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LIGHTLIKE HYPERSURFACES IN SEMI-RIEMMANIAN MANIFOLDS ADMITTING AFFINE CONFORMAL VECTOR FIELDS

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Abstract. Lightlike hypersurfaces with integrable screen distributions are very important as far as lightlike geometry is concerned. They include, among others, screen conformal and screen totally umbilic ones. In this paper, we show that any lightlike hypersurface of a semi-Riemannian manifold admitting a certain closed affine conformal vector field has an integrable screen distribution. Several examples are furnished in support of the main results.

1. Introduction

Lightlike submanifolds are very important and their numerous applications, particularly to mathematical physics-like in general relativity and electromagnetism motivated the study of lightlike geometry in semi-Riemannian manifolds. More precisely, lightlike submanifolds have been shown to represent different black hole horizons (see [3] and [4] for details). Among other motivations for investing in lightlike geometry by many physicists is the idea that the universe we are living in can be viewed as a 4-dimensional hypersurface embedded in $(4 + m)$-dimensional spacetime manifold, where m is any arbitrary integer. There are significant differences between lightlike geometry and Riemannian geometry as shown in [3] and [4], and many more references therein. Some of the pioneering work on this topic is due to Duggal-Bejancu [3], Duggal-Sahin [4] and Kupeli [15]. It is upon those books that many other researchers, including but not limited to [1, 5, 7–10, 12, 13] have extended their theories.

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Although a lot has been done on the geometry of lightlike submanifolds of semi-Riemannian manifolds, we remark that very little, see [3, page 259], efforts has been dedicated towards understanding what affine conformal vector fields, on semi-Riemmanian manifolds, can offer as far as characterising lightlike hypersurfaces. The present paper is directed towards achieving a characterisation of lightlike hypersurfaces in such spaces. The paper is arranged as follows. In Section 2, we quote some basic notions required in the rest of the paper. In Section 3, we prove some preliminary results on affine conformal vector fields, and Section 4 is dedicated to the main results of the study.

2. Preliminaries

An $(n+2)$-dimensional Lorentzian manifold $\overline{M}$ is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric $\overline{g}$, that is, $\overline{M}$ admits a smooth tensor field $\overline{g}$ of type $(0,2)$ such that, for each point $p \in \overline{M}$, the tensor $\overline{g}_p : T_p\overline{M} \times T_p\overline{M} \longrightarrow \mathbb{R}$ is a *non-degenerate* inner product of signature $(-,+,\ldots,+)$, where $T_p\overline{M}$ denotes the tangent vector space of $\overline{M}$ at p and $\mathbb{R}$ is the real number space. A non-zero vector field $v \in T_p\overline{M}$ is said to be timelike (resp., non-spacelike, null and spacelike) if it satisfies $\overline{g}_p(v,v) < 0$ (resp., ≤ 0, $= 0$ and > 0) [11]. Let $(\overline{M},\overline{g})$ be a $(n+2)$-dimensional semi-Riemannian manifold and let M be a hypersurface of $\overline{M}$. Let g be the induced tensor field by $\overline{g}$ on M. Then, M is called a *lightlike hypersurface* of $\overline{M}$ if g is of constant rank n [3]. Consider the vector bundle $TM^{\perp}$ whose fibers are defined by $T_xM^{\perp} = \{Y_x \in T_x\overline{M} : \overline{g}_x(X_x, Y_x) = 0 \text{ for all } X_x \in T_xM\}$, for any $x \in M$. Hence, a hypersurface M of $\overline{M}$ is lightlike if and only if $TM^{\perp}$ is a distribution of rank 1 on M. Let M be a lightlike hypersurface. We consider the complementary distribution $S(TM)$ to $TM^{\perp}$ in TM, which is called a *screen distribution*. It is well-known that $S(TM)$ is non-degenerate (see [3]). Thus, we have the decomposition $TM = S(TM) \perp TM^{\perp}$.

As $S(TM)$ is non-degenerate with respect to $\overline{g}$, we have $T\overline{M} = S(TM) \perp S(TM)^{\perp}$, where $S(TM)^{\perp}$ is the complementary vector bundle to $S(TM)$ in $T\overline{M}|_M$. Let (M,g) be a lightlike hypersurface of $(\overline{M},\overline{g})$. Then there exists a unique vector bundle $tr(TM)$, called the *lightlike transversal bundle* [3] of M with respect to $S(TM)$, of rank 1 over M such that for any non-zero section ξ of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique section N of $tr(TM)$ on $\mathcal{U}$ satisfying

$$\overline{g}(\xi, N) = 1, \quad \overline{g}(N,N) = \overline{g}(N,Z) = 0, \tag{2.1}$$

for any section Z of $S(TM)$. Consequently, we have the following decomposition of $T\overline{M}$

$$T\overline{M}|_M = S(TM) \perp \{TM^{\perp} \oplus tr(TM)\} = TM \oplus tr(TM).$$

Let ∇ and ∇^* denote the induced connections on M and $S(TM)$, respectively, and P be the projection of TM onto $S(TM)$, then the local Gauss-Weingarten equations of M and $S(TM)$ are the following [3]

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \quad \overline{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.2}$$

$$\nabla_X PY = \nabla^*_X PY + C(X,PY)\xi, \quad \nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{2.3}$$

for all $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(TM^\perp)$ and $N \in \Gamma(tr(TM))$, where $\overline{\nabla}$ is the Levi-Civita connection on $\overline{M}$. In the above setting, B is the local second fundamental form of M and C is the local second fundamental form on $S(TM)$. A_N and A^*_ξ are the shape operators on TM and $S(TM)$ respectively, while τ is a 1-form on TM. The above shape operators are related to their local fundamental forms by

$$B(\xi, X) = 0, \quad g(A^*_\xi X, Y) = B(X,Y), \quad g(A_N X, PY) = C(X, PY),$$

for any $X, Y \in \Gamma(TM)$. Moreover, $\overline{g}(A^*_\xi X, N) = 0$ and $\overline{g}(A_N X, N) = 0$ for all $X \in \Gamma(TM)$. From these relations, we notice that A^*_ξ and A_N are both screen-valued operators. Moreover, it is easy to show that

$$(\nabla_X g)(Y,Z) = B(X,Y)\theta(Z) + B(X,Z)\theta(Y), \tag{2.4}$$

for all $X, Y, Z \in \Gamma(TM)$. Consequently, ∇ is generally *not* a metric connection with respect to g. However, the induced connection ∇^* on $S(TM)$ is a metric connection.

A lightlike hypersurface $(M, g, S(TM))$ of a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is *screen conformal* [4, Definition 2.2.1, p. 51] if the shape operators A_N and A^*_ξ of M and $S(TM)$, respectively, are related by $A_N = \psi A^*_\xi$, where ψ is a non-vanishing smooth function on a neighbourhood $\mathcal{U}$ in M. In particular, if ψ is a non-zero constant, M is called *screen homothetic*. When A_N and A^*_ξ are instead linked by $A_N = \psi_1 A^*_\xi + \psi_2 P$, for some smooth functions ψ_1 and ψ_2, then M is called *quasi screen conformal* [12]. It is easy to see that a quasi screen conformal lightlike hypersurface is screen conformal when $\psi_2 \equiv 0$. A semi-Riemannian manifold $(\overline{M}, \overline{g})$ of constant sectional curvature c is called a semi-Riemannian space form (see [11, p. 80]) and denoted by $\overline{M}(c)$. The curvature tensor field $\overline{R}$ of $\overline{M}(c)$ is given by

$$\overline{R}(X,Y)Z = c\{\overline{g}(Y,Z)X - \overline{g}(X,Z)Y\}, \quad \text{for all } X, Y, Z \in \Gamma(T\overline{M}). \tag{2.5}$$

3. Some Basic Results

A smooth vector field V on a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is said to be an *affine conformal* vector (ACV) field if there exists a smooth function $\rho : \overline{M} \to \mathbb{R}$, called the *potential*, on $\overline{M}$ that satisfies

$$(\pounds_V \overline{\nabla})(X,Y) = (X\rho)Y + (Y\rho)X - \overline{g}(X,Y)\,\mathrm{grad}\,\rho, \tag{3.1}$$

where $\pounds_V$ is the Lie derivative with respect V and the affinity tensor $(\pounds_V \overline{\nabla})$ of V defined by

$$(\pounds_V \overline{\nabla})(X,Y) = \pounds_V \overline{\nabla}_X Y - \overline{\nabla}_{\pounds_L X} Y - \overline{\nabla}_X \pounds_Z Y,$$

for all $X, Y \in \Gamma(T\overline{M})$. In particular, V is an *affine* vector field if ρ is constant, that is if $\pounds_V \overline{\nabla} = 0$. The following result is well-known for an ACV field V.

Theorem 3.1 ([2,3])**.** *A vector field V on $\overline{M}$ is an ACV if and only if*

$$\pounds_V \overline{g} = 2\rho \overline{g}(X,Y) + K, \quad \overline{\nabla} K = 0, \tag{3.2}$$

where K is a covariant constant ($\overline{\nabla} K = 0$) symmetric and therefore, Killing tensor (abbreviated K-tensor) of second order.

A sub case is the conformal killing vector (CKV) when $K = 0$ and $\rho_{;a} = X_a \rho \neq 0$, $a = 0, 1, 2, \dots, n+1$. This also includes *homothetic* vector fields (HV) and *killing* vector fields (KV) when $\rho_{;a} = 0$ and $\rho = 0$, respectively. See [2, p. 276] or [3, p. 264], and many more references cited therein, for more details.

Example 3.1 (K. L. Duggal [2]). Let $\overline{M}$ be a four-dimensional Einstein static fluid spacetime with metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + (1-r^2)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}, \phi^2)$$

and the fluid 4-velocity vector $u^a = \delta^a_a$, $a = 0, 1, 2, 3$. This spacetime admits a CKV

$$V_1^a = (1-r^2)^{1/2} \cos t \delta^1_0 - r(1-r^2)^{1/2} \sin t \delta^a_0$$

and a proper affine vector $V_2^a = t\delta^a_0$. As the spacetime metric is reducible, the combination $V^a = V_1^a + V_2^a$ is a proper ACV [2, p. 279] such that

$$\begin{aligned} V =& (t + (1-r^2)^{1/2} \cos t)\delta^a_0 - r(1-r^2)^{1/2} \sin t \delta^a_1, \\ \rho =& -(1-r^2)^{1/2} \sin t, \quad K_{ab} = -2t_{;a} t_{;b}. \end{aligned}$$

Utilising Koszul's formula [11, Theorem 11, p. 61], we have

$$2\overline{g}(\overline{\nabla}_X V, Y) = (\pounds_V g)(X,Y) + d\eta(X,Y), \tag{3.3}$$

for all $X, Y \in \Gamma(T\overline{M})$, where η is the 1-form dual to V, that is, $\eta(X) = \overline{g}(V, X)$, $X \in \Gamma(T\overline{M})$. Define a skew symmetric tensor field φ of type $(1,1)$ on $\overline{M}$ by

$$d\eta(X,Y) = 2\overline{g}(\varphi X, Y), \tag{3.4}$$

for all $X, Y \in \Gamma(T\overline{M})$. The skew symmetric tensor field φ in the above equation is called the *associate tensor* field [6] of the affine conformal vector field V. We say that V is a *closed affine conformal vector* field if η is closed, that is $d\eta = 0$. Also, define a symmetric tensor field A_K of type $(1,1)$ on $\overline{M}$ by

$$K(X,Y) = \overline{g}(A_K X, Y), \tag{3.5}$$

for all $X, Y \in \Gamma(T\overline{M})$, where K is the symmetric $(0,2)$ tensor of Theorem 3.1. Then, using (3.2)–(3.5), and the fact that $\overline{g}$ is nondegenerate, we get the following result.

Lemma 3.1. *A vector field V on a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is an ACV field if and only if*

$$\overline{\nabla}_X V = \rho X + \frac{1}{2} A_K X + \varphi X \quad \textit{and} \quad (\overline{\nabla}_X A_K) = 0, \tag{3.6}$$

for all $X \in \Gamma(T\overline{M})$, where A_K and φ are tensor fields of type $(1,1)$ on $\overline{M}$, in which A_K is symmetric and φ is skew-symmetric.

Proof. From (3.2)–(3.5), we have

$$\overline{g}(\overline{\nabla}_X V, Y) = \rho \overline{g}(X, Y) + \frac{1}{2}\overline{g}(A_K X, Y) + \overline{g}(\varphi X, Y),$$

from which the first relation of (3.6) follows by utilising the fact that $\overline{g}$ is non-degenerate. On the other hand, using the second condition of (3.2), that is $\overline{\nabla} K = 0$, together with (3.5), we get

$$X\overline{g}(A_K Y, Z) = \overline{g}(A_K \overline{\nabla}_X Y, Z) + \overline{g}(A_K Y, \overline{\nabla}_X Z), \tag{3.7}$$

for any $X, Y, Z \in \Gamma(T\overline{M})$. As $\overline{\nabla}$ is Levi-Civita, it then follows from (3.7) that

$$\overline{g}(\overline{\nabla}_X A_K Y, Z) + \overline{g}(A_K Y, \overline{\nabla}_X Z) = \overline{g}(A_K \overline{\nabla}_X Y, Z) + \overline{g}(A_K Y, \overline{\nabla}_X Z),$$

from which one gets

$$\overline{g}((\overline{\nabla}_X A_K)Y, Z) = 0, \quad \text{for all } X, Y, Z \in \Gamma(T\overline{M}). \tag{3.8}$$

Then (3.8) shows that $(\overline{\nabla} A_K) = 0$, as $\overline{g}$ is non-degenerate, which proves the second relation in (3.6), and completing the proof. □

Lemma 3.2. *Let V be an ACV field on a semi-Riemannian manifold $(\overline{M}, \overline{g})$, then the covariant derivative of φ satisfies*

$$(\overline{\nabla}_X \varphi)(Y) = \overline{R}(X, V)Y + (Y\rho)X - \overline{g}(X, Y)\,\mathrm{grad}\,\rho, \tag{3.9}$$

where $(\overline{\nabla}_X \varphi)(Y) = \overline{\nabla}_X \varphi Y - \varphi \overline{\nabla}_X Y$ for any $X, Y \in \Gamma(T\overline{M})$.

Proof. Note, from (3.4), that the smooth 2-form $\overline{g}(\varphi X, Y)$ is closed. Thus, a direct calculation gives

$$\overline{g}((\overline{\nabla}_X \varphi)(Y), Z) + \overline{g}((\overline{\nabla}_Y \varphi)(Z), X) + \overline{g}((\overline{\nabla}_Z \varphi)(X), Y) = 0, \tag{3.10}$$

for all $X, Y, Z \in \Gamma(T\overline{M})$. Then, using Lemma 3.1, we derive

$$\begin{aligned} \overline{R}(X, Y)V &= (X\rho)Y - (Y\rho)X + \frac{1}{2}(\overline{\nabla}_X A_K)Y - \frac{1}{2}(\overline{\nabla}_Y A_K)X + (\overline{\nabla}_X \varphi)Y - (\overline{\nabla}_Y \varphi)X \\ &= (X\rho)Y - (Y\rho)X + (\overline{\nabla}_X \varphi)Y - (\overline{\nabla}_Y \varphi)X, \end{aligned} \tag{3.11}$$

in which we have used the fact that $\overline{\nabla} A_K = 0$ (see second relation of (3.6)). Substituting (3.11) in (3.10) and noting that $\overline{\nabla}\varphi$ is skew-symmetric, we get

$$\overline{g}(\overline{R}(X, Y)V - (X\rho)Y + (Y\rho)X, Z) + \overline{g}((\overline{\nabla}_Z \varphi)X, Y) = 0,$$

which reduces to

$$\overline{g}(\overline{R}(Z, V)X + (X\rho)Z - \overline{g}(X, Z)\mathrm{grad}\rho - (\overline{\nabla}_Z \varphi)X, Y) = 0, \tag{3.12}$$

for all $X, Y, Z \in \Gamma(T\overline{M})$. Finally, our result follows from (3.12) using the non-degeneracy of $\overline{g}$, which completes the proof. □

Lemma 3.3. *Let V be an ACV on a semi-Riemannian manifold $(\overline{M}, \overline{g})$. Then Hessian of the function $\alpha := \overline{g}(V, V)$ is given by*

$$\begin{aligned}\mathrm{Hess}_\alpha(X, Y) =& -2\overline{g}(\overline{R}(X, V)V, Y) - 2(V\rho)\overline{g}(X, Y)\\ &+ 2\overline{g}(\overline{\nabla}_X V, \overline{\nabla}_Y V) + 2(X\rho)\eta(Y) + 2(Y\rho)\eta(X),\end{aligned} \tag{3.13}$$

for all $X, Y \in \Gamma(T\overline{M})$.

Proof. By virtue of (3.6), we derive

$$\begin{aligned}\overline{g}(\overline{\nabla}_X \overline{\nabla}_Y V - \overline{\nabla}_{\overline{\nabla}_X Y} V, V) &= \overline{g}((X\rho)Y + \frac{1}{2}(\overline{\nabla}_X A_K)Y + (\overline{\nabla}_X \varphi)Y, V)\\ &= \overline{g}((X\rho)Y + (\overline{\nabla}_X \varphi)Y, V),\end{aligned} \tag{3.14}$$

for any $X, Y \in \Gamma(T\overline{M})$, in which we have used the fact $\overline{\nabla} A_K = 0$. On the other hand,

$$\overline{g}(\overline{\nabla}_X \overline{\nabla}_Y V, V) = \frac{1}{2} X(Y\alpha) - \overline{g}(\overline{\nabla}_X V, \overline{\nabla}_Y V), \tag{3.15}$$

$$\overline{g}(\overline{\nabla}_{\overline{\nabla}_X Y} V, V) = \frac{1}{2}(\overline{\nabla}_X Y)\alpha. \tag{3.16}$$

Replacing (3.15) and (3.16) in (3.14), leads to

$$\mathrm{Hess}_\alpha(X, Y) = 2\overline{g}(\overline{\nabla}_X V, \overline{\nabla}_Y V) + 2(X\rho)\eta(Y) + 2\overline{g}(\overline{\nabla}_X \varphi)Y, V). \tag{3.17}$$

Hence, the result follows from (3.17) and Lemma 3.2. □

4. Main Results

Consider a complementary vector bundle E of $TM^\perp$ in $S(TM)^\perp$ and take $V \in \Gamma(E_{|\mathcal{U}})$. Then $\overline{g}(V, \xi) \neq 0$ on $\mathcal{U}$ otherwise $S(TM)^\perp$ will be degenerate at a point of $\mathcal{U}$. Define on $\mathcal{U}$, a vector field

$$N = \frac{1}{\overline{g}(V, \xi)} \left\{ V - \frac{\overline{g}(V, V)}{2\overline{g}(V, \xi)} \xi \right\}, \tag{4.1}$$

where $V \in \Gamma(E_{|\mathcal{U}})$, such that $\overline{g}(V, \xi) \neq 0$. It is easy to see that N, given by (4.1), satisfies (2.1). See more details in [4, p. 45] on the construction of N.

The vector field V, appearing in (4.1), is fundamental to the study of lightlike hypersyrfaces, and submanifolds in general. Its choice on $\overline{M}$ determines, to some extent, the geometry of the underlying lightlike hypersurface. For example, it has been proved in [4, Theorem 2.3.5, p. 63] that if E admits a *covariant constant timelike* vector field V, then with respect to a section $\xi \in TM^\perp$, $(M, g, S(TM))$ is screen conformal. Thus, M can admit an integrable unique screen distribution. A concrete example in this category include the lightlike Monge hypersurface (see Example 6 in [4, p. 62]). Thus, we ask the following general question.

Problem 1. Classify lightlike hypersurfaces $(M, g, S(TM))$ of a semi-riemannian manifold $(\overline{M}, \overline{g})$ relative to the geometry of the vector field $V \in \Gamma(E_{|\mathcal{U}})$.

We partially respond to the above problem by considering a lightlike hypersurface $(M, g, S(TM))$ of a semi-Riemannian manifold $(\overline{M}, \overline{g})$, admitting an *affine conformal vector* field (ACV), $V \in \Gamma(E_{|\mathfrak{U}})$. To that end, let us set

$$\alpha := \overline{g}(V, V) \quad \text{and} \quad \beta := \overline{g}(V, \xi). \tag{4.2}$$

Then, using (4.2), we see that (4.1) give V as

$$V = \beta N + \frac{\alpha}{2\beta}\xi. \tag{4.3}$$

Then, we have the following result.

Theorem 4.1. *Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\overline{M}, \overline{g})$, admitting a closed ACV field, $V \in \Gamma(E_{|\mathfrak{U}})$ given by (4.3). Then, M admits an integrable screen distribution and, therefore, locally isometric to product manifold $\xi_c \times M^*$, where ξ_c is a lightlike curve tangent to $TM^{\perp}$ and M^* a leaf of its screen distribution. Moreover, M is quasi screen conformal lightlike hypersurface if $A_K \circ P = 0$ or $A_K = 0$.*

Proof. First note that when V is a closed ACV field, then $\varphi = 0$ which follows from (3.4). It then follows from Lemma 3.1 that

$$\overline{\nabla}_X V = \rho X + \frac{1}{2}A_K X \quad \text{and} \quad (\overline{\nabla}_X A_K) = 0. \tag{4.4}$$

Using (4.3) and (4.4), together with the Weingarten formulae (4.16) and (2.3), we get

$$\begin{aligned} & -\beta A_N X - \frac{\alpha}{2\beta}A^*_\xi X + \left\{X\left(\frac{\alpha}{2\beta}\right) - \frac{\alpha}{2\beta}\tau(X)\right\}\xi + \{X(\beta) + \beta\tau(X)\}N \\ & = \rho X + \frac{1}{2}A_K X, \end{aligned} \tag{4.5}$$

for any $X \in \Gamma(TM)$. Taking the inner product of (4.5) with $Y \in \Gamma(S(TM))$, one gets

$$\beta C(X, Y) + \frac{\alpha}{2\beta}B(X, Y) = -\rho g(X, Y) - \frac{1}{2}K(X, Y), \tag{4.6}$$

for any $X \in \Gamma(TM)$ and $Y \in \Gamma(S(TM))$. As B and K are symmetric, it follows from (4.6) that C is symmetric on $S(TM)$ too. Hence, by a direct calculation, using (2.3), we get

$$\theta([X, Y]) = C(X, Y) - C(Y, X) = 0,$$

for all $X, Y \in \Gamma(S(TM))$, from which we conclude that $S(TM)$ is an integrable distribution over M. Then, the product assertions follows from [5, Remark 5, p. 215]. Finally, when $A_K Y = 0$ for any $Y \in \Gamma(S(TM))$, then $K(X, Y) = 0$ for any $X \in \Gamma(TM)$. This shows that $\beta C(X, Y) + \frac{\alpha}{2\beta}B(X, Y) = -\rho g(X, Y)$. The case $A_K = 0$ follows in the similar manner. Hence, M is locally quasi screen conformal, which completes the proof. □

Example 4.1 (Lightlike hypersurface of generalised Robertson-Walker space). Consider (F, g_F) to be an $(n+1)$-dimensional, connected, Riemmanian manifold, $(I, -dt^2)$ an open interval of $\mathbb{R}$ with its usual metric reversed, and $f = e^\lambda$ (> 0) a smooth function on I. A Generalized Robertson-Walker (GRW) spacetime with base $(I, -dt^2)$, and fibre (F, g_F) and warping function f is the product manifold $\overline{M}(k, f) = I \times_f F$ endowed with the Lorentz metric

$$\overline{g} = -\pi_I^* dt^2 + (f \circ \pi_I)^2 \pi_F^* g_F \equiv -dt^2 + f^2(t) g_F, \tag{4.7}$$

where π_I and π_F are the natural projections of $I \times F$ onto I and F, respectively, and k the constant sectional curvature of F. The the GRW metric (4.7) can be rewritten as

$$\overline{g} = f^2(t)\{-f^{-2}(t)dt^2 + g_F\} = f^2(s)\{-ds^2 + g_F\}, \tag{4.8}$$

where the variable t is changed by s, define by $ds = dt/f(t)$. Thus, the warped metric $\overline{g}$ is conformal to the product metric $\tilde{g} = -ds^2 + F_F$. One of the consequences of this simple fact is: the vector field $V = f\partial t$ is parallel for $\tilde{g}$. That is $\tilde{\nabla} V = 0$, where $\tilde{\nabla}$ is the Levi-Civita connection for g. So, this vector filed is conformal for any metric conformal to $\tilde{g}$. Thus, for $\overline{g}$, we have

$$\pounds_V \overline{g} = 2\rho \overline{g}, \tag{4.9}$$

where $\rho = f' \circ \pi_I \equiv f'$. From [11, Corollary 8, p. 344], we get

$$\overline{\nabla}_X V = f' X, \quad \text{for all } X \in \Gamma(T\overline{M}). \tag{4.10}$$

It then follows from (4.9), (4.10), (3.4) and Lemma 3.1 that $V = f\partial t$ is CKV and the 1-form η dual to V is closed, that is V is a closed CKV vector field. Next, consider a lightlike hypersurface (M, g) of $(\overline{M}, \overline{g})$. Along M, consider the timelike section $V = f\partial t \in \Gamma(T\overline{M})$ such that $\overline{g}(V, \xi) = 1$, where $\xi \in \Gamma(TM^\perp)$. This means that V is not tangent to M. Therefore, the vector bundle H spanned by V and ξ is non-degenerate on M. The complementary orthogonal vector bundle $S(TM)$ to H in $T\overline{M}$ is a non-degenerate distribution on M and is complementary to $TM^\perp$. Thus, $S(TM)$ is a screen distribution on M. The unique lightlike transversal vector bundle $tr(TM)$ is spanned by $N = V + \frac{1}{2} f^2 \xi$. By direct calculation, using (4.10), we have

$$A_N X - \frac{1}{2} f^2 A_\xi^* X = -f' P X, \quad \tau(X) = 0, \quad X f = -(\ln f)' \theta(X), \tag{4.11}$$

for all $X \in \Gamma(TM)$. Then from the first relation in (4.11) we see that M is a quasi screen conformal lightlike hypersurface.

When $\overline{M}$ has constant curvature c, we have the following.

Theorem 4.2. *Let $(M, g, S(TM))$ be a lightlike hypersurface of an $(n+2)$-dimensional semi-Riemannian manifold $(\overline{M}(c), \overline{g})$ of constant curvature c, admitting a closed ACV* *field* $V \in \Gamma(E_{|\mathfrak{U}})$, *given by* (4.3). *Then,* $\overline{M}$ *is flat if and only if* V *is an affine vector field. Moreover, the function* $\alpha := \overline{g}(V,V)$ *satisfies the differential equations*

$$\alpha V(V\alpha) - (V\alpha)^2 + 2c\alpha^3 - \alpha VK(V,V) + (V\alpha)K(V,V) = 0. \tag{4.12}$$

Proof. From (3.4), the closure of η implies that $\varphi = 0$. Therefore, Lemma 3.2 leads to

$$\overline{R}(X,V)Y + (Y\rho)X - \overline{g}(X,Y)\mathrm{grad}\rho = 0, \tag{4.13}$$

for all $X, Y \in \Gamma(T\overline{M})$. As $\overline{M}$ has constant curvature c, (4.13) and (2.5) leads to

$$c\{\overline{g}(V,Y)\overline{g}(X,Z) - \overline{g}(X,Y)\overline{g}(V,Z)\} + (Y\rho)\overline{g}(X,Z) - (Z\rho)\overline{g}(X,Y) = 0,$$

for any $X, Y, Z \in \Gamma(T\overline{M})$. From the above relation, one gets

$$\{Y\rho + c\overline{g}(V,Y)\}X = \{X\rho + c\overline{g}(V,X)\}Y. \tag{4.14}$$

Then it follows from (4.14) that

$$X\rho + c\overline{g}(V,X) = 0, \quad \text{for all } X \in \Gamma(T\overline{M}), \tag{4.15}$$

which proves the first assertion in the theorem. Letting $X = V$ in (4.15) and using the obvious fact that $\rho = \frac{V\alpha - K(V,V)}{2\alpha}$ (comes from the first relation in (3.6) of Lemma 3.1), we get

$$V\left(\frac{V\alpha - K(V,V)}{\alpha}\right) + 2c\alpha = 0,$$

from which (4.12) follows by differentiation, which end the proof. □

Example 4.2. For $\overline{M}(c) = \overline{M}(k,f)$, the GRW of Example 4.1, we have $\rho = f'$, $V = f\partial t$, $\alpha = \overline{g}(V,V) = -f^2$ and $A_K = 0$. Then, from these quantities, we have

$$V\alpha = -2f^2 f' \quad \text{and} \quad V(V\alpha) = -2f^2\{2(f')^2 + ff''\}. \tag{4.16}$$

Replacing (4.16) in (4.12), we get

$$2f'' - 2cf = 0. \tag{4.17}$$

Multiplying (4.17) by f' leads to

$$\frac{d}{dt}((f')^2 - cf^2) = 0. \tag{4.18}$$

Integrating (4.2) gives

$$(f')^2 + k = cf^2, \tag{4.19}$$

where k is a some constant. It then follows from (4.19) that

$$c = \frac{(f')^2 + k}{f^2}. \tag{4.20}$$

Indeed, relation (4.20) gives the constant sectional curvature of a GRW manifold as seen in [11, Corollary 9, p. 345]. The parameter k represents the constant sectional curvature of F.

Corollary 4.1. *Let $(M, g, S(TM))$ be a lightlike hypersurface of an $(n+2)$ dimensional semi-Riemannian manifold $(\overline{M}(c), \overline{g})$ of constant curvature c, admitting a closed affine vector field $V \in \Gamma(E_{|\mathcal{U}})$, given by (4.3). Then $\overline{M}$ is a flat.*

When the smooth function $\alpha = \overline{g}(V, V)$ has a critical point on $\overline{M}$, we prove the following result, analogous to [14, Theorem 3.1, p. 98] for projective vector fields.

Theorem 4.3. *Let $(M, g, S(TM))$ be a lightlike hypersurface of an $(n+2)$-dimensional Lorentzian manifold, $(\overline{M}, \overline{g})$, admitting a timelike ACV field $V_p \in \Gamma(E_{|\mathcal{U}})$, given by (4.3). Assume that $\alpha := \overline{g}(V, V)$ attains a local maximum at $p \in \overline{M}$. Then*

$$\overline{g}(\overline{R}(X, V_p)V_p, X) + (V_p\rho)\overline{g}(X, X) \geq 0, \tag{4.21}$$

for all $X \in T_p\overline{M}$ orthogonal to V_p. Hence,

$$\overline{\mathrm{R}}\mathrm{ic}(V_p, V_p) + (n+1)V_p\rho \geq 0, \tag{4.22}$$

where $\overline{\mathrm{R}}\mathrm{ic}$ is the Ricci tensor of $\overline{M}$. Furthermore, the sectional curvature, $\kappa(\pi)$, of any non-degenerate plane π containing V_p satisfies

$$\kappa(\pi) + \frac{V_p\rho}{\alpha} \leq 0. \tag{4.23}$$

Moreover, if the equality holds for all such planes, then V is an affine vector field, that is ρ is constant. The underlying lightlike hypersurface M has an integrable screen distribution $S(TM)$, and therefore locally isometric to product manifold $\xi_c \times M^*$, where ξ_c is a lightlike curve tangent to $TM^\perp$ and M^* a leaf of its screen distribution. In case $A_N\xi = 0$, then M is locally screen conformal.

Proof. For the function $\alpha = \overline{g}(V, V)$ having a critical point means that $Y\alpha = 0$, for any $Y \in T_p\overline{M}$. This means that $\overline{g}(\overline{\nabla}_Y V, V_p) = 0$. Since V_p is timelike, it then follows that

$$\overline{g}(\overline{\nabla}_Y V, \overline{\nabla}_Y V) \geq 0, \quad \text{for all } Y \in T_p\overline{M}. \tag{4.24}$$

On the other hand, $(\mathrm{Hess}_\alpha)_p$ must be negative semi-definite if p is assumed to be a local maximum. Therefore, from (3.3) and (4.24), we get

$$\overline{g}(\overline{R}(X, V_p)V_p, X) + (V_p\rho)\overline{g}(X, X) \geq \overline{g}(\overline{\nabla}_X V, \overline{\nabla}_X V) \geq 0, \tag{4.25}$$

for all $X \in T_p\overline{M}$, orthogonal to V_p at $p \in \overline{M}$. Then (4.21) and (4.22) follows directly from (4.25). Furthermore, as V is timelike, we divide (4.25) by $\alpha\overline{g}(X, X)$ to get (4.23). If equality holds for all such planes, it easy to see, from (4.24), that

$$\overline{\nabla}_X V = 0, \tag{4.26}$$

for all X orthogonal to V_p. Thus, from (3.6) of Lemma 3.1 and (4.26), we get

$$2\rho X + A_K X + 2\varphi X = 0. \tag{4.27}$$

Applying the second condition of Lemma 3.1 to (4.27), we get

$$(\overline{\nabla}_Y \varphi)(X) + (Y\rho)X = 0, \tag{4.28}$$

for all $Y \in \Gamma(T\overline{M})$. The inner product of (4.28) with X and noting that $(\overline{\nabla}_Y \varphi)$ is skew-symmetric, leads to $Y\rho = 0$. Hence, V is an affine vector field. On the other hand, we know that V is orthorgonal to any $X \in \Gamma(S(TM))$. Hence, using (4.26), (4.3), (4.16) and (2.3), we derive

$$\beta A_N X + \frac{\alpha}{2\beta} A^*_\xi X - \left\{ X\left(\frac{\alpha}{2\beta}\right) - \frac{\alpha}{2\beta}\tau(X)\right\}\xi - \{X(\beta) + \beta\tau(X)\}N = 0,$$

for all $X \in \Gamma(S(TM))$. It then follows that

$$\beta A_N X + \frac{\alpha}{2\beta} A^*_\xi X = 0, \tag{4.29}$$

and $X(\beta) + \beta\tau(X) = 0$, and thus A_N is symmetric on $S(TM)$. Thus, $S(TM)$ is integrable and therefore a product manifold by Remark 5 of [5, p. 215]. Finally if $A_N\xi = 0$, we see, from (4.29) that M is locally screen conformal, which completes the proof. □

The following is a direct consequence of Theorem 4.3.

Corollary 4.2. *Under the assumptions of Theorem* 4.3*, there exist no any Einstein manifold* $\overline{M}^{n+2}$*,* $n \geq 1$*, that is* $\overline{\mathrm{Ric}} = \gamma\overline{g}$*, such that* $\alpha := \overline{g}(V, V)$ *attains a maximum,* $\gamma > 0$ *and* $V\rho \leq 0$.

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