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KONTSEVICH GRAPHONS

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Abstract. The article applies graph functions to extend the Kontsevich differential graded Lie algebraic formalism (in Deformation Quantization) to infinite Kontsevich graphs on the basis of the Connes-Kreimer Hopf algebraic renormalization and the theory of noncommutative differential geometry.

1. Introduction

The motivation of this work has been inspired from the recent progresses about the mathematical foundations of the Connes-Kreimer renormalization theory of gauge field theories under two different settings. The one setting concerns finding a new interpretation of the BPHZ Hopf algebraic perturbative renormalization in the context of the Kontsevich Deformation Quantization theory. In this direction, the Hopf-Birkhoff factorization of Feynman rules characters has been described in terms of the Baker-Campbell-Hausdorff formula and the Kontsevich's bi-differential symplectic operator for quantum deformations [5, 12, 16]. The other setting concerns finding some new applications of the theory of graphons in dealing with large Feynman diagrams (namely, infinite Feynman graphs) as sparse graphs generated by sequences of expansions of Feynman diagrams. In this direction, solutions of combinatorial Dyson-Schwinger equations in Quantum Field Theory have been described in terms of graph limits of sequences of random graphs derived from graphon models [17–19]. In addition, in arXiv:1811.05333: *A mathematical perspective on the phenomenology of non-perturbative Quantum Field Theory*, 2020, The MPIM Preprint Series 2018 (65),

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the author has addressed some recent applications of graphon models in Quantum Field Theory.

Thanks to the combination of these topics, in this work we aim to show the existence of a new class of infinite Kontsevich graphs generated by sequences of finite Kontsevich's admissible graphs. These infinite graphs allow us to extend the Hochschild-Kontsevich products to a non-perturbative setting. One immediate consequence of this investigation is the formulation of a new class of non-commutative differential calculi which can encode some geometric information (such as quantized motion integral equations) about the evolution of sequences of Kontsevich's admissible graphs. Our main task in this work is to formulate a new non-perturbative modification of the Kontsevich deformation theory via infinite combinatorial tools and the Connes-Kreimer renormalization Hopf algebra. We first apply the theory of graphons for sparse graphs [1–3, 9, 14, 15] to determine a new compact Hausdorff sub-space of graph functions namely, the space of Kontsevich graphons equipped with the cut-distance topology. This topological space can encode the convergent limits of sequences of finite Kontsevich's admissible graphs. Thanks to the Kreimer's renormalization coproduct and Kontsevich graphons, we explain the structure of a new topological Hopf algebra $H^{\mathrm{cut}}_{\mathrm{Kont}}$ on Kontsevich's admissible graphs which is closely related to the structure of a new topological Hopf algebra $\mathcal{S}^{\mathrm{Kont}}_{\mathrm{graphon}}$ on Kontsevich graphons. Then we apply this Hopf algebraic setting together with the BPHZ perturbative renormalization to build a new noncommutative differential calculus machinery on Kontsevich's admissible graphs on the basis of the Nijenhuis property of the minimal subtraction map as the renormalization scheme. This study enables us to formulate a new class of quantized motion integrals associated to Kontsevich's admissible graphs. This formalism can be modified for Kontsevich graphons which leads us to obtain a new non-perturbative version of Kontsevich $\star$-products. Finally, we lift the Maurer-Cartan equations onto the level of Kontsevich graphons and their corresponding infinite Kontsevich graphs.

The Connes-Kreimer renormalization Hopf algebra of Feynman diagrams in Quantum Field Theory is derived from the Bogoliubov-Zimmermann forest formula in perturbative renormalization [4, 10, 11]. This Hopf algebra has been applied by Ionescu in arXiv:hep-th/0307062: *Perturbative Quantum Field Theory and configuration space integrals*, 2003 and [8] to build a differential graded Hopf algebra of the Kontsevich's graph complex. In this work, we determine a new class of graphon models for Kontsevich's admissible graphs namely, Kontsevich graphons and then we apply these graphon representations to build a new topological Hopf algebra on the space of Kontsevich's admissible graphs. We equip also the space of Kontsevich graphons with a new compact Hausdorff topological Hopf algebra structure where objects in the boundary region enable us to determine a new collection of infinite Kontsevich graphs. These infinite graphs can be studied in terms of graphon models. The resulting topological Hopf algebra might be useful to search for a completion of the differential graded Hopf algebra of the Kontsevich's graph complex with respect to the cut-distance topology.

Deformation Quantization focuses on the construction of a mathematical model for the description of quantum systems under Dirac's correspondence principle. The model is actually based on quantizing the space of observables on a Poisson manifold in terms of defining a new associative multiplication as a deformation of pointwise multiplication in the direction of the Poisson bracket. The Kontsevich approach has provided a universal deformation quantization for any open domain in $\mathbb{R}^d$ via a graphical representation for bi-differential operators [12, 13]. In this work we apply our new topological Hopf algebraic setting to formulate a new non-perturbative generalization for Deformation Quantization. For this purpose we explain the construction of a new class of noncommutative differential calculi on Kontsevich's admissible graphs originated from the Connes-Kreimer renormalization theory of gauge field theories [4, 21] and the theory of noncommutative differential geometry [6]. We show that the Connes-Kreimer Renormalization Group can provide a new class of quantized integrable systems which can encode the evolution of sequences of Kontsevich's admissible graphs. We then extend this study to the level of Kontsevich graphons which enable us to formulate a new non-perturbative generalization for the Kontsevich $\star$-products in Deformation Quantization. These quantized star type of products are actually the results of the quantization of Poisson structures which are generated by the minimal subtraction map in the BPHZ renormalization theory. Furthermore, we formulate a new version of the Maurer-Cartan equations on infinite Kontsevich graphs in terms of their graphon models.

2. Graphons

We can study a dense or sparse graph in terms of the ratio between the number of its edges and the maximal number of possible edges. Passing from discrete graphs to dense graphs requires to apply sequences of edge weighted graphs such that their vertex sets tend to a continuum set of vertices. The notion of convergence for an arbitrary sequence of graphs with the growing number of vertices can be formulated via graph functions or graphons. At first, the theory of graphons has been initiated in infinite combinatorics for the study of dense graphs derived from sequences of finite weighted graphs with growing density values. The basic idea was to build a convergent limit for any sequence of this type in terms of the behavior of subgraph densities. Homomorphism densities play the fundamental rules for the construction of graph limits in this setting. However this theory has been developed immediately for the study of graph limits of sequences of finite sparse graphs in the context of random graphs and measure theoretic tools. The basic idea in this setting was to generate non-zero graph limits from sequences of graphs with almost zero densities. [1–3, 9, 14, 15]

The convergence of a sequence of pixel pictures can provide the most fundamental example for graphons. It is possible to generate different pixel picture presentations (as labeled graphons) for a graph in terms of the rescaling of the ground measure space or relabeling procedures. However we can encapsulate all these pixel picture presentations into a suitable isomorphic class to achieve the notion of uniqueness for this class of graph limits. Graphons, as analytic objects in infinite combinatorics, can be redefined in terms of a class of graph functions.

Definition 2.1. For a given measure space or a probability space (J, μ_J), a graphon is a symmetric bounded measurable function such as $W : J \times J \to [a, b] \subset \mathbb{R}$. It is called a bigraphon if we remove the symmetric property.

In the standard graphon models, we can work on the closed interval $J = [0, 1]$ equipped with the Lebesgue measure as the ground measure space to build graphons. In this setting, invertible Lebesgue measure preserving transformations on $[0, 1]$ such as ρ can generate relabeled versions of a given graphon. In other words, a relabeled graphon W^ρ is defined by $W^\rho(x, y) := W(\rho(x), \rho(y))$.

In general, graphons W_1, W_2 are called weakly isomorphic (or weakly equivalent), if there exist μ_J-measure preserving transformations σ_1, σ_2 on J such that $W_1^{\sigma_1}$ and $W_2^{\sigma_2}$ are the same almost everywhere. We can define an equivalence class $[W]$, known as unlabeled graphon class, which contains all relabeled graphons and weakly isomorphic versions with respect to a fixed graphon W.

We can define the cut-norm (as a semi-norm) on the space of labeled graphons. It is given by

$$\|W^\rho\|_{\text{cut}} := \sup_{A,B \subsetneq J} \left| \int_{A \times B} W^\rho(x, y) d\mu_J(x) d\mu_J(y) \right|. \tag{2.1}$$

This semi-norm is the key tool to define graph limits where we need to work on the space of unlabeled graphon classes to define the notion of unique convergence for the space of finite graphs. The cut-norm (2.1) gives us a metric structure on the space of unlabeled graphon classes. It is defined by

$$d_{\text{cut}}([W_1], [W_2]) := \inf_{\rho_1, \rho_2} \|W_1^{\rho_1} - W_2^{\rho_2}\|_{\text{cut}}, \tag{2.2}$$

such that the resulting topological space is compact and Hausdorff [9, 14].

Lemma 2.1. *Each finite simple weighted graph can determine a unique unlabeled graphon class.*

Proof. We consider labeled graphons on the closed interval $[0, 1]$ equipped with the Lebesgue measure. Each finite simple weighted graph $G = (V, E)$ can determine a class of labeled graph functions generated via its corresponding adjacency matrix A_G. They are pixel picture presentations. The set of vertices V can be seen as a finite probability space with the uniform measure and the set of edges E as the indicator of adjacency. Then we define the labeled graph function W_G^σ by fixing a partition σ on the closed interval such as dividing $[0, 1]$ into $|V|$ equal sub-intervals I_is. Now define $W_G^\sigma(x, y) := a_{ij} \in A_G$ for $x \in I_i$ and $y \in I_j$. Up to the weakly isomorphic relation, now we can associate an unlabeled graphon class $[W_G^\sigma]$ to the graph G which contains all possible labeled graph functions W_G^σ which are equivalent in terms of relabeling via invertible measure preserving transformations or they are weakly isomorphic. $\square$

The metric (2.2) is the key tool for the study of the behavior of extremely large graphs or complex networks whenever the vertex set of these graphs goes to infinity. In this setting, we can check that two graphons are weakly isomorphic if they have zero cut-distance from each other. Graphons generated by relabeling are weakly isomorphic. The graphon corresponding to the empty graph (i.e., 0-graphon) is identified by the class $[W^{\sigma}_{\mathbb{I}}]$ of graph functions such that $\int_{[0,1]\times[0,1]} W^{\sigma}_{\mathbb{I}}(x,y)dxdy = 0$. Graph limits can be interpreted as objects of the boundary region of the topological space of all finite graphs with respect to the cut-distance topology [9, 14, 15].

The theory of graphons has also been developed for the study of sparse graphs where we need to renormalize graph functions or rescale the base measure of the ground measure space to build non-zero graphons via the convergent limits of sequences of sparse graphs with weak densities [1–3, 15]. We recently applied this class of graphon models to formulate an analytic generalization for Feynman diagrams in Quantum Field Theory. These graphon models have led us to find some new combinatorial tools in dealing with Dyson-Schwinger equations as fixed point equations of Green's functions. It is then shown that non-perturbative solutions of quantum motions in gauge field theories can be described in terms of cut-distance convergent limits of sequences of random graphs generated by graphon representations of Feynman diagrams and their formal expansions [17–19].

3. Topological Hopf Algebra Structures on Kontsevich's Admissible Graphs and their Graphon Models

In this part, we study the fundamental elements of Deformation Quantization namely, Kontsevich's admissible graphs, Hochschild-Kontsevich products and their connection to the Connes-Kreimer insertion operator on Feynman diagrams. We then define Kontsevich graphons which are useful to study graph limits of Kontsevich's admissible graphs. We then equip the space of finite Kontsevich's admissible graphs and the space of their corresponding graphon models with the cut-distance topology together with some new Hopf algebra structures derived from the Connes-Kreimer renormalization Hopf algebra of Feynman diagrams. The Hopf algebra of Kontsevich's admissible graphs is topologically completed via the topology of graphons which can lead us to formulate the concept of convergence for sequences of these graphs. Our study provides a new class of infinite Kontsevich graphs which can be described in terms of convergent limits of sequences of random graphs derived from graphon models.

Definition 3.1. A Kontsevich's admissible graph is a simple oriented graph which contains two classes of totally ordered disjoint sets of vertices called internal and boundary vertices. Boundary vertices are leaves while there are no multiple edges or self-loops in the graph. There is also a total order on the set of all edges.

Remark 3.1. A Kontsevich's admissible graph can be presented via a disk such that internal vertices live inside the disk and boundary vertices live on the boundary region of the disk.

Definition 3.2. Let $\mathcal{G}^{p,q}$ be the set of isomorphism classes of all Kontsevich's admissible graphs such as K with q internal vertices such that $v(K) - e(K) - 1 = p$. Set $\mathfrak{g}^{\bullet,\bullet}$ as the bigraded vector space generated by $\bigcup_{p,q=0}^{\infty} \mathcal{G}^{p,q}$.

A subgraph G of K is called a normal subgraph if the quotient graph $H = K/G$ as the result of collapsing the subgraph G to a vertex v_G is itself a graph in $\mathfrak{g}^{\bullet,\bullet}$. Each normal subgraph G should be a full subgraph which means that every edge of K connecting two vertices of G is an edge of G [7, 12].

Remark 3.2. We can describe K as an extension of H by G in terms of inserting the graph G into a vertex of H. This process can be summarized by the notation $G \hookrightarrow K \twoheadrightarrow H$ such that the extension is called internal or boundary with respect to the type of that vertex which G is inserted into.

Definition 3.3. We can define two different Hochschild-Kontsevich products on $\mathfrak{g}^{\bullet,\bullet}$ in terms of types of vertices. They are given by

$$H \bullet G := \sum_{G \hookrightarrow K \twoheadrightarrow H, \text{ internal}} \pm K, \qquad H \circ G := \sum_{G \hookrightarrow L \twoheadrightarrow H, \text{ boundary}} \pm L \tag{3.1}$$

such that $\bullet$ is a $(0, -1)$ degree product and $\circ$ is a bigraded product.

Feynman diagrams in Quantum Field Theory are finite oriented labeled graphs which contains two classes of edges namely, internal and external edges. Each internal edge has begining and ending points while each external edge has only begining or ending point. Decorations in each Feynman diagram can encode fundamental data of physical systems such as conservation of momenta while vertices encode interactions among elementary particles (i.e., edges). Each Feynman diagram is a simplified model for a complicated iterated ill-defined integral which exists in the Green's functions of the physical theory. In Connes-Kreimer theory, we can describe the perturbative renormalization machinery in terms of a factorization algorithm on Feynman diagrams originated from the insertion operator. Rebuilding Feynman diagrams from the components of this factorization might not be unique in gauge field theories where we need to apply some new shuffle type products on Feynman diagrams or some identities among Feynman diagrams to generate a uniqueness [10, 11, 20].

Lemma 3.1. *The Hochschild-Kontsevich products $\bullet$ and $\circ$ can determine a pre-Lie operator on the set of Feynman diagrams.*

Proof. In terms of types of vertices and types of edges, we can glue Feynman diagrams to obtain a new diagram or decompose a complicated Feynman diagram into its primitive components. For any given Feynman diagrams Γ_1, Γ_2, suppose there exists a vertex $v_i \in \Gamma_1$ such that $f_{v_i} \sim \Gamma_2^{[1],\mathrm{ext}}$. Then we can define the insertion of Γ_2 inside Γ_1 via v_i in terms of the formula

$$\Gamma_1 *_{v_i} \Gamma_2 := \Gamma_1/\{v_i\} \cup \Gamma_2/\Gamma_2^{[1],\mathrm{ext}}, \tag{3.2}$$

which is a new graph such that for each edge $e_j \in f_i$, $\{v_{e_j}\}$ contains only one vertex of Γ_2. The sum over all possible vertices which have equivalent type with $\Gamma_2^{[1],\mathrm{ext}}$ gives us the insertion of Γ_2 inside Γ_1. We have

$$\Gamma_1 *_{\mathrm{ins}} \Gamma_2 := \sum_{v \in \Gamma_1, f_v \sim \Gamma_2^{[1],\mathrm{ext}}} \Gamma_1 *_v \Gamma_2, \tag{3.3}$$

which is known as the Connes-Kreimer insertion operator and it provides a pre-Lie algebra structure on Feynman diagrams. The commutator with respect to the insertion operator defines a Lie algebra structure on Feynman diagrams which leads us to build the Connes-Kreimer renormalization Hopf algebra [4, 10, 21]. The insertion operator $*_{\mathrm{ins}}$ is a non-homogeneous product which can be described as a combination of the Hochschild-Kontsevich products $\bullet$ and $\circ$ (3.1). □

We can formulate an analytic generalization for Kontsevich's admissible graphs in the context of the theory of graphons.

Lemma 3.2. *Any Kontsevich's admissible graph K can determine a unique unlabeled (bi)graphon class $[W_K]$.*

Proof. We need to update Lemma 2.1. We choose the closed interval $[0, 1]$ equipped with the Lebesgue measure as the ground measure space. Thanks to Definition 3.1, we can build the adjacency matrix A_K corresponding to the graph K. This matrix can be presented by a pixel picture P_K presentation built by the scaling of $[0, 1]^2$ where 1's in A_K turn into black squares and 0's in A_K turn into white squares. This class of presentations can be encoded by choosing partitions σ on $[0, 1]$ together with symmetric bounded Lebesgue measurable maps W_K^σ defined on $[0, 1]^2$.

We call $[W_K]$ the unlabeled Kontsevich graphon class corresponding to the graph K. This class contains all relabeled Kontsevich graphons corresponding to K and all other Kontsevich graphons which are weakly isomorphic to W_K. □

Definition 3.4. A sequence $\{K_n\}_{n\geq 0}$ of finite Kontsevich's admissible graphs is called convergent when n tends to infinity, if the corresponding sequence $\{[W_{K_n}]\}_{n\geq 0}$ of unlabeled Kontsevich graphon classes converges to a non-zero unlabeled Kontsevich graphon class $[W_\infty]$ with respect to the cut-distance topology.

The non-zero graph limit W_∞ can be built by rescaling methods explained in [1–3, 15] which enable us to renormalize the canonical graphons.

Definition 3.5. The Kontsevich's admissible graph generated by the information of the Kontsevich graphon W_∞ is an infinite graph K_{W_∞}. It contains infinite number of internal or boundary vertices or (infinite) number of edges. We call K_{W_∞} an infinite Kontsevich graph.

Kontsevich graphons are useful to study the asymptotic behavior of growing sequences of Kontsevich normal subgraphs with respect to the cut distance topology.

Proposition 3.1. *We can lift products $\circ$ and $\bullet$ onto the level of Kontsevich graphons.*

Proof. Let $\{K_n\}_{n\geq 0}$ be a sequence of Kontsevich's admissible graphs which is cut-distance convergent to the unlabeled Kontsevich graphon class $[W_{K_\infty}]$ with the corresponding infinite Kontsevich graph K_∞. Let $\{G_n\}_{n\geq 0}$ be another sequence of Kontsevich's admissible graphs such that for each n, G_n is a normal subgraph of K_n. Let the sequence $\{G_n\}_{n\geq 0}$ is cut-distance convergent to the unlabeled Kontsevich graphon class $[W_{G_\infty}]$ with the corresponding infinite Kontsevich graph G_∞.

We can build a new sequence $\{H_n\}_{n\geq 0} := \{K_n/G_n\}_{n\geq 0}$ of quotient graphs which is cut-distance convergent to the infinite Kontsevich graph H_∞. Thanks to Kontsevich graphon representations W_{K_∞}, W_{G_∞} and W_{K_∞/G_∞}, we can show that $W_{H_\infty} \in [W_{K_\infty/G_\infty}]$. Therefore, $H_\infty = K_\infty/G_\infty$. Now for each n, we can define

$$H_n \bullet G_n = \sum_{G_n \hookrightarrow K_n \twoheadrightarrow H_n,\ \text{internal}} \pm K_n, \quad H_n \circ G_n = \sum_{G_n \hookrightarrow K_n \twoheadrightarrow H_n,\ \text{boundary}} \pm K_n. \tag{3.4}$$

As the result, we can define $H_\infty \bullet G_\infty$ as the infinite Kontsevich graph corresponding to the cut-distance convergent limit of the sequence $\{H_n \bullet G_n\}_{n\geq 0}$ and define $H_\infty \circ G_\infty$ as the infinite Kontsevich graph corresponding to the cut-distance convergent limit of the sequence $\{H_n \circ G_n\}_{n\geq 0}$. □

Definition 3.6. The bigraded vector space $\mathfrak{g}^{\bullet,\bullet}$ (i.e., Definition 3.2) together with the cut-distance topology give us a topological vector space. We present this new space with $\mathfrak{g}^{\bullet,\bullet}_{\text{cut}}$ such that its objects have graphon representations determined by Lemma 3.2, Definition 3.4 and Definition 3.5.

Remark 3.3. $H_\infty \bullet G_\infty$ or $H_\infty \circ G_\infty$ could have infinite terms in their series. The compactness of the topology of graphons enables us to describe these infinite series in terms of objects in the boundary of the space $\mathfrak{g}^{\bullet,\bullet}_{\text{cut}}$.

Ionescu in arXiv:hep-th/0307062: *Perturbative Quantum Field Theory and configuration space integrals*, 2003 and [8] has applied the Kreimer's renormalization coproduct to build a differential graded Hopf algebra structure on Kontsevich's graph complex. Thanks to our explained graphon models, now we can formulate a new topological Hopf algebra structure on Kontsevich's admissible graphs which can be completed in terms of the cut-distance topology.

Proposition 3.2. *The completion map with respect to normal subgraphs together with the graphon representations of Kontsevich's admissible graphs can determine a topological Hopf algebra structure on $\mathfrak{g}^{\bullet,\bullet}_{\text{cut}}$.*

Proof. Thanks to the Connes-Kreimer renormalization Hopf algebra of Feynman diagrams, the structure of a differential graded Hopf algebra on Kontsevich's admissible graphs has been explained in [8]. We work on the free commutative algebra generated by Kontsevich's admissible graphs over the field $\mathbb{Q}$ or $\mathbb{R}$ such that the empty graph is its unit. For any given Kontsevich's admissible graph K, define

$$\Delta(K) = \mathbb{I} \otimes K + K \otimes \mathbb{I} + \sum_{G} G \otimes K/G, \tag{3.5}$$

as a coproduct such that the sum is over all non-trivial normal subgraphs of K and $\mathbb{I}$ is the empty graph. Terms in this expansion are in an one to one correspondence with all possible internal or boundary extensions of normal subgraphs of the original graph.

The counit is defined by $\varepsilon(\mathbb{I}) = 1$ and $\varepsilon(K) = 0$ for $K \neq \mathbb{I}$. If we apply the graduation parameter on Kontsevich's admissible graphs given by Definition 3.1 and Definition 3.2, then we can define an antipode recursively. This completes the construction of the renormalization Hopf algebra of Kontsevich's admissible graphs.

Now we plan to topologically complete this Hopf algebra in terms of graphon representations of Kontsevich's admissible graphs (i.e., Lemma 3.2 and Definition 3.4). It is enough to show the continuity of the coproduct and antipode with respect to the topology of graphons.

We work on the free commutative algebra generated by unlabeled Kontsevich graphon classes over the field $\mathbb{Q}$ or $\mathbb{R}$ such that $[W_{\mathbb{I}}]$ corresponding to the empty graph is its unit. Thanks to the coproduct (3.5), for any unlabeled Kontsevich graphon class $[W_K]$ corresponding to a finite graph K, its coproduct is given by

$$\Delta([W_K]) = [W_{\mathbb{I}}] \otimes [W_K] + [W_K] \otimes [W_{\mathbb{I}}] + \sum [W_G] \otimes [W_{K/G}], \tag{3.6}$$

such that the sum is controlled by Kontsevich graphons associated to non-trivial normal subgraphs of K. This coproduct is a bounded and linear map which makes it a continuous map with respect to the cut-distance topology.

In addition, let K_∞ be an infinite Kontsevich graph as the graph limit of the sequence $\{K_n\}_{n\geq 0}$ of finite Kontsevich's admissible graphs. Let $[W_\infty]$ be the unique unlabeled Kontsevich graphon class corresponding to K_∞. This means that the sequence $\{[W_{K_n}]\}_{n\geq 1}$ is cut-distance convergent to $[W_\infty]$. Thanks to the continuity of the coproduct (3.6), $\Delta([W_\infty])$ can be defined as the cut-distance convergent limit of the sequence $\{\Delta([K_n])\}_{n\geq 0}$.

The counit is defined by $\varepsilon([W_{\mathbb{I}}]) = 1$ and $\varepsilon([W_K]) = 0$, for $K \neq \mathbb{I}$. We can also define the antipode map on unlabeled Kontsevich graphon classes recursively in terms of the cut-distance convergent limit of a sequence of antipodes of finite Kontsevich graphs. The compactness of the cut-distance topology is enough to observe that the defined coproduct and antipode are bounded. The linearity and boundary condition guarantee the continuity of the coproduct and antipode.

We use the notation $\mathcal{S}^{\mathrm{Kont}}_{\mathrm{graphon}}$ for the resulting topological Hopf algebra of unlabeled Kontsevich graphon classes. We also use the notation $H^{\mathrm{cut}}_{\mathrm{Kont}}$ for the resulting topological Hopf algebra of Kontsevich's admissible graphs which is generated by $\mathfrak{g}^{\bullet,\bullet}_{\mathrm{cut}}$ as a vector space.

Any linear combination $\alpha_1 K_1 + \cdots + \alpha_n K_n$ of Kontsevich's admissible graphs can generate a Kontsevich graphon class in $\mathcal{S}^{\text{Kont}}_{\text{graphon}}$. The corresponding labeled Kontsevich graphon $W_{\alpha_1 K_1 + \cdots + \alpha_n K_n}$ can be determined in terms of the normalizing or rescaling methods used on each $W_{\alpha_i K_i}$. In other words, for each $1 \leq i \leq n$, we first project the labeled Kontsevich graphon $W_{\alpha_i K_i}$ into the subinterval I_i of $[0,1]$ where $\{I_i\}_i$ is a partition for $[0,1]$. We present the resulting labeled graphons with $W_{\tilde{\alpha}_i \tilde{K}_i}$. Then we can define

$$W_{\tilde{\alpha}_1 \tilde{K}_1 + \cdots + \tilde{\alpha}_n \tilde{K}_n} := \frac{W_{\tilde{\alpha}_1 \tilde{K}_1} + \cdots + W_{\tilde{\alpha}_n \tilde{K}_n}}{\| W_{\tilde{\alpha}_1 \tilde{K}_1} + \cdots + W_{\tilde{\alpha}_n \tilde{K}_n} \|_{\text{cut}}}. \tag{3.7}$$

Thanks to the correspondences $K \mapsto [W_K]$ and $\{K_n\}_{n \geq 0} \mapsto K_{[W_\infty]}$, we can complete the Hopf algebra of Kontsevich's admissible graphs and formulate a surjective topological Hopf algebra homomorphism

$$\Psi_{\text{Kont}} : \mathcal{S}^{\text{Kont}}_{\text{graphon}} \to H^{\text{cut}}_{\text{Kont}}. \tag{3.8}$$ □

Thanks to this study, now it is possible to define the notion of distance between Kontsevich's admissible graphs via their graphon representations.

Definition 3.7. The distance between Kontsevich's admissible graphs K_1 and K_2 is defined in terms of the cut-distance between their corresponding unlabeled Kontsevich graphon classes. In other words, thanks to the metric (2.2), we have

$$d(K_1, K_2) := d_{\text{cut}}([W_{K_1}], [W_{K_2}]). \tag{3.9}$$

Corollary 3.1. *A sequence of Kontsevich's admissible graphs is convergent if and only if it is a cut-distance Cauchy sequence.*

Corollary 3.2. *For a given Kontsevich graphon W_∞, there exists a sequence of finite random graphs which is cut-distance convergent to W_∞.*

Proof. For each n, we can define a finite random graph $G(W_\infty, n)$ which contains n points $x_1, \ldots, x_n$ from the Kontsevich graphon W_∞ such that the existence of an edge between x_i and x_j is determined by the probability $W_\infty(x_i, x_j)$. Thanks to [9, 19], we can show that the sequence $\{G(W_\infty, n)\}_{n \geq 0}$ is cut-distance convergent to W_∞. □

Infinite polydifferential operators can be described in terms of multiplication of functions and infinite vector fields which act as polyderivations on infinite functions. We can define these operators as the cut-distance convergent limit of sequences of finite operators. In this setting, the multiplication of infinite functions is represented by the Kontsevich graphon $b_{0,\infty}$ with no internal vertices and infinite (countable) boundary vertices. The resulting Kontsevich graphon is actually the cut-distance convergent limit of Kontsevich's admissible graphs which belong to $\mathcal{G}^{m-1,0}$ when m tends to infinity. In addition, ∞-vector field with infinite polyderivations is represented by the Kontsevich graphon $b_{1,\infty}$ with one internal vertex and infinite countable boundary vertices with infinite countable edges.

4. Noncommutative Differential Calculi on Kontsevich's Admissible Graphs and their Graphon Models via the Renormalization Map

In [12] it is shown that the Hopf-Birkhoff factorization of Feynman rules characters in the Connes-Kreimer perturbative renormalization process can be interpreted as a deformation of the pointwise multiplication of some exponential functions under the Kontsevich product. In this part we plan to work on the space of linear functionals on the topological Hopf algebra of Kontsevich's admissible graphs or Kontsevich graphons with values in the algebra $A_{\rm dr}$ of Laurent series with finite pole parts equipped with the minimal subtraction map to build a new class of differential graded Lie algebras and Poisson structures with respect to deformed versions of the convolution product.

The Rota-Baxter algebra $(A_{\rm dr}, R_{\rm ms})$ determines a class of deformed convolution products on the space $L(H^{\rm cut}_{\rm Kont}, A_{\rm dr})$ of linear maps given by

$$\phi_1 \circ_\lambda \phi_2 := \mathcal{R}_\lambda(\phi_1) * \phi_2 + \phi_1 * \mathcal{R}_\lambda(\phi_2) - \mathcal{R}_\lambda(\phi_1 * \phi_2), \tag{4.1}$$

such that $\mathcal{R}_\lambda := \mathcal{R} - \lambda(\mathrm{Id} - \mathcal{R})$, where $\mathcal{R}$ is the extension of $R_{\rm ms}$ on $L(H^{\rm cut}_{\rm Kont}, A_{\rm dr})$ and λ is a real number.

The convolution product $*$ is defined in terms of the coproduct (3.5) on Kontsevich's admissible graphs. In other words, for any $\phi_1, \phi_2 \in L(H^{\rm cut}_{\rm Kont}, A_{\rm dr})$ and any Kontsevich's admissible graph K, we have

$$\phi_1 * \phi_2(K) := \phi_1(\mathbb{I})\phi_2(K) + \phi_1(K)\phi_2(\mathbb{I}) + \sum_G \phi_1(G)\phi_2(K/G), \tag{4.2}$$

such that G are non-trivial normal subgraphs of K.

Let an infinite Kontsevich graph K_∞ is the result of the cut-distance convergent limit of a sequence $\{K_n\}_{n\geq 1}$ of finite Kontsevich's admissible graphs. Thanks to the continuity of the coproduct (3.5) with respect to the cut-distance topology and Proposition 3.1, we can show that the sequence $\{\sum_{G_n} \phi_1(G_n)\phi_2(K_n/G_n)\}_{n\geq 1}$ is cut-distance convergent to $\sum_{G_\infty} \phi_1(G_\infty)\phi_2(K_\infty/G_\infty)$. This means that we can extend the convolution product $*$ on infinite Kontsevich graphs where $\phi_1 * \phi_2(K_\infty)$ can be defined as the convergent limit of the sequence $\{\phi_1 * \phi_2(K_n)\}_{n\geq 1}$.

The associative products $\circ_\lambda$ on $L(H^{\rm cut}_{\rm Kont}, A_{\rm dr})$ are actually the direct consequence of the Nijenhuis property of the map $\mathcal{R}_\lambda$. The non-cocommutativity of $H^{\rm cut}_{\rm Kont}$ ensures that each product $\circ_\lambda$ is noncommutative. Therefore we can define a new Lie bracket $[\cdot,\cdot]_\lambda$ via the commutator with respect to $\circ_\lambda$.

Proposition 4.1. *There exists a noncommutative differential calculus on* $H^{\wedge\lambda}_{\rm Kont} := (L(H^{\rm cut}_{\rm Kont}, A_{\rm dr}), \circ_\lambda)$.

Proof. Set $Z(H^{\wedge\lambda}_{\rm Kont})$ as the center of the algebra and $\mathrm{Der}^\lambda_{\rm Kont}$ as the space of all linear maps $\theta : H^{\wedge\lambda}_{\rm Kont} \to H^{\wedge\lambda}_{\rm Kont}$ which obey the Leibniz rule. The Lie bracket $[\cdot,\cdot]_\lambda$, which satisfies the Jacobi identity, can determine the corresponding Poisson bracket $\{\cdot,\cdot\}_\lambda$. For each $\phi \in H^{\wedge\lambda}_{\rm Kont}$, define $\psi \mapsto \{\phi, \psi\}_\lambda$ as the corresponding Hamiltonian derivation. Set $\mathrm{Ham}^\lambda_{\rm Kont}$ as the $Z(H^{\wedge\lambda}_{\rm Kont})$-module generated by all Hamiltonian derivations.

Thanks to the theory of noncommutative differential geometry on the basis of the space of derivations [6], for $n \geq 1$, define $\Omega^n_{\mathrm{Kont},\lambda}$ as the space of all $Z(H^{\wedge\lambda}_{\mathrm{Kont}})$-multilinear anti-symmetric maps from $\mathrm{Ham}^{\lambda}_{\mathrm{Kont}} \times \cdots^{n} \times \mathrm{Ham}^{\lambda}_{\mathrm{Kont}}$ to $H^{\wedge\lambda}_{\mathrm{Kont}}$. We have the differential graded algebra $(\Omega^{\bullet}_{\mathrm{Kont},\lambda}, d_\lambda)$ such that the degree one anti-derivative differential operator d_λ is given by

$$\begin{aligned} d_\lambda \omega(\theta_0, \ldots, \theta_n) := &\sum_{k=0}^{n} (-1)^k \theta_k \omega(\theta_0, \ldots, \hat{\theta}_k, \ldots, \theta_n) \\ &+ \sum_{0 \leq r < s \leq n} (-1)^{r+s} \omega([\theta_r, \theta_s]_\lambda, \theta_0, \ldots, \hat{\theta}_r, \ldots, \hat{\theta}_s, \ldots, \theta_n). \qquad \square \end{aligned}$$

Corollary 4.1. *There exists a new class of integrable systems which can geometrically evaluate Kontsevich's admissible graphs.*

Proof. We apply the renormalization map $R_{\mathrm{ms}} : A_{\mathrm{dr}} \to A_{\mathrm{dr}}$ and work on the noncommutative deRham complex derived from Proposition 4.1. We have

$$\mathrm{DR}^{\bullet}_{\mathrm{Kont},\lambda} := \frac{\Omega^{\bullet}_{\mathrm{Kont},\lambda}}{[\Omega^{\bullet}_{\mathrm{Kont},\lambda}, \Omega^{\bullet}_{\mathrm{Kont},\lambda}]_\lambda}. \tag{4.3}$$

The deformed Lie bracket $[\cdot,\cdot]_\lambda$ allows us to define a class of $Z(H^{\wedge\lambda}_{\mathrm{Kont}})$-bilinear anti-symmetric non-degenerate closed 2-forms for the presentation of the Poisson bracket $\{\cdot,\cdot\}_\lambda$. For any derivations $\theta_1 = \sum u_i \circ_\lambda \mathrm{ham}(f_i)$, $\theta_2 = \sum v_j \circ_\lambda \mathrm{ham}(h_j)$, define the symplectic form

$$\omega_\lambda(\theta_1, \theta_2) = \sum_{i,j} u_i \circ_\lambda v_j \circ_\lambda [f_i, h_j]_\lambda, \tag{4.4}$$

such that $\{f_1, \ldots, f_n, h_1, \ldots, h_m\} \subsetneq H^{\wedge\lambda}_{\mathrm{Kont}}$, $\{u_1, \ldots, u_n, v_1, \ldots, v_m\} \subsetneq Z(H^{\wedge\lambda}_{\mathrm{Kont}})$.

If θ^λ_f is the symplectic vector field associated to the symplectic form ω_λ, then we have

$$\{f, g\}_\lambda = \omega_\lambda(\theta^\lambda_f, \theta^\lambda_g), \tag{4.5}$$

as the quantization of the Poisson structure on Kontsevich's admissible graphs and Kontsevich graphons in the direction of the minimal subtraction scheme.

Thanks to [4, 21], we can build the Connes-Kreimer Renormalization Group $\{F_t\}_t$ of the topological Hopf algebra $H^{\mathrm{cut}}_{\mathrm{Kont}}$ of Kontsevich's admissible graphs. This is a 1-parameter subgroup of the Lie group $\mathrm{Hom}(H^{\mathrm{cut}}_{\mathrm{Kont}}, A_{\mathrm{dr}})$ of characters. Then we can check that $\{F_t, F_s\}_0 = 0$. $\square$

Remark 4.1. Thanks to the surjective homomorphism Ψ_{Kont} (3.8), Proposition 4.1 and Corollary 4.1, we can build a noncommutative differential calculus on $\mathcal{S}^{\mathrm{Kont},\wedge\lambda}_{\mathrm{graphon}}$ and then we can show that the Connes-Kreimer Renormalization Group of $\mathcal{S}^{\mathrm{Kont}}_{\mathrm{graphon}}$ can determine a new class of integrable systems.

Corollary 4.2. *The Kontsevich's Deformation Quantization* [7, 12] *can be lifted onto the level of Kontsevich graphons.*

Proof. For the algebra

$$S^{\wedge_\lambda}_{\text{Kont}} := (L(\mathcal{S}^{\text{Kont}}_{\text{graphon}}, A_{\text{dr}}), \circ_\lambda), \tag{4.6}$$

we work on the Lie algebra $\text{der}^\lambda_{\text{Kont}}$ of all derivations $\rho : \mathcal{S}^{\text{Kont}}_{\text{graphon}} \to A_{\text{dr}}$. This space is generated by infinitesimal characters such as $\rho_{[W]}$ corresponding to each Kontsevich graphon $[W]$. Let $\mathcal{A}^d$ as the space of functions with the domain $\text{der}^\lambda_{\text{Kont}} \times \cdots^d \times \text{der}^\lambda_{\text{Kont}}$ and with the images in $\mathcal{S}^{\text{Kont}}_{\text{graphon}}$.

For any Kontsevich graphon $[W_K]$ corresponding to the graph $K \in H^{\text{cut}}_{\text{Kont}}$, we can define the bi-differential operator $B_{[W_K],\lambda} : \mathcal{A}^d \times \mathcal{A}^d \longrightarrow \mathcal{A}^d$ in terms of the differential operator d_λ (determined by the Poisson structure $\{\cdot,\cdot\}_\lambda$) and derivations ρ_K.

Set G_n, $n \geq 0$, as the collection of all Kontsevich graphs with $n+2$ vertices $\{1,\dots,n\} \cup \{X, Y\}$ and $2n$ edges such that for each vertex k, there exist two edges staring at k. We can now define a new $\star$-product on $\mathcal{A}^d$ as the Kontsevich's quantization of $\circ_\lambda$. For any functions $F, G \in \mathcal{A}^d$, $F \star_\lambda G$ is defined as the convergent limit of the sequence

$$\left\{ \sum_{j=0}^{n} \epsilon^j \sum_{L \in G_j} \omega_K(L) B_{[W_K],\lambda}(F,G) \right\}_{n \geq 0}, \tag{4.7}$$

with respect to the cut-distance topology defined on Kontsevich graphons when n tends to infinity. □

The quantization $F \star_\lambda G$ can contain an infinite formal expansion of growing Kontsevich's admissible graphs which can not be handled by the perturbative setting. Therefore we name it a non-perturbative generalization of the standard Kontsevich's Deformation Quantization. Thanks to the compactness of the topology of graphons [9, 14], we can search for cut-distance graph limits for these infinite expansions.

5. Maurer-Cartan Equations on Kontsevich Graphons

In this section, we aim to formulate a new generalization of the Maurer-Cartan equations for infinite Kontsevich graphs (i.e., Definition 3.5) generated as the graph limits of sequences of finite Kontsevich's admissible graphs.

The commutator with respect to the operation $\circ$ gives a Lie algebraic structure on $\mathfrak{g}^{\bullet;\bullet}_{\text{cut}}$. This Lie bracket is actually obtained as an extension of the Hochschild-Kontsevich Lie bracket with respect to the cut-distance topology. It determines the differential operator d_1 of degree $(1,0)$. In addition, we can also extend the Kontsevich's vertical differential operator on $\mathfrak{g}^{\bullet;\bullet}_{\text{cut}}$ to define the differential operator d_2 on infinite Kontsevich's admissible graphs. For a given infinte Kontsevich graph $K_{[U_\infty]}$ corresponding to the unlabeled Kontsevich graphon class $[U_\infty]$, $d_2(K_{[U_\infty]})$ is the result of the cut-distance convergent limit of the sequence $\{d_2(K_n)\}_{n\geq 0}$, where for each n

$$d_2(K_n) := \sum_{e \hookrightarrow G \twoheadrightarrow K_n,\ \text{internal}} \pm G = K_n \bullet e, \tag{5.1}$$

which is expanding the internal vertices of K_n by the insertion of an additional edge. d_2 is a differential operator of degree $(0,1)$.

Proposition 5.1. *There exists a Hochschild-Kontsevich differential graded Lie algebra on Kontsevich graphons.*

Proof. Set $\mathfrak{g}^n_{\text{cut}} := \oplus_{p+q=n}\mathfrak{g}^{p,q}$ as the graded vector space equipped with the cut-distance topology. We can show that differential operators d_1, d_2 commute on the total complex $\mathfrak{g}^\bullet_{\text{cut}}$ and therefore $d := d_1 \pm d_2$ is a total differential operator which is compatible with the graded Lie bracket $[\cdot,\cdot]_\circ$ induced by $\circ$. □

Now we can formulate the Maurer-Cartan equations on an infinite generalization of Kontsevich's admissible graphs.

Corollary 5.1. *There exists a modified version of the Maurer-Cartan equation on infinite Kontsevich graphs.*

Proof. The topological Hopf algebra $H^{\text{cut}}_{\text{Kont}}$ of Kontsevich's admissible graphs (built by Proposition 3.2) and the noncommutative differential calculus (i.e., Proposition 4.1) can be applied to associate the 1-form

$$\alpha_{MC}(K) = \sum_G S(G) d_\lambda \theta_{K/G}, \tag{5.2}$$

such that S is the antipode of $H^{\text{cut}}_{\text{Kont}}$, the sum is taken over all normal subgraphs G of K and $\theta_{K/G}$ is the infinitesimal characters with respect to Kontsevich's admissible quotient graphs K/G. We can check that

$$\alpha_{MC}(KL) = \alpha_{MC}(K)\varepsilon(L) + \varepsilon(K)\alpha_{MC}(L). \tag{5.3}$$

Therefore, a general presentation of the Maurer-Cartan equation has the form

$$d_\lambda \alpha_{MC}(K) = -\sum_G \alpha_{MC}(G)\alpha_{MC}(K/G). \tag{5.4}$$

Now suppose $\{K_n\}_{n\geq 0}$ be a sequence of finite Kontsevich's admissible graphs which satisfy the equation (5.4) for each $n \geq 0$ and the sequence is cut-distance convergent to the Kontsevich graphon W_∞. Then it can be seen that the infinite Kontsevich graph $K_{[W_\infty]}$ is also a solution for (5.4). □

It is possible to define a new morphism $\bar{\mathcal{U}}$ of differential graded Lie algebras (as a generalization of the map $\mathcal{U}$ given in [7]) between $\mathfrak{g}^{\bullet;\bullet}_{\text{cut}}$ and the Chevalley-Eilenberg complex $\text{CE}^{\bullet;\bullet}_{\text{cut}}(T^\bullet_{\text{poly}}, D^\bullet_{\text{poly}})$ equipped with the cut-distance topology. This enables us to formulate the Maurer-Cartan equations on the complex of Kontsevich graphons in the language of morphisms between $T^\bullet_{\text{poly}}$ and $D^\bullet_{\text{poly}}$.

6. Conclusion

The main achievement of this work is to provide some new mathematical tools for the study of the Kontsevich Deformation Quantization under a non-perturbative setting. We applied graphon models for the study of the space of Kontsevich's admissible graphs to formulate some new topological Hopf algebra structures $H^{\rm cut}_{\rm Kont}$ on these graphs and $\mathcal{S}^{\rm Kont}_{\rm graphon}$ on their corresponding graphon models. Then we worked on the basis of this Hopf algebraic setting to build a new class of noncommutative differential calculi on Kontsevich's admissible graphs originated from the BPHZ perturbative renormalization. This study has led us to determine a new class of quantized integrable systems which can geometrically describe the evolution of sequences of Kontsevich's admissible graphs. In addition, thanks to the topologically completion of our Hopf algebra model, we formulated the Kontsevich's Deformation Quatization for Kontsevich graphons which has led us to obtain a non-perturbative generalization for deformation quantization procedure. Furthermore, we have obtained a new modified version of the Maurer-Cartan equations on infinite Kontsevich graphs.

As the final note, the topological Hopf algebras $H^{\rm cut}_{\rm Kont}$ and $\mathcal{S}^{\rm Kont}_{\rm graphon}$ are also useful to work on combinatorial Dyson-Schwinger equations on Kontsevich's admissible graphs in the context of Hochschild type of equations. Solutions of these equations can be described in terms of random graphs generated by Kontsevich graphons. This study can be useful to find some new interconnections between combinatorial Dyson-Schwinger equations and the non-perturbative generalization of the Kontsevich Deformation Quantization.

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