]\,, & [0,3,p+7]\,, \\
[2,6,p+1]\,, & [0,7,p+2]\,, & [1,3,p+3]\,, & [2,4,p+4]\,, & [0,5,p+5]\,, & [1,6,p+6]\,, & [2,7,p+7]\,.
\end{array}$$

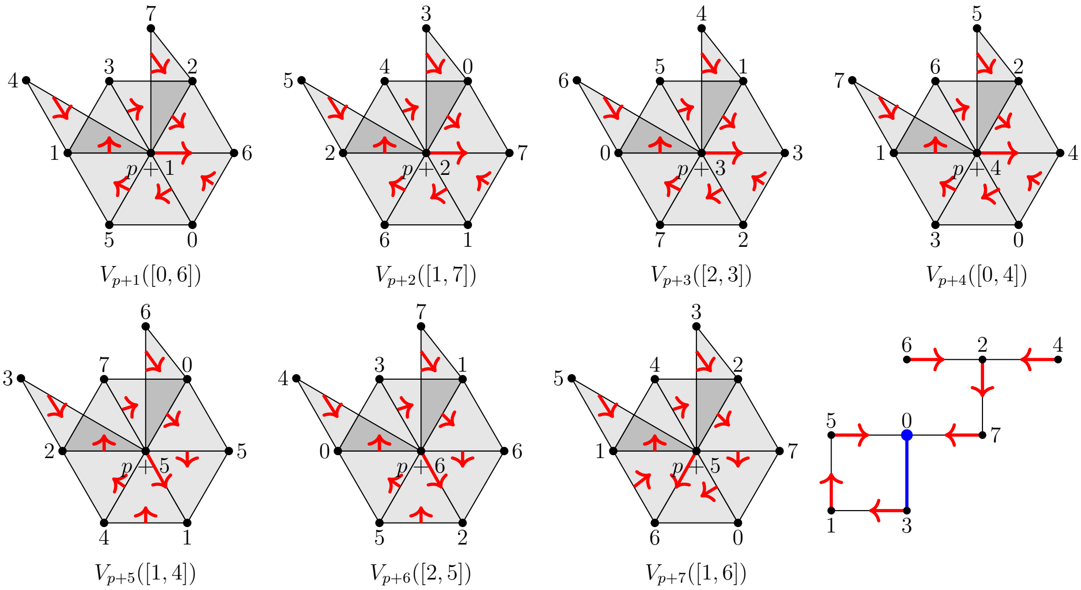


FIGURE 11. Gradient vector field C

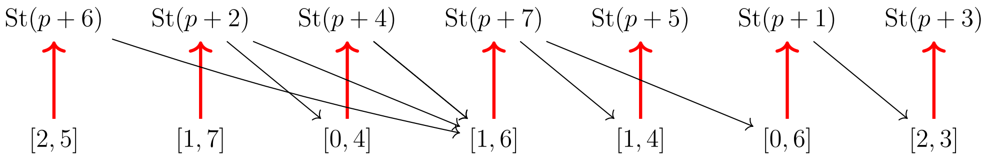


FIGURE 12. Digraph Flow(C)

Let us define discrete vector field on $\mathrm{St}(p+1,\dots,p+7)$ as it follows

$$\begin{aligned} C =& V_{p+1}([2,6] \cup V_{p+2}([0,7]) \cup V_{p+3}([1,3]) \cup V_{p+4}([2,4]) \\ &\cup V_{p+5}([1,7]) \cup V_{p+6}([2,3]) \cup V_{p+7}([0,4]) \\ &\cup \{\{[3],[0,3]\},\{[0],[0,5]\},\{[6],[0,6]\},\{[4],[1,4]\},\{[1],[1,6]\}, \\ &\{[2],[2,5]\},\{[7],[2,7]\}\}. \end{aligned}$$

Discrete vector field C is well-defined (see Figure 13). Figure 14 shows that digraph Flow(C) is acyclic. In addition, Figure 13 shows that there are no non-trivial closed C-paths consisting of 0-simplices and 1-simplices. Therefore, C is acyclic on $\mathrm{St}(p+1,\dots,p+7)$ and $\mathcal{C}(\mathrm{St}(p+1,\dots,p+7),V) = \{[5]\}$.

It follows that

$$K_{3,5} \setminus \mathrm{St}(p+1,\dots,p+7) = \{[1,5]\}.$$

As $[1,5] \in F_{8p \pmod n} \subset \mathrm{St}(8)$ we conclude that $[1,5] \in \mathrm{St}(8) \cap K_{3,5} = \mathrm{Lk}(8)$. Hence,

$$\mathrm{Lk}(8,\dots,p) \setminus \mathrm{St}(p+1,\dots,p+7) = [1,5].$$

From the previous considerations, we conclude

$$\mathcal{C}(K_\emptyset,V) = \mathcal{C}(\mathrm{St}(p+1,\dots,p+7),C) \cup \{[1,5]\} = \{[5],[1,5]\}.$$

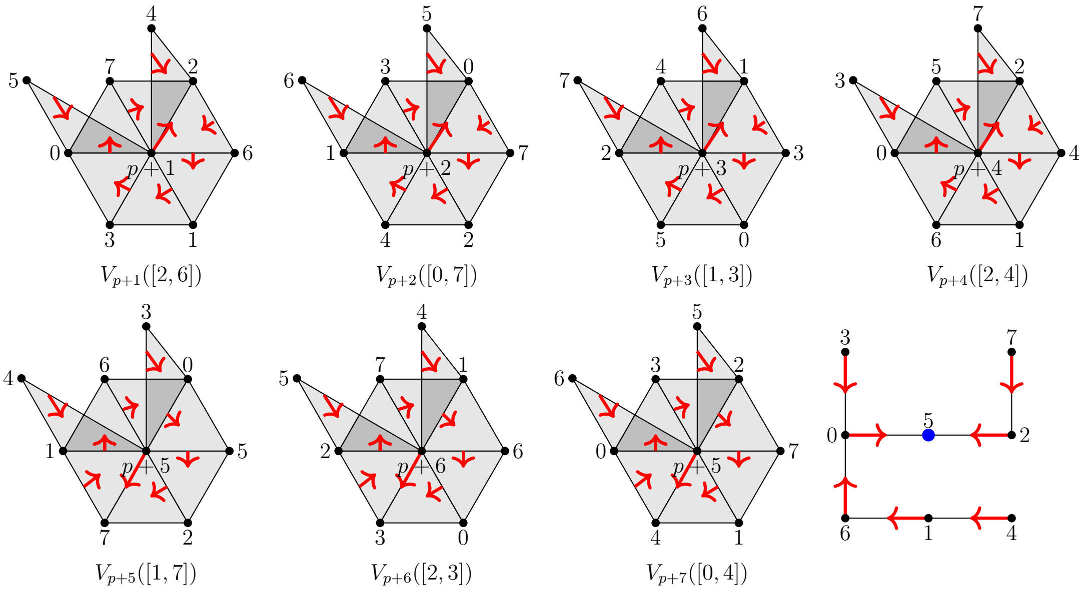


FIGURE 13. Gradient vector field C

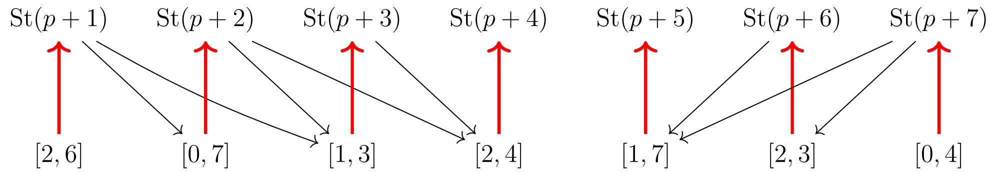


FIGURE 14. Digraph Flow(C)

$\square$

5. Proof of Theorem 1.4

In order to prove Theorem 1.4, we will need next theorem which points out an interesting feature of the complex $K_\emptyset$. Namely, under some conditions, complex $K_\emptyset$ is completely determined by its subcomplex $\mathrm{St}(p+1,\dots,p+7)$.

Theorem 5.1. *Let $n = 3\cdot 5\cdot p$, where $p \geq 7$ is a prime and $p \equiv k \pmod{15}$. If $k \in \{2,4,7,8,11,13\}$, then complex $K_\emptyset$ is homotopy equivalent to its subcomplex* $\mathrm{St}(p+1,\dots,p+7)$.

Proof. Obviously, $K_\emptyset = \mathrm{St}(p+1,\dots,p+7)$ when $p=7$, therefore we consider $p>7$. Let C be an acyclic discrete vector field on $\mathrm{St}(p+1,\dots,p+7)$. Then, by Lemma 3.1, $V = (\bigcup_{i=8}^{p} S_i)\cup C$ is an acyclic vector field on $K_\emptyset$. In order to prove this theorem, we show that there are no critical simplices in $K_\emptyset \setminus \mathrm{St}(p+1,\dots,p+7)$ with respect to V and $\mathrm{St}(p+1,\dots,p+7) \not\rightarrow K_\emptyset \setminus \mathrm{St}(p+1,\dots,p+7)$. Then the theorem follows by Theorem 2.3.

Recall that S_i is an acyclic vector field on $\mathrm{St}(i)$ for $i \in \{8,\dots,p\}$ (see Figure 2). Additionally, all 1-simplices from the set $\mathrm{Lk}(i)$, $i \in \{8,\dots,p\}$, are unmatched with respect to S_i. Consequently,

$$\mathrm{St}(p+1,\dots,p+7) \not\to K_\emptyset \setminus \mathrm{St}(p+1,\dots,p+7).$$

It follows that $\alpha \in K_\emptyset \setminus \mathrm{St}(p+1,\dots,p+7)$ is critical with respect to V, when $\alpha \in \mathrm{Lk}(i) \setminus \mathrm{St}(p+1,\dots,p+7)$ for some $i \in \{8,\dots,p\}$. However, $\mathrm{St}(p+1,\dots,p+7)$ is built of facets $F_{j \pmod{15p}}$, where $j \in \{dp-i\}_{d=\overline{8,15},\, i=\overline{1,7}}$. There are 15 numbers which are distinct modulo 15 between numbers $\{dp-i\}_{j=\overline{8,15},\, i=\overline{1,7}}$ when $k \in \{2,4,7,8,11,13\}$ (see Table 1). Thus, it follows that

$$K_{3,5} \subset \mathrm{St}(p+1,\dots,p+7).$$

TABLE 1

k	15 numbers among $\{dp-i\}_{j=\overline{8,15},\, i=\overline{1,7}}$ which are distinct modulo 15
2	$8p-1$, $9p-2$, $9p-1$, $10p-2$, $10p-1$, $11p-2$, $11p-1$, $12p-2$,$12p-1$, $8p-7$, $8p-6$, $8p-5$, $8p-4$, $8p-3$, $8p-2$
4	$8p-2$, $8p-1$, $9p-4$, $9p-3$, $9p-2$, $9p-1$, $10p-4$, $10p-3$, $10p-2$, $10p-1$, $8p-7$, $8p-6$, $8p-5$, $8p-4$, $8p-3$
7	$9p-3$, $9p-2$, $9p-1$, $10p-7$, $8p-7$, $8p-6$, $8p-5$, $8p-4$, $8p-3$, $8p-2$, $8p-1$, $9p-7$, $9p-6$, $9p-5$, $9p-4$
8	$8p-3$, $8p-2$, $8p-1$, $10p-1$, $9p-7$, $9p-6$, $9p-5$, $9p-4$, $9p-3$, $9p-2$, $9p-1$, $8p-7$, $8p-6$, $8p-5$, $8p-4$
11	$10p-5$, $10p-4$, $9p-7$, $9p-6$, $9p-5$, $9p-4$, $8p-7$, $8p-6$, $8p-5$, $8p-4$, $8p-3$, $8p-2$, $8p-1$, $10p-7$, $10p-6$
13	$12p-6$, $11p-7$, $11p-6$, $10p-7$, $10p-6$, $9p-7$, $9p-6$, $8p-7$, $8p-6$, $8p-5$, $8p-4$, $8p-3$, $8p-2$, $8p-1$, $12p-7$

As $\mathrm{Lk}(i) \subset K_{3,5}$, we can conclude that

$$\mathrm{Lk}(i) \subseteq \mathrm{St}(p+1,\dots,p+7),$$

for all $i \in \{8,\dots,p\}$. Therefore, there are no simplices in $K_\emptyset \setminus \mathrm{St}(p+1,\dots,p+7)$ which are critical with respect to V. $\square$

Remark 5.1. It is not always true that $K_{3,5} \subset \mathrm{St}(p+1,\dots,p+7)$. Namely, if $p \equiv 1 \pmod{15}$ then $[0,3] \in F_{15p \pmod{n}} \subset \mathrm{St}(8)$ and $[0,3] \notin \mathrm{St}(p+1,\dots,p+7)$ (see Case 1 in the proof of Theorem 1.3). Therefore,

$$[0,3] \in \mathrm{Lk}(8) \setminus \mathrm{St}(p+1,\dots,p+7) \subseteq K_{3,5} \setminus \mathrm{St}(p+1,\dots,p+7).$$

Similarly, when $p \equiv 14 \pmod{15}$, it follows that $[1,5] \in F_{8p \pmod{n}} \subset \mathrm{St}(8)$. On the other hand, $[1,5] \notin \mathrm{St}(p+1,\dots,p+7)$ (see Case 4 in the proof of Theorem 1.3), so it follows that

$$[1,5] \in \mathrm{Lk}(8) \setminus \mathrm{St}(p+1,\dots,p+7) \subseteq K_{3,5} \setminus \mathrm{St}(p+1,\dots,p+7).$$

We are now prove Theorem 1.4.

Proof of Theorem 1.4. According to the above theorem and consideration from the beginning of the previous section, it is enough to examine the smallest possible cases: $p = 19$, $p = 7$, $p = 23$ and $p = 11$.

In [8], it was calculated that $\pi_1(K_\emptyset) = \langle a, b \mid ab^2a^{-1}b^{-2}a^{-1}bab^{-1}a^{-1}b^{-1}\rangle$ when $p = 7$. Further, it was proved that $\pi_1(K_\emptyset)$ is not commutative, and consequently, $K_\emptyset \not\simeq \mathbb{S}^1$. Here, we will use a similar idea for the remaining three cases.

Using Algorithm 1 from [8] for computing the fundamental group, for the listed maximal spanning trees, we obtain the following results.

p	maximal tree	$\pi_1(K_\emptyset)$
19	{[0,3], [0,4], [0,5], [0,6], [0,7], [0,8], [0,9], [0,10], [0,11], [0,12], [0,13], [0,14], [0,15], [0,16], [0,17], [0,18], [0,19], [0,20], [0,21], [0,22], [0,23], [0,24], [0,25], [0,26], [1,3], [2,3]}	$\langle a, b \mid a^{-1}b^{-1}a^{-2}b^{-1}ab = 1\rangle$
23	{[2,3], [2,4], [2,5], [2,6], [2,7], [2,8], [2,9], [2,10], [2,11], [2,12], [2,13], [2,14], [2,15], [2,16], [2,17], [2,18], [2,19], [2,20], [2,21], [2,22], [2,23], [2,24], [2,25], [2,26], [2,27], [2,28], [2,29], [2,30], [0,3], [1,3]}	$\langle a, b \mid b^{-1}a^{-1}b^2ab^{-1}aba^{-1}b^{-1}a = 1\rangle$
11	{[0,3], [0,4], [0,5], [0,6], [0,7], [0,8], [0,9], [0,10], [0,11], [0,12], [0,13], [0,14], [0,15], [0,16], [0,17], [0,18], [1,3], [2,3]}	$\langle a, b \mid a^{-1}bab^{-2} = 1\rangle$

Note that $\langle a, b \mid a^{-1}b^{-1}a^{-2}b^{-1}ab = 1\rangle$ and $\langle a, b \mid a^{-1}bab^{-2} = 1\rangle$ are distinct presentations of the same group. Namely, starting with $\langle a, b \mid a^{-1}b^{-1}a^{-2}b^{-1}ab = 1\rangle$ and letting $x = a^{-1}b^{-1}a^{-1}, y = a^{-1}$ we obtain new presentation $\langle x, y \mid x^2y^{-1}x^{-1}y = 1\rangle$ for the same group. This group is Baumslag-Solitar group $\mathrm{BS}(1, 2)$ (for more details see [2]). The group $\mathrm{BS}(1, 2)$ is not commutative. Namely, we can define an epimorphism $f : \mathrm{BS}(1, 2) \to \mathrm{S}_3$, such that $f(x) = (123)$, $f(y) = (23)$. Since permutation group S_3 is not commutative, $\mathrm{BS}(1, 2)$ cannot be commutative. Consequently, $\pi_1(K_\emptyset) \not\simeq \mathrm{S}^1$ when $p = 11$ and $p = 19$.

Now, we consider $\pi_1(K_\emptyset)$ when $p = 23$. Letting $x = b^{-1}a^{-1}$, relation

$$b^{-1}a^{-1}b^2ab^{-1}aba^{-1}b^{-1}a = 1$$

transforms into relation $xb^2x^{-1}b^{-2}x^{-1}bxb^{-1}x^{-1}b^{-1} = 1$. Thus, $\pi_1(K_\emptyset)$ is the same group for $p = 7$ and $p = 23$. Therefore, $K_\emptyset \not\simeq \mathbb{S}^1$ when $p = 23$. □

References

[1] R. M. Adin, *Counting colorful multi-dimensional trees*, Combinatorica **12** (1992), 247–260. `https://doi.org/10.1007/BF01285814`

[2] G. Baumslag and D. Solitar, *Some two-generator one-relator non-Hopfian groups*, Bull. Amer. Math. Soc. **68**(3) (1962), 199–201. `https://doi.org/10.1090/S0002-9904-1962-10745-9`

[3] E. D. Bolker, *Simplicial geometry and transportation polytopes*, Trans. Amer. Math. Coc. **217** (1976), 121–142. `https://doi.org/10.2307/1997562`
[4] R. Forman, *Morse theory for cell complexes*, Adv. Math. **134**(1) (1998), 90–145. `https://doi.org/10.1006/aima.1997.1650`
[5] R. Forman, *A user's guide to discrete Morse theory*, Sém. Lothar. Combin. **48** (2002), Article ID B48c, 35 pages.
[6] J. Jonsson, *Simplicial Complexes of Graphs*, Lecture Notes in Mathematics **1928**, Springer-Verlag, Berlin, 2008.
[7] G. Kalai, *Enumeration of Q-acyclic simplicial complexes*, Israel J. Math. **45** (1983), 337–351. `https://doi.org/10.1007/BF02804017`
[8] A. Kostić, N. Milošević and Z. Z. Petrović, *Note on the Cyclotomic Polynomial Topologically*, Exp. Math. (to appear). `https://doi.org/10.1080/10586458.2019.1680464`
[9] G. Musiker and V. Reiner, *The cyclotomic polynomial topologically*, J. Reine Angew. Math. **687** (2014), 113–132. `https://doi.org/10.1515/crelle-2012-0051`
[10] J. R. Munkres, *Elements of Algebraic Topology*, The Benjamin/Cummings Publishing Company, Menlo Park, California 1984.

[1]Department of Algebra and Mathematical Logic,
Faculty of Mathematics,
University of Belgrade,
Studentski trg 16, Belgrade
Email address: `alex@matf.bg.ac.rs`