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SHIFTED GEGENBAUER-GAUSS COLLOCATION METHOD FOR SOLVING FRACTIONAL NEUTRAL FUNCTIONAL-DIFFERENTIAL EQUATIONS WITH PROPORTIONAL DELAYS

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Abstract. In this paper, the shifted Gegenbauer-Gauss collocation (SGGC) method is applied to fractional neutral functional-differential equations with proportional delays. The technique we have used is based on shifted Gegenbauer polynomials and Gauss quadrature integration. The shifted Gegenbauer-Gauss method reduces solving the generalized fractional pantograph equation fractional neutral functional-differential equations to a system of algebraic equations. Reasonable numerical results are obtained by selecting few shifted Gegenbauer-Gauss collocation points. Numerical results demonstrate its accuracy, and versatility of the proposed techniques.

1. Introduction

Fractional differential operators have a long history, having been mentioned by Leibniz in a letter to L'Hospital in 1695. A history of the development of fractional differential operator can be founded in [26, 27]. One of the most recent works on the subject of the fractional calculus, i.e. the theory of derivatives and integrals of fractional (non-integer) order, is the book of Podlubny [31], which deals principally with fractional equations. Today, there are many works on fractional calculus (see for example [11, 36]).

For the past three centuries, this subject has been dealt with by the mathematicians and only in the last few years, this was pulled to several (applied) fields of engineering, science and economics [11]. However, the number of scientific and engineering problems

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involving fractional calculus is already very large and still growing and perhaps the fractional calculus will be the calculus of the twenty-first century. It was found that various, especially interdisciplinary applications can be elegantly modeled with the help of fractional derivatives. Fractional differentials and integral provide more accurate models of systems under considerations. Many authors have demonstrated applications of fractional calculus in the nonlinear oscillation of earthquakes [15], fluid-dynamic traffic model [16], to model frequency dependent damping behavior of many viscoelastic materials [4, 5], continuum and statistical mechanics [23], colored noise [24], solid mechanics [35], economics [6], bioengineering [20–22], anomalous transport [25], and dynamics of interfaces between nanoparticles and substrates [10].

The analytic results on the existence and uniqueness of solutions of the fractional differential equations have been investigated by many authors (see, for examples [31, 36]).

Spectral methods (see, for instance [14,30,33,38]) are one of the principal methods of discretization for the numerical solution of differential equations. The main advantage of these methods lies in their accuracy for a given number of unknowns. For smooth problems in simple geometries, they offer exponential rates of convergence/spectral accuracy. In contrast, finite-difference and finite-element methods yield only algebraic convergence rates. The three most widely used spectral versions are the Galerkin, collocation, and tau methods. Collocation methods [8, 9] have become increasingly popular for solving differential equations, also they are very useful in providing highly accurate solutions to nonlinear differential equations.

Neutral functional-differential equations play an important role in the mathematical modeling of several phenomena. It is well known that most of delay differential equations cannot be solved exactly. Therefore, numerical methods would be presented and developed to get approximate solutions of these equations. In this direction, Bhrawy et al. [3] proposed a new spectral collocation scheme based upon the generalized Laguerre polynomials and Gauss quadrature integration for solving generalized fractional pantograph equations. Anapali et al. [29] investigated a Taylor collocation method, which is based on collocation method for solving fractional pantograph equation. Rahimkhani et al. [28] defined a new functions called generalized fractional-order Bernoulli wavelet functions based on the Bernoulli wavelets to obtain the numerical solution of fractional order pantograph differential equations in a large interval. Furthermore, Yang and Huang [39] proposed a spectral Jacobi-collocation approximation for fractional order integrodifferential equations of Volterra type with pantograph delay. Recently, in [13], Ghasemi et al. proposed an approximate solution of a class of nonlinear fractional-order delay differential equation using Hilbert function space.

Our fundamental goal of this paper is to develop a suitable way to approximate the neutral fractional functional-differential equations with proportional delays on the interval $(0, L)$ using the shifted Gegenbauer polynomials, we propose the spectral shifted Gegenbauer-Gauss collocation (SGGC) method to find the solution $u_N(x)$. The shifted Gegenbauer spectral collocation (SGGC) approximation, which is more reliable, is employed to obtain approximate solution of neutral fractional functional-differential equations with proportional delays of order ν $(m-1<\nu<m)$ and m initial conditions. For suitable collocation points we use the $(N-m+1)$ nodes of the shifted Gegenbauer-Gauss interpolation on $(0,L)$. These equations together with initial conditions generate $(N+1)$ algebraic equations which can be solved. Finally, the accuracy of the proposed methods are demonstrated by test problems, numerical results are presented in which the usual exponential convergence behavior of spectral approximations is exhibited.

This paper is organized as follows. In the subsequent section, we present some definitions and properties of fractional calculus theory. In Section 3 we give an overview of shifted Gegenbauer polynomials and their relevant properties needed hereafter, and in Section 4, the way of constructing the collocation technique for neutral fractional functional-differential equations with proportional delays is described using the shifted Gegenbauer polynomials. In Section 6, we present some numerical results exhibiting the accuracy and efficiency of our numerical algorithms.

2. Basic Definitions

The two most commonly used definitions are the Riemann-Liouville operator and the Caputo operator [36]. We give some definitions and properties of fractional derivatives.

Definition 2.1. The Riemann-Liouville fractional integral operator of order ν, $\nu>0$, is defined as

$$J^{\nu} f(x)=\frac{1}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1} f(t)dt, \quad \nu>0,\, x>0,$$
$$J^0 f(x)=f(x).$$

Definition 2.2. The Caputo fractional derivative of order ν is defined as

$$D^{\nu} f(x)=J^{m-\nu}D^m f(x)=\frac{1}{\Gamma(m-\nu)}\int_0^x (x-t)^{m-\nu-1}\frac{d^m}{dt^m} f(t)dt,$$

$m-1<\nu<m$, $x>0$, D^m is the classical differential operator of order m.

For the Caputo derivative we have

$$D^{\nu} C=0, \quad C \text{ is a constant,} \tag{2.1}$$

(2.2)
$$D^{\nu} x^{\beta}=\begin{cases} 0, & \text{for } \beta\in\mathbb{N}_0 \text{ and } \beta<\lceil\nu\rceil, \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\nu)}\, x^{\beta-\nu}, & \text{for } \beta\in\mathbb{N}_0 \text{ and } \beta\ge\lceil\nu\rceil \text{ or } \beta\notin\mathbb{N} \text{ and } \beta>\lfloor\nu\rfloor, \end{cases}$$

where $\lceil\nu\rceil$ and $\lfloor\nu\rfloor$ are the ceiling and floor functions, respectively, while $\mathbb{N}=\{1,2,\ldots\}$ and $\mathbb{N}_0=\{0,1,2,\ldots\}$.

The Caputo's fractional differentiation is a linear operation, similar to the integer-order differentiation

$$D^{\nu}(\lambda f(x)+\mu g(x))=\lambda D^{\nu}f(x)+\mu D^{\nu}g(x), \tag{2.3}$$

where λ and μ are constants.

Remark 2.1. The reason for using the fractional derivative in the Caputo sense is that it is mathematically rigorous than the Riemann-Liouville sense (one can be referred to [19, 32]). Caputo definition is advantageous in many disciplines such as applied science and engineering [18]. Moreover, properties of the Caputo derivative are useful in translating the higher fractional-order differential systems into lower ones [17]. Some comparisons between Caputo and Riemann-Liouville operators can be found in [27]. It is a future work for us to study the Riemann-Liouville derivative in relation to solve some physical models.

3. Shifted Gegenbauer Polynomials Interpolation

In this section, we detail the properties of shifted Gegenbauer polynomials that will be used to construct the method.

3.1. **Shifted Gegenbauer polynomials.**

The Gegenbauer polynomial $C_i^{\alpha}(z)$, of degree $i\in\mathbb{Z}^+$, and associated with the parameter $\alpha>-\frac{1}{2}$, is a real-valued function, which appears as an eigensolution to a singular Sturm-Liouville problem in the finite domain $[-1,1]$ [37]. It is a Jacobi polynomial, $P_i^{(\alpha,\beta)}(x)$, with $\alpha=\beta$, and can be standardized so that

$$C_i^{\alpha}(z)=\frac{i!\Gamma(\alpha+\frac{1}{2})}{\Gamma(i+\alpha+\frac{1}{2})}P_i^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(z),\quad i=0,1,2,\dots$$

There exist useful relations between Legendre polynomials $L_i(z)$ and Chebyshev polynomials of the first kind and second kind, $T_i(z)$, $U_i(z)$, respectively, and the Gegenbauer polynomials $C_i^{\alpha}(z)$ as [37],

$$L_i(z)=C^{\frac{1}{2}}(z),\quad U_i(z)=C^{1}(z),$$

and

$$T_i(z)=\frac{i}{2}\lim_{\alpha\to 0}\alpha^{-1}C_i^{\alpha}(z),\quad i\geq 1.$$

The well-known Gegenbauer polynomials can be determined with the aid of the following recurrence formulae:

$$(i+1)C_{i+1}^{\alpha}(z)=2z(\alpha+i)C_i^{\alpha}(z)-(2\alpha+i-1)C_{i-1}^{\alpha}(z),\quad i=1,2,\dots,$$

where $C_0^{\alpha}(z)=1$ and $C_1^{\alpha}(z)=2\alpha z$. The weight function for the Gegenbauer polynomials is the even function $w^{\alpha}(z)=(1-z^2)^{\alpha-\frac{1}{2}}$. The Gegenbauer polynomials form a complete orthogonal basis polynomials in $L^2_{w^\alpha}[-1,1]$ and their orthogonality relation is given by the following weighted inner product:

$$(C_j^\alpha, C_i^\alpha)_{w^\alpha} = \int_{-1}^{1} C_j^\alpha(z) C_i^\alpha(z) w^\alpha(z) dz = \gamma_i^\alpha \delta_{ji},$$

where

$$\gamma_i^\alpha = \frac{2^{1-2\alpha}\ \pi\ \Gamma(i+2\alpha)}{\Gamma(i+1)\ (i+\alpha)\ \Gamma^2(\alpha)}$$

is the normalization factor and δ_{ji} is the Kronecker delta function. We denote the zeroes of the Gegenbauer polynomial $C_{i+1}^\alpha(z)$ (also called Gegenbauer-Gauss nodes) by z_k^α, $k = 0, \dots, i$. We also denote their corresponding Christoffel numbers by ϖ_k^α, $k = 0, \dots, i$, and define them by the following relation:

$$(\varpi_k^\alpha)^{-1} = \sum_{j=0}^{i} (\gamma_j^\alpha)^{-1} (C_j^\alpha(z_k))^2, \quad k = 0, 1, 2, \dots, i.$$

In order to use these polynomials on the interval $t \in [0, L]$ we defined the so-called shifted Gegenbauer polynomials by introducing the change of variable $z = \frac{2}{L}t - 1$. Let the shifted Gegenbauer polynomials $C_i^\alpha(\frac{2}{L}t - 1)$ be denoted by $G_i^\lambda(t)$. Then $G_i^\alpha(t)$ can be obtained as follows:

$$(i+1)G_{i+1}^\alpha(t) = 2\left(\frac{2}{L}t - 1\right)(\alpha+i)G_i^\alpha(t) - (2\alpha+i-1)G_{i-1}^\alpha(t), \quad i = 1, 2, \dots,$$

and

$$D^q G_i^\alpha(t) = \left(\frac{4}{L}\right)^q (\alpha)_q G_{i-q}^{\alpha+q}(t), \tag{3.1}$$

where $G_0^\alpha(t) = 1$ and $G_1^\alpha(t) = 2\alpha(\frac{2}{L}t - 1)$. The analytic form of the shifted Gegenbauer polynomials $G_i^\alpha(t)$ of degree i is given by

$$G_i^\alpha(t) = \sum_{k=0}^{i} \frac{(-1)^{i-k}\ (2\alpha)_{i+k}}{L^k\ k!\ (i-k)!\ (\alpha+\frac{1}{2})_k}\ t^k, \quad 0 \le t \le L, \tag{3.2}$$

where $(\theta)_i$ is the Pochhammer symbol, means $\theta(\theta+1)(\theta+2)\cdots(\theta+i-1)$ for $i \ge 1$ and $(\theta)_0 = 1$. Note that the values

$$G_i^\alpha(0) = (-1)^i \frac{\Gamma(i+2\alpha)}{i!\ \Gamma(2\alpha)} \quad \text{and} \quad G_i^\alpha(L) = \frac{\Gamma(i+2\alpha)}{i!\ \Gamma(2\alpha)}, \tag{3.3}$$

are fulfilled at the endpoints. Moreover, the relations (3.1) and (3.3) imply that the special value

$$D^q G_i^\alpha(0) = \frac{(-1)^{i-q} 2^{2q} (\alpha)_q \Gamma(i+2\alpha+q)}{L^q (i-q)! \Gamma(2\alpha+2q)}, \tag{3.4}$$

will be of important use later. The orthogonality condition is

$$\int_0^L G_j^\alpha(t) G_k^\alpha(t) w_L^\alpha(t) dt = \gamma_{L,k}^\alpha \delta_{jk},$$

where $w_L^\alpha(t) = (Lt - t^2)^{\alpha-\frac{1}{2}}$ is the weight function for the shifted Gegenbauer polynomials and $\gamma_{L,k}^\alpha = \frac{L^{2\alpha}\ \pi\ \Gamma(k+2\alpha)}{2^{4\alpha-1}\ \Gamma(k+1)\ (k+\alpha)\ \Gamma^2(\alpha)}$ is the normalization factor.

If $f(t)$ is a polynomial of degree n, then it may be expressed in terms of shifted Gegenbauer polynomials as

$$u(t) = \sum_{j=0}^{n} b_j G_j^\alpha(t),$$

where the coefficients b_j are given by

$$b_j = \frac{1}{\gamma_{L,j}^\alpha} \int_0^L u(t) G_j^\alpha(t) w_L^\alpha(t) dt, \quad j = 0, 1, 2, \dots \tag{3.5}$$

We denote the zeroes of the shifted Gegenbauer polynomial $G_{i+1}^\alpha(t)$ by $t_{L,k}^\alpha$, $k = 0, 1, 2, \dots, i$. We also denote their corresponding Christoffel numbers by $\varpi_{L,k}^\alpha$, $k = 0, 1, 2, \dots, i$.

Clearly,

$$\begin{aligned} t_{L,k}^\alpha =& \frac{L}{2}(t_k^\alpha + 1), \quad k = 0, \dots, i, \\ \varpi_{L,k}^\alpha =& \left(\frac{L}{2}\right)^{2\alpha} \varpi_k^\alpha, \quad k = 0, \dots, i. \end{aligned}$$

If we denote by $\mathbb{P}_i$, the space of all polynomials of degree at most i, $i \in \mathbb{Z}^+$, then for any $\phi \in \mathbb{P}_{i+1}$

$$\begin{aligned} \int_0^L \phi(t) w_L^\alpha dt &= \left(\frac{L}{2}\right)^{2\alpha} \int_{-1}^1 \phi\left(\frac{L}{2}(t+1)\right) w^\alpha(t) dt = \left(\frac{L}{2}\right)^{2\alpha} \sum_{j=0}^{n} \varpi_j^\alpha \phi\left(\frac{L}{2}(t_j^\alpha + 1)\right) \\ &= \sum_{j=0}^{n} \varpi_{L,j}^\alpha \phi(t_{L,j}^\alpha). \end{aligned}$$

Lemma 3.1. *The qth derivative of $G_k^\alpha(t)$ can be written as*

$$D^q G_k^\alpha(t) = \sum_{\substack{m=0 \\ (k+m-q)\ \ even}}^{k} C_q(k, m, \alpha) G_m^\alpha(t),$$

where

$$\begin{aligned} C_q(k, m, \alpha) =& \frac{2^{2q}\ k!}{L^q (q-1)!\ \Gamma(k+2\lambda)} \\ & \times \frac{(m+\alpha)\ \Gamma(m+2\alpha)\ \left(\frac{k-m+q-2}{2}\right)!\ \Gamma\left(\frac{k+m+q+2\alpha}{2}\right)}{m!\ \left(\frac{k-q-m}{2}\right)!\ \Gamma\left(\frac{k+m-q+2\alpha+2}{2}\right)}, \quad k \geq q. \end{aligned}$$

(For the proof see Doha [12]).

3.2. The fractional derivatives of $G_i^\alpha(t)$.

Lemma 3.2. *Let $G_i^\alpha(t)$ be a shifted Gegenbauer polynomial then*

$$D^{(\nu)}G^{\alpha}{}_i(t) = 0, \quad i = 0, 1, 2, \ldots, \lceil \nu \rceil - 1,\ \nu > 0.$$

Proof. Using (2.1)–(2.3) in (3.2) the lemma can be proved. □

Theorem 3.1. *The fractional derivative of order ν in the Caputo sense for the shifted Gegenbauer polynomials is given by*

$$D^{(\nu)}G^{\alpha}{}_i(t) = \sum_{j=0}^{\infty} S_\nu(i,j,\alpha) G_j^\alpha(t), \quad i = \lceil \nu \rceil, \lceil \nu \rceil + 1 \ldots, \tag{3.6}$$

where

$$S_\nu(i,j,\alpha) = \sum_{k=\lceil \nu \rceil}^{i} \theta_{i,j,k},$$

and

$$\begin{aligned}\theta_{i,j,k} =& \frac{(-1)^{i-k}\ 2^{4\alpha-1}\ \Gamma(j+1)\ (j+\alpha)\Gamma^2(\alpha)\ (2\alpha)_{i+k}}{\pi\ \Gamma(j+2\alpha)\ (i-k)!\ (\alpha+\frac{1}{2})_k\ \Gamma(k-\nu+1)} \\ &\times \sum_{l=0}^{j} \frac{(-1)^{j-l}\ L^{-\nu}\ (2\alpha)_{j+l}\ \Gamma(\alpha+\frac{1}{2})\ \Gamma(k+l+\alpha-\nu+\frac{1}{2})}{l!\ (j-l)!\ (\alpha+\frac{1}{2})_l\ \Gamma(2\alpha+k+l-\nu+1)}, \quad j = 0, 1, \ldots,\end{aligned}$$

where $D^{(\nu)}$ is the fractional derivative of order ν in the Caputo sense.

Proof. The analytic form of the shifted Gegenbauer polynomials $G_i^\alpha(t)$ of degree i is given by (3.2). Using (2.2), (2.3) and (3.2) we have

(3.7)

$$\begin{aligned}D^{(\nu)}G_i^\alpha(t) &= \sum_{k=0}^{i} \frac{(-1)^{i-k}\ (2\alpha)_{i+k}}{L^k\ k!\ (i-k)!\ (\alpha+\frac{1}{2})_k}\ D^\nu t^k \\ &= \sum_{k=\lceil \nu \rceil}^{i} \frac{(-1)^{i-k}\ (2\alpha)_{i+k}\ \Gamma(k+1)}{L^k\ k!\ (i-k)!\ (\alpha+\frac{1}{2})_k\ \Gamma(k-\nu+1)}\ t^{k-\nu}, \quad i = \lceil \nu \rceil,\ \lceil \nu \rceil + 1, \ldots\end{aligned}$$

Now approximate $t^{k-\nu}$ by $(N+1)$ terms of shifted Gegenbauer series, so we have

$$t^{k-\nu} \simeq \sum_{j=0}^{N} b_{k,j} G_j^\alpha(t), \tag{3.8}$$

where $b_{k,j}$ is given from (3.5) with $u(t) = t^{k-\nu}$ and

$$\begin{aligned}b_{k,j} =& \frac{2^{4\alpha-1}\ \Gamma(j+1)\ (j+\alpha)\ \Gamma^2(\alpha)}{\pi\ \Gamma(j+2\alpha)} \\ &\times \sum_{l=0}^{j} \frac{(-1)^{j-l}\ L^{k-\nu}\ (2\alpha)_{j+l}\ \Gamma(\alpha+\frac{1}{2})\ \Gamma(k+l+\alpha-\nu+\frac{1}{2})}{l!\ (j-l)!\ (\alpha+\frac{1}{2})_l\ \Gamma(2\alpha+k+l-\nu+1)}, \quad j = 0, 1, \ldots\end{aligned}$$

Employing (3.7)–(4.1), yields

$$D^{\nu}G_i^{\alpha}(t)=\sum_{j=0}^{\infty}S_{\nu}(i,j,\alpha)G_j^{\alpha}(t),\quad i=\lceil\nu\rceil,\ \lceil\nu\rceil+1,\ldots, \tag{3.9}$$

where $S_{\nu}(i,j,\alpha)$ is given as in (3.6) and this proves the theorem. □

Remark 3.1. It is to be noted here that if $\nu=q\in\mathbb{N}$, then Lemma 3.1 may be obtained as a special case of Theorem 3.1.

4. Shifted Gegenbauer Polynomials Interpolation Approximation

In this section, we use the shifted Gegenbauer-Gauss collocation method to solve numerically the following model problem:

$$(u(t)+a(t)u(p_m t))^{(\nu)}=\beta u(t)+\sum_{n=0}^{m-1}b_n(t)D^{\gamma_n}u(p_n t)+f(t),\quad t\geq 0, \tag{4.1}$$

with the initial conditions

$$\sum_{n=0}^{m-1}c_{in}u^{(n)}(0)=\lambda_i,\quad i=0,1,\ldots,m-1. \tag{4.2}$$

Here, a and b_n, $n=0,1,\ldots,m-1$, are given analytical functions, $m-1<\nu\leq m$, $0<\gamma_0<\gamma_1<\cdots<\gamma_{m-1}<\nu$ and β, p_n, c_{in}, λ_i denote given constants with $0<p_n<1$, $n=0,1,\ldots,m$.

By using the shifted Gegenbauer-Gauss collocation method, we can approximate the fractional neutral functional-differential equations with proportional delays, without any artificial boundary and variable transformation. Let us first introduce some basic notation that will be used in the sequel. We set

$$S_N(0,L)=\operatorname{Span}\{G_0^{\alpha}(t),G_1^{\alpha}(t),\ldots,G_N^{\alpha}(t)\},$$

and we define the discrete inner product and norm as follows:

$$(u,v)_{w_{L,N}^{\alpha}}=\sum_{j=0}^{N}u(t_{L,j}^{\alpha})\,v(t_{L,j}^{\alpha})\,\varpi_{L,j}^{\alpha},\quad \|u\|_{w_{L,N}^{\alpha}}=\sqrt{(u,u)_{w_{L,N}^{\alpha}}},$$

where $t_{L,j}^{\alpha}$ and $\varpi_{L,j}^{\alpha}$ are the nodes and the corresponding weights of the shifted Gegenbauer-Gauss quadrature formula on the interval $(0,L)$, respectively. Obviously,

$$(u,v)_{w_{L,N}^{\alpha}}=(u,v)_{w_L^{\alpha}},\quad \text{for all } u,v\in S_{2N+1}.$$

Thus, for any $u\in S_N(0,L)$, the norms $\|u\|_{w_{L,N}^{\alpha}}$ and $\|u\|_{w_L^{\alpha}}$ coincide.

Associating with this quadrature rule, we denote by $I_N^{G_T^{\alpha}}$ the shifted Gegenbauer-Gauss interpolation,

$$I_N^{G_T^{\alpha}}u(t_{L,k}^{\alpha})=u(t_{L,k}^{\alpha}),\quad 0\leq k\leq N.$$

The shifted Gegenbauer-Gauss collocation method for solving (4.1) and (4.2) is to seek $u_N(t) \in S_N(0, L)$, such that

$$D^{\nu}(u(t^{\alpha}_{L,k}) + a(t^{\alpha}_{L,k})u(p_m t^{\alpha}_{L,k})) = \beta u(t^{\alpha}_{L,k}) + \sum_{n=0}^{m-1} b_n(t^{\alpha}_{L,k}) D^{\gamma_n} u(p_n t^{\alpha}_{L,k}) + f(t^{\alpha}_{L,k}),$$

where $k = 0, 1, \ldots, N-m$, $\sum_{n=0}^{m-1} c_{in} u^{(n)}(0) = \lambda_i$, $i = 0, 1, \ldots, m-1$.

We now derive the algorithm for solving (4.1) and (4.2). To do this, let

$$u_N(t) = \sum_{h=0}^{N} a_h G^{\alpha}_h(t), \quad \mathbf{a} = (a_0, a_1, \ldots, a_N)^T. \tag{4.3}$$

We first approximate $D^{\nu}u(t)$ and $D^{\gamma_n}u(t)$, as (4.3). By substituting these approximation in (4.1), we get

$$\begin{aligned}\left(\sum_{h=0}^{N} a_h G^{\alpha}_h(t) + a(t)\sum_{h=0}^{N} a_h G^{\alpha}_h(p_m t)\right)^{(\nu)} =& \beta \sum_{h=0}^{N} a_h G^{\alpha}_h(t) + \sum_{n=0}^{m-1}\sum_{h=0}^{N} a_h b_n(t) D^{\gamma_n} G^{\alpha}_h(p_n t) \\ &+ f(t).\end{aligned} \tag{4.4}$$

Making use of (3.6), we deduce that

$$\begin{aligned}\left(\sum_{h=0}^{N} a_h G^{\alpha}_h(t) + a(t)\sum_{h=0}^{N} a_h G^{\alpha}_h(p_m t)\right)^{(\nu)} =& \beta \sum_{h=0}^{N} a_h G^{\alpha}_h(t) \\ &+ \sum_{n=0}^{m-1}\sum_{h=0}^{N}\sum_{f=0}^{M} a_h b_n(t) S_{\gamma_n}(h, f) G^{\alpha}_f(p_n t) + f(t).\end{aligned} \tag{4.5}$$

Also, by substituting (4.3) in (4.2) we obtain

$$\sum_{n=0}^{m-1}\sum_{f=0}^{M} a_{in} D^{(n)} G^{\alpha}_f(0) = \lambda_i. \tag{4.6}$$

Now, we collocate (4.5) at the $(N-m+1)$ shifted Gegenbauer-Gauss interpolation points, yields

$$\begin{aligned}&\left(\sum_{h=0}^{N} a_h G^{\alpha}_h(t^{\alpha}_{L,k}) + a(t^{\alpha}_{L,k})\sum_{h=0}^{N} a_h G^{\alpha}_h(p_m t^{\alpha}_{L,k})\right)^{(\nu)} \\ =&\beta \sum_{h=0}^{N} a_h G^{\alpha}_h(t^{\alpha}_{L,k}) + \sum_{n=0}^{m-1}\sum_{h=0}^{N}\sum_{f=0}^{M} a_h b_n(t^{\alpha}_{L,k}) S_{\gamma_n}(h, f) G^{\alpha}_f(p_n t^{\alpha}_{L,k}) + f(t^{\alpha}_{L,k}).\end{aligned} \tag{4.7}$$

Next (4.6), after using (3.4), can be written as

$$\sum_{n=0}^{m-1}\sum_{f=0}^{M} (-1)^{f-n} c_{in} \frac{2^{2n}\ (\alpha)_n\ \Gamma(f+2\alpha+n)}{L^n\ (f-n)!\ \Gamma(2\alpha+2n)} = \lambda_i. \tag{4.8}$$

Finally, (4.7), with relation (4.8) generate $(N+1)$ set of algebraic equations which can be solved for the unknown coefficients a_j, $j = 0, 1, 2, \ldots, N$, by using any standard solver technique.

5. Convergence and Error Analysis

This section is dedicated to investigating the convergence and error analysis of the suggested Gegenbauer expansion. In this regard, we follow Abd-Elhameed and Youssri [2], the following two theorems are stated and proved. In what follows by the notation $A \lesssim B$, we means that there exists a generic constant Υ such that $A \leq \Upsilon\, B$. The following three lemmas are of important use in sequel.

Lemma 5.1 ([1]).

$$\int G_j^{\alpha}(t) w_L^{\alpha}(t) dt = -\frac{\alpha\ 2^{1-2\alpha}\ L^{2\alpha}}{j(j+\alpha)} G_{j-1}^{\alpha+1}(t).$$

Lemma 5.2 ([10]). *For $\alpha > 0$ one have:*

$$|G_j^{\alpha}(t)| \lesssim \frac{(Lt - t^2)^{\frac{-\alpha}{2}}}{j^{\frac{3}{2}}}.$$

Lemma 5.3 ([34]).

$$\lim_{n\to\infty} \frac{\Gamma(n+\alpha)}{(n-1)!\ n^{\alpha}} = 1.$$

Theorem 5.1. *If $u(t)$ is expanded in a series of shifted Gegenbauer polynomial, has a bounded first derivative, then we have the following estimate: for all $\alpha > 0$ and $j > 1:\ |b_j| \lesssim j^{-2\alpha-\frac{3}{2}}$.*

Proof. Integration by parts on the right hand side of (3.5) and based on Lemma 5.1 we have

$$|b_j| \lesssim \frac{|\int_0^L u'(t)\ G_{j-1}^{\alpha+1}(t)\ dt|}{j^2\ \gamma_{L,j}^{\alpha}}. \tag{5.1}$$

Based on Lemma 5.2 and by hypothesis of the theorem we have

$$\left|\int_0^L u'(t)\ G_{j-1}^{\alpha+1}(t)\ dt\right| \lesssim \frac{M}{j^{\frac{3}{2}}}, \tag{5.2}$$

where M is the upper bound of $u'(t)$. Application of Lemma 5.3 will yield

$$|\gamma_{L,j}^{\alpha}| = \mathcal{O}(j^{2\alpha-2}). \tag{5.3}$$

Joining (5.1), (5.2) and (5.3), we have

$$b_j \lesssim j^{-\frac{3}{2}-2\alpha},$$

which completes the proof of the theorem. □

Theorem 5.2. *If* $u(t) = \sum_{j=0}^{\infty} b_j\, G_j^{\alpha}(t)$, $u_N(t) = \sum_{j=0}^{N} b_j\, G_j^{\alpha}(t)$ *are the exact and approximate solutions of* (4.1), *respectively and* $u(t)$ *satisfies the hypothesis of Theorem* 5.1, *then we have the following error estimate:*

$$\|u - u_N\|w_{L,N}^{\alpha} \lesssim 1/N^{2+\alpha}.$$

Proof.

$$\begin{aligned}\| u - u_N \|^2_{w^{\alpha}_{L,N}} &= \int_0^L (u - u_N)^2 w_L^{\alpha}\, dt \\ &= \left(\sum_{j=N+1}^{\infty} b_j\, G_j^{\alpha},\ \sum_{j=N+1}^{\infty} b_j\, G_j^{\alpha} \right)_{w_L^{\alpha}} \\ &= \sum_{j=N+1}^{\infty} b_j^2\, \gamma_{L,j}^{\alpha}.\end{aligned}$$

Now, based on the estimate in Theorem 5.1 and the estimate in (5.3), we get

$$\| u - u_N \|^2_{w^{\alpha}_{L,N}} \lesssim \sum_{j=N+1}^{\infty} j^{-5-2\alpha} \lesssim N^{-4-2\alpha},$$

which completes the proof of the theorem. □

6. Numerical results

In order to show the effectiveness of shifted Gegenbauer-Gauss collocation method for solving fractional neutral functional-differential equations with proportional delays, we present some numerical examples. The absolute errors in the given tables are the values of $|u(x) - u_N(x)|$ at selected points.

Example 6.1 ([7]). Consider the following fractional neutral functional-differential equation with proportional delay

$$u^{\frac{1}{2}}(t) = -u(t) + \frac{1}{3}u\left(\frac{t}{4}\right) + \frac{1}{2}u^{\frac{1}{2}}\left(\frac{t}{4}\right) + g(t), \quad u(0) = 1,\ t \in [0, 5], \tag{6.1}$$

where

$$g(t) = -\frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-x)^{-\frac{1}{2}} \sin(x)\, dx + \cos(t) - \frac{1}{3}\cos\left(\frac{t}{4}\right) + \frac{1}{2\Gamma(\frac{1}{2})}\int_0^t (t-x)^{-\frac{1}{2}} \sin\left(\frac{x}{4}\right) dx,$$

and the exact solution is given by $u(t) = \cos(t)$.

In Table 1, we list the absolute errors obtained by the shifted Gegenbauer-Gauss collocation method, with different values of α at $N = 22$. The outcomes are contrasted with the outcome of the modifed generalized Laguerre-Gauss collocation (MGLC) method [7]. It is clear from this table that, the solutions got by our technique are superior in examination with modifed generalized Laguerre-Gauss collocation scheme [7]. In order to compare the present method with the analytic solution, the resulting graph of (6.1) is shown in Figure 1.

TABLE 1. Comparison of the absolute errors at $N = 22$ for Example 6.1.

x	MGLC method [7]			Our method		
	$\alpha = 1$, $\beta = 2$	$\alpha = 4$, $\beta = 3$	$\alpha = 5$, $\beta = 5$	$\alpha = \frac{1}{2}$	$\alpha = 1$	$\alpha = \frac{3}{2}$
0	$7.993.10^{-15}$	$0.000.10^{-00}$	$2.220.10^{-16}$	$2.220.10^{-16}$	$0.00.10^{-00}$	$0.00.10^{-00}$
0.5	$5.164.10^{-4}$	$7.041.10^{-4}$	$1.746.10^{-4}$	$1.663.10^{-5}$	$2.466.10^{-5}$	$3.757.10^{-5}$
1	$5.066.10^{-4}$	$5.763.10^{-4}$	$1.416.10^{-4}$	$1.592.10^{-5}$	$2.466.10^{-5}$	$3.757.10^{-5}$
1.5	$3.521.10^{-4}$	$4.93.10^{-4}$	$1.257.10^{-4}$	$4.765.10^{-6}$	$1.804.10^{-5}$	$3.067.10^{-5}$
2	$2.793.10^{-4}$	$4.268.10^{-4}$	$1.059.10^{-4}$	$3.517.10^{-7}$	$1.684.10^{-5}$	$2.572.10^{-5}$
2.5	$4.480.10^{-4}$	$3.862.10^{-4}$	$8.856.10^{-5}$	$1.276.10^{-5}$	$1.684.10^{-5}$	$2.440.10^{-5}$
3	$2.269.10^{-4}$	$3.437.10^{-4}$	$8.429.10^{-5}$	$1.585.10^{-5}$	$1.694.10^{-5}$	$2.264.10^{-5}$
3.5	$1.998.10^{-4}$	$3.113.10^{-4}$	$7.249.10^{-5}$	$5.960.10^{-8}$	$1.066.10^{-5}$	$1.873.10^{-5}$
4	$5.164.10^{-4}$	$3.037.10^{-4}$	$6.271.10^{-5}$	$1.364.10^{-5}$	$5.364.10^{-6}$	$1.522.10^{-5}$
4.5	$1.141.10^{-3}$	$2.841.10^{-4}$	$1.159.10^{-4}$	$3.051.10^{-5}$	$4.649.10^{-6}$	$1.657.10^{-5}$

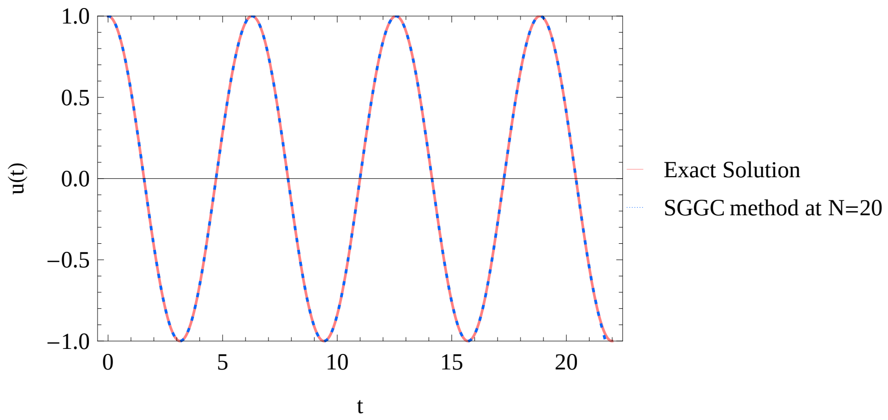

FIGURE 1. Graph of exact solution and approximate solution for $\alpha = \frac{5}{2}$ at $N = 20$ for Example 6.1.

Example 6.2. Consider the following fractional neutral functional-differential equation with proportional delay

$$u^{\frac{1}{2}}(t) = -u(t) + \frac{1}{4}u\left(\frac{t}{3}\right) + \frac{1}{3}u^{\frac{1}{2}}\left(\frac{t}{3}\right) + g(t), \quad u(0) = 0,\, t \in [0,1], \tag{6.2}$$

where

$$g(t) = \frac{\Gamma(q+1)}{\Gamma(q+\frac{1}{2})}t^{q-\frac{1}{2}} + t^q - \frac{1}{4}\left(\frac{t}{3}\right)^q - \frac{1}{3}\frac{\Gamma(q+1)}{\Gamma(q+\frac{1}{2})}\left(\frac{t}{3}\right)^{q-\frac{1}{2}},$$

and the exact solution is given by $u(t) = t^q$, $q \geq \lceil \frac{1}{2} \rceil$.

In Table 2, we list the absolute errors obtained by the shifted Gegenbauer-Gauss collocation method, with several values of t, q and at $N = 16$.

TABLE 2. Absolute errors using SGGC method at $N = 16$ for Example 6.2.

t	$q=1$	$q=1.3$	$q=1.5$	$q=1.7$	$q=1.9$
0.1	$1.259.10^{-4}$	$1.570.10^{-4}$	$7.967.10^{-5}$	$2.889.10^{-5}$	$4.584.10^{-6}$
0.2	$8.676.10^{-5}$	$1.784.10^{-4}$	$1.070.10^{-4}$	$4.379.10^{-5}$	$9.350.10^{-6}$
0.3	$3.542.10^{-4}$	$8.152.10^{-5}$	$3.702.10^{-5}$	$1.451.10^{-5}$	$1.102.10^{-6}$
0.4	$1.037.10^{-4}$	$2.138.10^{-5}$	$3.047.10^{-6}$	$1.577.10^{-6}$	$1.465.10^{-6}$
0.5	$3.993.10^{-4}$	$8.114.10^{-5}$	$5.435.10^{-5}$	$2.070.10^{-5}$	$5.960.10^{-6}$
0.6	$2.855.10^{-4}$	$1.182.10^{-4}$	$7.906.10^{-5}$	$3.327.10^{-5}$	$8.494.10^{-6}$
0.7	$2.324.10^{-4}$	$5.947.10^{-5}$	$3.263.10^{-5}$	$1.503.10^{-5}$	$2.164.10^{-6}$
0.8	$5.255.10^{-4}$	$3.819.10^{-5}$	$9.847.10^{-6}$	$4.389.10^{-6}$	$1.710.10^{-6}$
0.9	$2.723.10^{-4}$	$4.922.10^{-6}$	$1.289.10^{-5}$	$6.527.10^{-6}$	$3.031.10^{-6}$
1.0	$1.124.10^{-4}$	$5.547.10^{-5}$	$3.204.10^{-5}$	$1.146.10^{-5}$	$2.783.10^{-6}$

TABLE 3. Absolute errors using SGGC method at $N = 16$ for Example 6.3.

t	$\alpha=\frac{1}{2}$	$\alpha=1$	$\alpha=\frac{3}{2}$
0.0	$4.609.10^{-16}$	$7.844.10^{-16}$	$2.387.10^{-17}$
0.1	$6.161.10^{-6}$	$8.530.10^{-6}$	$1.075.10^{-5}$
0.2	$3.563.10^{-5}$	$4.959.10^{-5}$	$6.287.10^{-5}$
0.3	$9.834.10^{-5}$	$1.380.10^{-4}$	$1.758.10^{-4}$
0.4	$2.028.10^{-4}$	$2.857.10^{-4}$	$3.651.10^{-4}$
0.5	$3.579.10^{-3}$	$5.054.10^{-4}$	$6.457.10^{-4}$
0.6	$5.777.10^{-4}$	$8.090.10^{-4}$	$1.032.10^{-3}$
0.7	$8.657.10^{-4}$	$1.208.10^{-3}$	$1.541.10^{-3}$
0.8	$1.227.10^{-3}$	$1.714.10^{-3}$	$2.186.10^{-3}$
0.9	$1.671.10^{-3}$	$2.338.10^{-3}$	$2.981.10^{-3}$
1.0	$2.071.10^{-3}$	$2.971.10^{-3}$	$3.834.10^{-3}$

Example 6.3. Consider the following fractional neutral functional-differential equation with proportional delay

$$u^{\frac{5}{2}}(t) = u(t) + u^{\frac{1}{2}}\left(\frac{t}{3}\right) + u^{\frac{3}{2}}\left(\frac{t}{4}\right) + u^{\frac{5}{2}}\left(\frac{t}{5}\right) + g(t), \quad t \in [0,1], \tag{6.3}$$

subject to

$$u(0) = 0, \quad u'(0) = 0, \quad u''(0) = 0,$$

where

$$g(t) = \frac{32}{\sqrt{\pi}}t^{\frac{3}{2}} - t^4 - \frac{128}{945\sqrt{3\pi}}t^{\frac{7}{2}} - \frac{2}{5\sqrt{\pi}}t^{\frac{5}{2}} - \frac{32}{5\sqrt{5\pi}}x^{\frac{3}{2}},$$

and the exact solution is given by $u(t) = t^4$.

In Table 3, we list the absolute errors obtained by the shifted Gegenbauer-Gauss collocation method, with several values of α and at $N = 16$. Meanwhile, Figure 1

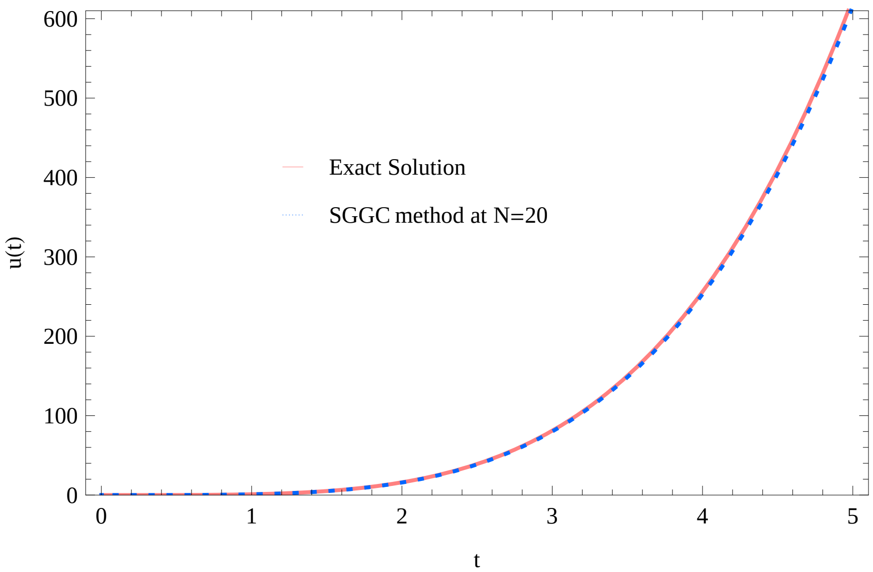

FIGURE 2. Graph of exact solution and approximate solution for $\alpha = \frac{5}{2}$ at $N = 20$ for Example 6.3.

presents the SGGC solution with $\alpha = \frac{5}{2}$ at $N = 20$ and exact solution, which are found to be in excellent agreement.

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REFERENCES

[1] W. M. Abd-Elhameed and Y. H. Youssri, *New ultraspherical wavelets spectral solutions for fractional Riccati differential equations*, Abstr. Appl. Anal. **2014** (2014), 8 pages.
[2] W. M. Abd-Elhameed and Y. H. Youssri, *New spectral solutions of multi-term fractional-order initial value problems with error analysis*, CMES-Comp. Model. Eng. Sci. **105** (2015), 375–398.
[3] A. H. Bhrawy, A. A. AL-Zahrani, Y. A. Alhamed and D. Baleanu, *A new generalized Laguerre-Gauss collocation scheme for numerical solution of generalized fractional pantograph equations*, Rom. Journ. Phys. **59** (2014), 646–657.
[4] R. L. Bagley and P. J. Torvik, *A theoritical asis for the appliation of fractional calculus to viscoelastiity*, J. Rheology **27** (1983), 201–210.
[5] R. L. Bagley and P. J. Torvik, *Fractional calculus in the transient analysis of viscoelastially damped struturesy*, AIAA J. **23** (1985), 918–925.
[6] R. Baillie, *Long memory processes and fractional integration in econometrices*, J. Econometrices **73** (1996), 5–59.
[7] A. H. Bhrawy, A. Al Zahrani, D. Baleanu and Y. Alhamed, *A modified generalized laguerre-gauss collocation method for fractional neutral functional-differential equations on the half-line*, Abstr. Appl. Anal. **2014** (2014), 7 pages.
[8] E. H. Doha, A. H. Bhrawy, D. Baleanu and R. M. Hafez, *Efficient Jacobi-Gauss collocation method for solving initial value problems of Bratu type*, Comp. Math. Math. Phys. **53** (2013), 1292–1302.

[9] E. H. Doha, A. H. Bhrawy, D. Baleanu and R. M. Hafez, *A new Jacobi rational-Gauss collocation method for numerical solution of generalized pantograph equations*, Appl. Numer. Math. **77** (2014), 43–54.

[10] T. S. Chow, *Fractional dynamics of interfaces etween soft-nanoparticls and rough sustrates*, Phys. Lett. A **342** (2005), 148–155.

[11] S. Das, *Functional Fractional Calculus for System Identification and Controls*, Springer, New York, 2008.

[12] E. H. Doha, *The coefficients of differentiated expansions and derivatives of ultraspherical polynomials*, J. Comp. Math. Appl. **19** (1991), 115–122.

[13] M. Ghasemi, M. Fardi and R. K. Ghaziani, *Numerical solution of nonlinear delay differential equations of fractional order in reproducing kernel Hilbert space*, Appl. Math. Comput. **268** (2015), 815–831.

[14] B. Fornberg, *A practical guide to pseudospectral methods*, Cambridge University Press, Cambridge, 1998.

[15] J. H. He, *Nonlinear oscillation with fractional derivatives and its applications* in: *International Conference on Vibrating Engineering'98*, Dalia, Chinan, 1998.

[16] J. H. He, *Some applications of nonlinear fractional differential equations and thier approximations bull*, Sci. Technol. **15** (1999), 86–90.

[17] M. Ishteva, *Properties and applications of the Caputo fractional operator*, PhD thesis, Department of Mathematics, Universitat Karlsruhe (TH), Sola, Bulgaria, 2005.

[18] C. Li and W. Deng, *Remarks on fractional derivatives*, Appl. Math. Comput. **187** (2007), 777–784.

[19] C. Li and Z. Zhao, *Introduction to fractional integrability and differentiability*, The European Physical Journal Special Topics **193** (2011), 5–26.

[20] R. Magin, *Frational calculus in bioengineering*, Crit. Rev. Biomed. Eng. **32** (2004), 1–104.

[21] R. Magin, *Frational calculus in bioengineering-part* 2, Crit. Rev. Biomed. Eng. **32** (2004), 105–193.

[22] R. Magin, *Frational calculus in bioengineering-part* 3, Crit. Rev. Biomed. Eng. **32** (2004), 194–377.

[23] F. Mainardi, *Fractional calculus: some basic prolems in continuum and statical mechanis*, in: A. Carpinteri and F. Mainardi (Eds.), *Fratals and Fractional Calculus in Continuum Mechanics*, Springer, Verlag, New York, 1997, 291–348.

[24] B. Mandelbort, *Some noises with* $1/f$ *spectrum, a bridge between direct current and white noise*, IEEE Trans. Inform. Theory **13** (1967), 289–298.

[25] R. Metzler and J. Klafter, *The restaurant at the end of the random walk: Recent developments in the description of anomalos transport by fractional dyuamics*, J. Phys. A **37** (2004), 161–208.

[26] K. S. Miller and B. Ross, *An Introduaction to the Frational Calculus and Fractional Differential Equations*, Wiley, New York, 1993.

[27] K. B. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, New York, 1974.

[28] P. Rahimkhani, Y. Ordokhani and E. Babolian, *Numerical solution of fractional pantograph differential equations by using generalized fractional-order Bernoulli wavelet*, J. Comput. Appl. Math. **309** (2017), 493–510.

[29] A. Anapali, Y. Öztürk and M. Gülsu, *Numerical approach for solving fractional pantograph equation*, IJCA **113** (2015), 45–52.

[30] R. Peyret, *Spectral Methods for Incompressible Viscous Flow*, New York, Springer, 2002.

[31] I. Podluny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[32] C. Li D. Qian and Y. Chen, *On Riemann-Liouville and Caputo derivatives*, Discrete Dynamics in Nature and Society **2011** (2011), 15 pages.

[33] C. Canuto, M. Y. Hussaini, A. Quarteroni and T. A. Zang, *Spectral Methods in Fluid Dynamics*, New York, Springer, 1988.

[34] E. D. Rainville, *Special functions*, Chelsea Pub Co., New York, 1971.
[35] Y. A. Rossikhin and M. V. Shitikova, *Applications of fractional calculus to dynamic prolems of linear and nonlinear hereditary mechanics of solids*, Appl. Mech. Rev. **50** (1997), 15–67.
[36] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, San Diego, 2006.
[37] G. Szegö, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, Providence, Rhode Island, 1975.
[38] L. N. Trefethen, *Spectral Methods in MATLAB*, Philadelphia, PA: SIAM, 2000.
[39] Y. Yang and Y. Huang, *Spectral-collocation methods for fractional pantograph delay-integrodifferential equations*, Adv. Math. Phys. **2013** (2013), 14 pages.

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