

SOME EXTENSIONS OF A THEOREM OF PAUL TURÁN CONCERNING POLYNOMIALS

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ABSTRACT. In this paper, certain new results concerning the maximum modulus of the polar derivative of a polynomial with restricted zeros are obtained. These estimates strengthen some well known inequalities for polynomial due to Turán, Dubinin and others.

1. INTRODUCTION

In scientific disciplines like physics, engineering, computer science, biology, physical chemistry, economics, and other applied areas, experimental observations and investigations when translated into mathematical language are called mathematical models. The solution of these models could lead to problems of estimating how large or small the maximum modulus of the derivative of an algebraic polynomial can be in terms of the maximum modulus of that polynomial. Bounds for such type of problems are of some practical importance. Since, there are no closed formulae for precise evaluation of these bounds and whatever is available in literature is in the form of approximations. However for practical purposes, nobody ever needs exacts bounds and mathematicians must only indicate methods for obtaining approximate bounds. These approximate bounds, when computed efficiently, are quite satisfactory for the needs of investigators and scientists. Therefore there is always a desire to look for better and improved bounds than those available in literature. It is this aspiration of obtaining more refined and revamped bounds that has inspired our work in this article. In this paper, we have generalized and refined some well known results concerning the polynomials due to Turán [16], Dubinin [6] and others. To begin with let \mathcal{P}_n denote the linear space of all polynomials of the form $P(z) = \sum_{\nu=0}^n a_{\nu}z^{\nu}$ of degree $n \geq 1$

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and let $P'(z)$ be the derivative of $P(z)$. Then concerning the lower bound for the maximum of $|P'(z)|$ in terms of maximum of $|P(z)|$ for class of polynomials $P \in \mathcal{P}_n$ not vanishing outside unit disc, Turán [16] showed that

$$(1.1) \quad \max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|.$$

Equality in inequality (1.1) holds for those polynomials $P \in \mathcal{P}_n$ which have all their zeros on $|z| = 1$. As an extension of (1.1), Govil [8] proved that if $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \leq k, k \geq 1$, then

$$(1.2) \quad \max_{|z|=1} |P'(z)| \geq \frac{n}{1+k^n} \max_{|z|=1} |P(z)|.$$

In literature, there exist several generalizations and extensions of (1.1) and (1.2) (see [1, 2, 4, 5, 13–15]). Dubinin [6] refined inequality (1.1) by proving that if all the zeros of $P \in \mathcal{P}_n$ lie in $|z| \leq 1$, then

$$(1.3) \quad \max_{|z|=1} |P'(z)| \geq \frac{1}{2} \left(n + \frac{\sqrt{|a_n|} - \sqrt{|a_0|}}{\sqrt{|a_n|}} \right) \max_{|z|=1} |P(z)|.$$

The polar derivative $D_\alpha P(z)$ of $P \in \mathcal{P}_n$ with respect to the point $\alpha \in \mathbb{C}$ is defined by

$$D_\alpha P(z) := nP(z) + (\alpha - z)P'(z).$$

The polynomial $D_\alpha P(z)$ is of degree at most $n - 1$ and it generalizes the ordinary derivative $P'(z)$ of $P(z)$ in the sense that

$$\lim_{\alpha \rightarrow \infty} \frac{D_\alpha P(z)}{\alpha} = P'(z),$$

uniformly for $|z| \leq R, R > 0$.

A. Aziz [1], Aziz and Rather ([4, 5]) obtained several sharp estimates for maximum modulus of $D_\alpha P(z)$ on $|z| = 1$ and among other things they extended inequality (1.2) to the polar derivative of a polynomial by showing that if $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq k, k \geq 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$

$$(1.4) \quad \max_{|z|=1} |D_\alpha P(z)| \geq n \left(\frac{|\alpha| - k}{1 + k^n} \right) \max_{|z|=1} |P(z)|.$$

2. MAIN RESULTS

In this paper, we obtain certain refinements and generalizations of inequalities (1.1), (1.2), (1.3) and (1.4). We first prove the following result.

Theorem 2.1. *If $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \leq k, k \geq 1$, then for every $\alpha \in \mathbb{C}$, with $|\alpha| \geq k$*

$$(2.1) \quad \max_{|z|=1} |D_\alpha P(z)| \geq \frac{|\alpha| - k}{1 + k^n} \left(n + \frac{\sqrt{k^n |a_n|} - \sqrt{|a_0|}}{\sqrt{k^n |a_n|}} \right) \left(\max_{|z|=1} |P(z)| + \frac{|a_{n-1}| \phi(k)}{k} \right) + \psi(k) |na_0 + \alpha a_1|,$$

where $\phi(k) = \left(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\right)$ or $\frac{(k-1)^2}{2}$ and $\psi(k) = 1 - \frac{1}{k^2}$ or $1 - \frac{1}{k}$ according as $n > 2$ or $n = 2$.

Remark 2.1. Since all the zeros of $P(z)$ lie in $|z| \leq k, k \geq 1$, it follows that $\sqrt{|a_0|} \leq \sqrt{k^n|a_n|}$. In view of this, inequality (2.1) refines inequality (1.4).

If we divide the two sides of inequality (2.1) by $|\alpha|$ and let $|\alpha| \rightarrow \infty$, we get the following result.

Corollary 2.1. *If $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \leq k, k \geq 1$, then*

$$(2.2) \quad \max_{|z|=1} |P'(z)| \geq \frac{1}{1+k^n} \left(n + \frac{\sqrt{k^n|a_n|} - \sqrt{|a_0|}}{\sqrt{k^n|a_n|}} \right) \left(\max_{|z|=1} |P(z)| + \frac{|a_{n-1}|}{k} \phi(k) \right) + \psi(k)|a_1|,$$

where $\phi(k) = \left(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\right)$ or $\frac{(k-1)^2}{2}$ and $\psi(k) = 1 - \frac{1}{k^2}$ or $1 - \frac{1}{k}$ according as $n > 2$ or $n = 2$.

The result is best possible and equality in inequality (2.2) holds for $P(z) = z^n + k^n$.

Remark 2.2. As before, it can be easily seen that inequality (2.2) refines inequality (1.2). Further for $k = 1$, inequality (2.2) reduces to inequality (1.3).

Next, we present the following result which is generalisation of Theorem 2.1 and in particular, includes refinement of inequality (1.2) as a special case.

Theorem 2.2. *If all the zeros of $P \in \mathcal{P}_n$ lie in $|z| \leq k, k \geq 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k, 0 \leq l < 1$,*

$$(2.3) \quad \max_{|z|=1} |D_\alpha P(z)| \geq \frac{n}{1+k^n} \left\{ (|\alpha| - k) \max_{|z|=1} |P(z)| + (|\alpha| + 1/k^{n-1})lm \right\} + \frac{(|\alpha| - k)}{k^n(k^n + 1)} \left(\frac{\sqrt{k^n|a_n|} - lm - \sqrt{|a_0|}}{\sqrt{k^n|a_n|} - lm} \right) \left(k^n \max_{|z|=1} |P(z)| - lm \right) + \frac{(|\alpha| - k)|a_{n-1}| \phi(k)}{k(1+k^n)} \left(n + \frac{\sqrt{k^n|a_n|} - lm - \sqrt{|a_0|}}{\sqrt{k^n|a_n|} - lm} \right) + \psi(k)|na_0 + \alpha a_1|,$$

where $m = \min_{|z|=k} |P(z)|, \phi(k) = \left(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\right)$ or $\frac{(k-1)^2}{2}$ and $\psi(k) = 1 - \frac{1}{k^2}$ or $1 - \frac{1}{k}$ according as $n > 2$ or $n = 2$.

If we divide both sides of inequality (2.3) by $|\alpha|$ and let $|\alpha| \rightarrow \infty$, we get the following result.

Corollary 2.2. *If all the zeros of $P \in \mathcal{P}_n$ lie in $|z| \leq k$, $k \geq 1$, then for $0 \leq l < 1$*

$$\begin{aligned}
 \max_{|z|=1} |P'(z)| &\geq \frac{n}{1+k^n} \left\{ \max_{|z|=1} |P(z)| + lm \right\} + \psi(k)|a_1| \\
 (2.4) \qquad &+ \frac{1}{k^n(k^n+1)} \left(\frac{\sqrt{k^n|a_n|-lm}-\sqrt{|a_0|}}{\sqrt{k^n|a_n|-lm}} \right) \left(k^n \max_{|z|=1} |P(z)| - lm \right) \\
 &+ \frac{|a_{n-1}|}{k(1+k^n)} \left(n + \frac{\sqrt{k^n|a_n|-lm}-\sqrt{|a_0|}}{\sqrt{k^n|a_n|-lm}} \right) \phi(k),
 \end{aligned}$$

where $m = \min_{|z|=k} |P(z)|$, $\phi(k) = \left(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2} \right)$ or $\frac{(k-1)^2}{2}$ and $\psi(k) = 1 - \frac{1}{k^2}$ or $1 - \frac{1}{k}$ according as $n > 2$ or $n = 2$.

Remark 2.3. For $l = 0$, Corollary 2.2 reduces to Corollary 2.1 and for $k = 1$, inequality (2.4) refines inequality (1.3).

3. LEMMAS

We need the following lemmas for the proof of our theorems. The first lemma is due to Dubinin [6].

Lemma 3.1. *If $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \leq 1$, then*

$$(3.1) \qquad |P'(z)| \geq \frac{1}{2} \left(n + \frac{\sqrt{|a_n|} - \sqrt{|a_0|}}{\sqrt{|a_n|}} \right) |P(z)|, \quad \text{for } |z| = 1.$$

The next lemma is special case of a result due to Aziz and Rather [3, 4].

Lemma 3.2. *If $P \in \mathcal{P}_n$ and $P(z)$ has its all zeros in $|z| \leq 1$, then for $|z| = 1$*

$$|Q'(z)| \leq |P'(z)|,$$

where $Q(z) = z^n \overline{P(1/\bar{z})}$.

Lemma 3.3. *If all the zeros of $P \in \mathcal{P}_n$ lie in a circular region C and w is any zero of $D_\alpha P(z)$, the polar derivative of $P(z)$, then at most one of the points w and α may lie outside C .*

The above lemma is due to Laguerre (see [10]). The following lemma is due to Frappier, Rahman and Ruscheweyh [7].

Lemma 3.4. *If $P(z)$ is a polynomial of degree at most $n \geq 1$, then for $R \geq 1$*

$$(3.2) \qquad \max_{|z|=R} |P(z)| \leq R^n \max_{|z|=1} |P(z)| - (R^n - R^{n-2})|P(0)|, \quad \text{if } n \geq 2,$$

and

$$(3.3) \qquad \max_{|z|=R} |P(z)| \leq R \max_{|z|=1} |P(z)| - (R - 1)|P(0)|, \quad \text{if } n = 1.$$

Next lemma is the famous result of P. D. Lax [9].

Lemma 3.5. *If $P \in \mathcal{P}_n$ does not vanish in $|z| < 1$, then*

$$|P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|, \quad \text{for } |z| = 1.$$

We also need the following lemma.

Lemma 3.6. *If $P(z) = a_n \prod_{j=1}^n (z - z_j)$ is a polynomial of degree $n \geq 2$ having no zero in $|z| < 1$, then for $R \geq 1$*

$$(3.4) \quad \max_{|z|=R} |P(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1} |P(z)| - \left(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2} \right) |P'(0)|, \quad \text{if } n > 2,$$

and

$$(3.5) \quad \max_{|z|=R} |P(z)| \leq \frac{R^2 + 1}{2} \max_{|z|=1} |P(z)| - \frac{(R - 1)^2}{2} |P'(0)|, \quad \text{if } n = 2.$$

Proof of Lemma 3.6. For each θ , $0 \leq \theta < 2\pi$, we have

$$(3.6) \quad P(Re^{i\theta}) - P(e^{i\theta}) = \int_1^R e^{i\theta} P'(te^{i\theta}) dt,$$

which gives with the help of (3.2) of Lemma 3.4 and Lemma 3.5 for $n > 2$

$$\begin{aligned} |P(Re^{i\theta}) - P(e^{i\theta})| &\leq \int_1^R |P'(te^{i\theta})| dt \\ &\leq \frac{n}{2} \int_1^R t^{n-1} dt \max_{|z|=1} |P(z)| - \int_1^R (t^{n-1} - t^{n-3}) dt |P'(0)| \\ &= \frac{R^n - 1}{2} \max_{|z|=1} |P(z)| - \left(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2} \right) |P'(0)|. \end{aligned}$$

Consequently for $n > 2$ and $0 \leq \theta < 2\pi$, we have

$$\begin{aligned} |P(Re^{i\theta})| &\leq |P(Re^{i\theta}) - P(e^{i\theta})| + |P(e^{i\theta})| \\ &\leq \frac{R^n + 1}{2} \max_{|z|=1} |P(z)| - \left(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2} \right) |P'(0)|, \end{aligned}$$

which immediately leads to (3.4). Similarly we can prove inequality (3.5) by using inequality (3.3) of Lemma 3.4. This proves Lemma 3.6. \square

Finally we require the following lemma.

Lemma 3.7. *If $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq k$, where $k \geq 1$, then for $0 \leq l < 1$*

$$(3.7) \quad \begin{aligned} \max_{|z|=k} |P(z)| &\geq \frac{2k^n}{1 + k^n} \max_{|z|=1} |P(z)| + l \left(\frac{k^n - 1}{k^n + 1} \right) \min_{|z|=k} |P(z)| \\ &\quad + \frac{2k^{n-1} |a_{n-1}|}{k^n + 1} \left(\frac{k^n - 1}{n} - \frac{k^{n-2} - 1}{n-2} \right), \quad \text{if } n > 2, \end{aligned}$$

and

$$(3.8) \quad \max_{|z|=k} |P(z)| \geq \frac{2k^2}{1+k^2} \max_{|z|=1} |P(z)| + l \left(\frac{k^2-1}{k^2+1} \right) \min_{|z|=k} |P(z)| + \frac{k(k-1)^2|a_1|}{k^2+1}, \quad \text{if } n = 2.$$

Proof of Lemma 3.7. Since all the zeros of $P \in \mathcal{P}_n$ lie in $|z| \leq k, k \geq 1$, therefore, all the zeros of $g(z) = P(kz)$ lie in $|z| \leq 1$ and hence all the zeros of $f(z) = z^n \overline{g(1/\bar{z})} = z^n \overline{P(k/\bar{z})}$ lie in $|z| \geq 1$. Moreover, $m = \min_{|z|=k} |P(z)| = \min_{|z|=1} |f(z)|$, so that

$$m|z^n| \leq |f(z)|, \quad \text{for } |z| = 1.$$

We show that for $\lambda \in \mathbb{C}$ with $|\lambda| < 1, f(z) + \lambda m z^n \neq 0$ in $|z| < 1$. This is trivially true if $m = 0$. Henceforth we suppose that $m \neq 0$, so that all the zeros of $f(z)$ lie in $|z| > 1$. By the maximum modulus theorem

$$(3.9) \quad m|z^n| < |f(z)|, \quad \text{for } |z| < 1.$$

Now if there is point $z = z_0$ with $|z_0| < 1$, such that $f(z_0) + \lambda m z_0^n = 0$, then

$$|f(z_0)| = |\lambda| |z_0^n| m < |z_0^n| m,$$

a contradiction to inequality (3.9). Hence, it follows that the polynomial $T(z) = f(z) + \lambda m z^n$ does not vanish in $|z| < 1$. Applying inequality (3.4) of Lemma 3.6 to the polynomial $T(z)$, with $R = k \geq 1$ and $n > 2$, we get for $|z| = 1$,

$$|f(kz) + \lambda k^n m z^n| \leq \frac{k^n + 1}{2} |f(z) + \lambda m z^n| - \left(\frac{k^n - 1}{n} - \frac{k^{n-2} - 1}{n-2} \right) |f'(0)|.$$

Which implies, for $n > 2$,

$$(3.10) \quad |f(kz) + \lambda m k^n z^n| \leq \frac{k^n + 1}{2} (|f(z)| + |\lambda m|) - \left(\frac{k^n - 1}{n} - \frac{k^{n-2} - 1}{n-2} \right) |f'(0)|.$$

Choosing argument of λ suitably in the left hand side of inequality (3.10), we get for $n > 2$

$$|f(kz)| + |\lambda| m k^n \leq \frac{k^n + 1}{2} (|f(z)| + |\lambda| m) - \left(\frac{k^n - 1}{n} - \frac{k^{n-2} - 1}{n-2} \right) k^{n-1} |a_{n-1}|.$$

Replacing $f(z)$ by $z^n \overline{P(k/\bar{z})}$, we obtain for $n > 2$ and $|z| = 1$

$$\begin{aligned} k^n \max_{|z|=1} |P(z)| + |\lambda| m k^n &\leq \frac{k^n + 1}{2} \left(\max_{|z|=k} |P(z)| + |\lambda| m \right) \\ &\quad - \left(\frac{k^n - 1}{n} - \frac{k^{n-2} - 1}{n-2} \right) k^{n-1} |a_{n-1}|, \end{aligned}$$

which on simplification yields inequality (3.7). In a similar manner we can prove inequality (3.8) by applying inequality (3.5) of Lemma 3.6 instead of inequality (3.4) to the polynomial $T(z)$. This proves Lemma 3.7. □

4. Proof of the Theorems

Proof of Theorem 2.1. Let $f(z) = P(kz)$. Since $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq k$ where $k \geq 1$, therefore, $f \in \mathcal{P}_n$ and $f(z)$ has all its zeros in $|z| \leq 1$. If $Q(z) = z^n f(1/\bar{z})$, then it is easy to verify that

$$(4.1) \quad |Q'(z)| = |nf(z) - zf'(z)|, \quad \text{for } |z| = 1.$$

Combining (4.1) with Lemma 3.2, we get

$$(4.2) \quad |f'(z)| \geq |nf(z) - zf'(z)|, \quad \text{for } |z| = 1.$$

Now for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$, we have for $|z| = 1$,

$$|D_{\alpha/k}f(z)| = |nf(z) + (\alpha/k - z)f'(z)| \geq |\alpha/k||f'(z)| - |nf(z) - zf'(z)|,$$

which gives with the help of (4.2)

$$(4.3) \quad |D_{\alpha/k}f(z)| \geq \left(\frac{|\alpha| - k}{k}\right) |f'(z)|.$$

Consequently,

$$(4.4) \quad \max_{|z|=k} |D_{\alpha}P(z)| \geq (|\alpha| - k) \max_{|z|=k} |P'(z)|.$$

Again since all the zeros of $f(z) = P(kz)$ lie in $|z| \leq 1$, therefore, using Lemma 3.1, we have

$$|f'(z)| \geq \frac{1}{2} \left(n + \frac{\sqrt{k^n|a_n|} - \sqrt{|a_0|}}{\sqrt{k^n|a_n|}} \right) |f(z)|, \quad \text{for } |z| = 1.$$

Replacing $f(z)$ by $P(kz)$, we obtain

$$k \max_{|z|=1} |P'(kz)| \geq \frac{1}{2} \left(n + \frac{\sqrt{k^n|a_n|} - \sqrt{|a_0|}}{\sqrt{k^n|a_n|}} \right) \max_{|z|=1} |P(kz)|,$$

which implies

$$(4.5) \quad \max_{|z|=k} |P'(z)| \geq \frac{1}{2k} \left(n + \frac{\sqrt{k^n|a_n|} - \sqrt{|a_0|}}{\sqrt{k^n|a_n|}} \right) \max_{|z|=k} |P(z)|.$$

Combining inequality (4.4) and inequality (4.5), we have

$$(4.6) \quad \max_{|z|=k} |D_{\alpha}P(z)| \geq \frac{(|\alpha| - k)}{2k} \left(n + \frac{\sqrt{k^n|a_n|} - \sqrt{|a_0|}}{\sqrt{k^n|a_n|}} \right) \max_{|z|=k} |P(z)|.$$

Further since $D_{\alpha}P(z)$ is a polynomial of degree at most $n - 1$, using inequality (3.2) of Lemma 3.4, we have for $n > 2$

$$\max_{|z|=R} |D_{\alpha}P(z)| \leq R^{n-1} \max_{|z|=1} |D_{\alpha}P(z)| - (R^{n-1} - R^{n-3})|na_0 + \alpha a_1|.$$

Using this inequality and inequality (3.7) of Lemma 3.7 with $l = 0$ and $R = k \geq 1$ in (4.6), we have for $n > 2$

$$\begin{aligned} & k^{n-1} \max_{|z|=1} |D_\alpha P(z)| - (k^{n-1} - k^{n-3})|na_0 + \alpha a_1| \\ & \geq \frac{(|\alpha| - k)}{2k} \left(n + \frac{\sqrt{k^n |a_n|} - \sqrt{|a_0|}}{\sqrt{k^n |a_n|}} \right) \\ & \quad \times \left\{ \frac{2k^n}{1 + k^n} \max_{|z|=1} |P(z)| + \frac{2k^{n-1}|a_{n-1}|}{k^n + 1} \left(\frac{k^n - 1}{n} - \frac{k^{n-2} - 1}{n - 2} \right) \right\}, \end{aligned}$$

which on simplification gives

$$\begin{aligned} \max_{|z|=1} |D_\alpha P(z)| & \geq \frac{|\alpha| - k}{1 + k^n} \left(n + \frac{\sqrt{k^n |a_n|} - \sqrt{|a_0|}}{\sqrt{k^n |a_n|}} \right) \\ & \quad \times \left\{ \max_{|z|=1} |P(z)| + \frac{|a_{n-1}|}{k} \left(\frac{k^n - 1}{n} - \frac{k^{n-2} - 1}{n - 2} \right) \right\} \\ & \quad + (1 - 1/k^2)|na_0 + \alpha a_1|, \quad \text{if } n > 2. \end{aligned}$$

The above inequality is equivalent to the inequality (2.1) for $n > 2$. For $n = 2$, the result follows on similar lines by using inequality (3.3) of Lemma 3.4 and inequality (3.8) of Lemma 3.7 in the inequality (4.6). This completes the proof of Theorem 2.1. \square

Proof of Theorem 2.2. By hypothesis $P \in \mathcal{P}_n$ has all zeros in $|z| \leq k$, $k \geq 1$. If $P(z)$ has a zero on $|z| = k$, then $m = \min_{|z|=k} |P(z)| = 0$ and result follows from Theorem 2.1. Henceforth, we suppose that $P(z)$ has all its zeros in $|z| < k$, $k \geq 1$, so that $m > 0$. Now if $f(z) = P(kz)$, then $f \in \mathcal{P}_n$ and $f(z)$ has all zeros in $|z| < 1$ and $m = \min_{|z|=k} |P(z)| = \min_{|z|=1} |f(z)|$. This implies

$$m \leq |f(z)|, \quad \text{for } |z| = 1.$$

By the Rouché's Theorem, we conclude that for every $\lambda \in \mathbb{C}$ with $|\lambda| < 1$, the polynomial $g(z) = f(z) - \lambda m z^n$ has all zeros in $|z| < 1$. Applying inequality (4.3) to the polynomial $g(z)$, it follows for $|z| = 1$ and $|\alpha| \geq k$

$$|D_{\alpha/k} g(z)| \geq \left(\frac{|\alpha| - k}{k} \right) |g'(z)|.$$

Since all the zeros of $g(z)$ lie in $|z| < 1$, using Lemma 3.1, we obtain for $|z| = 1$ and $|\alpha| \geq k$

$$|D_{\alpha/k} g(z)| \geq \frac{1}{2} \left(\frac{|\alpha| - k}{k} \right) \left(n + \frac{\sqrt{|k^n a_n - \lambda m|} - \sqrt{|a_0|}}{\sqrt{|k^n a_n - \lambda m|}} \right) |g(z)|.$$

Using the fact that the function $S(x) = \frac{x - \sqrt{|a_0|}}{x}$, $x > 0$, is non-decreasing function of x and $|k^n a_n - \lambda m| \geq k^n |a_n| - |\lambda| m > 0$, we get for every $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ and $|z| = 1$

$$(4.7) \quad |D_{\alpha/k} g(z)| \geq \frac{1}{2} \left(\frac{|\alpha| - k}{k} \right) \left(n + \frac{\sqrt{k^n |a_n| - |\lambda| m} - \sqrt{|a_0|}}{\sqrt{k^n |a_n| - |\lambda| m}} \right) |g(z)|.$$

Replacing $g(z)$ by $f(z) - \lambda m z^n$ in (4.7), we get for $|z| = 1$ and $|\alpha| \geq k$

$$(4.8) \quad \begin{aligned} & \left| D_{\alpha/k} f(z) - \frac{nm\alpha\lambda}{k} z^{n-1} \right| \\ & \geq \frac{1}{2} \left(\frac{|\alpha| - k}{k} \right) \left(n + \frac{\sqrt{k^n |a_n| - |\lambda| m} - \sqrt{|a_0|}}{\sqrt{k^n |a_n| - |\lambda| m}} \right) (|f(z) - \lambda m|). \end{aligned}$$

Since all the zeros of $f(z) - \lambda m z^n = g(z)$ lie in $|z| < 1$ and $|\alpha/k| \geq 1$, it follows by Lemma 3.3 that all the zeros of

$$D_{\alpha/k}(f(z) - m\lambda z^n) = D_{\alpha/k} f(z) - \frac{nm\alpha\lambda}{k} z^{n-1}$$

lie in $|z| < 1$. This implies that

$$(4.9) \quad |D_{\alpha/k} f(z)| \geq \frac{nm|\alpha|}{k} |z|^{n-1}, \quad \text{for } |z| \geq 1.$$

In view of this inequality, choosing argument of λ in the left hand side of inequality (4.8) such that

$$\left| D_{\alpha/k} f(z) - \frac{nm\alpha\lambda}{k} z^{n-1} \right| = |D_{\alpha/k} f(z)| - \frac{nm|\alpha||\lambda|}{k}, \quad \text{for } |z| = 1,$$

we get for $|z| = 1$ and $|\alpha| \geq k$

$$\begin{aligned} & |D_{\alpha/k} f(z)| - \frac{nm|\alpha||\lambda|}{k} \\ & \geq \frac{1}{2} \left(\frac{|\alpha| - k}{k} \right) \left(n + \frac{\sqrt{k^n |a_n| - |\lambda| m} - \sqrt{|a_0|}}{\sqrt{k^n |a_n| - |\lambda| m}} \right) (|f(z)| - |\lambda| m), \end{aligned}$$

which on simplification yields

$$\begin{aligned} |D_{\alpha/k} f(z)| & \geq \frac{1}{2} \left(\frac{|\alpha| - k}{k} \right) \left(n + \frac{\sqrt{k^n |a_n| - |\lambda| m} - \sqrt{|a_0|}}{\sqrt{k^n |a_n| - |\lambda| m}} \right) |f(z)| \\ & \quad - \frac{1}{2} \left(\frac{|\alpha| - k}{k} \right) \left(\frac{\sqrt{k^n |a_n| - |\lambda| m} - \sqrt{|a_0|}}{\sqrt{k^n |a_n| - |\lambda| m}} \right) |\lambda| m + \frac{n}{2} \left(\frac{|\alpha| + k}{k} \right) |\lambda| m. \end{aligned}$$

This implies for $|z| = 1$ and $|\alpha| \geq k$

(4.10)

$$\begin{aligned} \max_{|z|=k} |D_\alpha P(z)| &\geq \frac{1}{2} \left(\frac{|\alpha| - k}{k} \right) \left(n + \frac{\sqrt{k^n |a_n| - |\lambda|m - \sqrt{|a_0|}}}{\sqrt{k^n |a_n| - |\lambda|m}} \right) \max_{|z|=k} |P(z)| \\ &\quad - \frac{1}{2} \left(\frac{|\alpha| - k}{k} \right) \left(\frac{\sqrt{k^n |a_n| - |\lambda|m - \sqrt{|a_0|}}}{\sqrt{k^n |a_n| - |\lambda|m}} \right) |\lambda|m + \frac{n}{2} \left(\frac{|\alpha| + k}{k} \right) |\lambda|m. \end{aligned}$$

Moreover, since $D_\alpha P(z)$ is a polynomial of degree at most $n - 1$, applying inequality (3.2) of Lemma 3.4 and inequality (3.7) of Lemma 3.7 with $R = k \geq 1$, we obtain for $|\alpha| \geq k$, $0 \leq l < 1$ and $|z| = 1$

$$\begin{aligned} &k^{n-1} \max_{|z|=1} |D_\alpha P(z)| - (k^{n-1} - k^{n-3}) |na_0 + \alpha a_1| \\ &\geq \frac{1}{2} \left(\frac{|\alpha| - k}{k} \right) \left(n + \frac{\sqrt{k^n |a_n| - lm - \sqrt{|a_0|}}}{\sqrt{k^n |a_n| - lm}} \right) \\ &\quad \times \left\{ \frac{2k^n}{1 + k^n} \max_{|z|=1} |P(z)| + \frac{k^n - 1}{k^n + 1} lm + \frac{2k^{n-1} |a_{n-1}|}{k^n + 1} \left(\frac{k^n - 1}{n} - \frac{k^{n-2} - 1}{n - 2} \right) \right\} \\ &\quad - \frac{1}{2} \left(\frac{|\alpha| - k}{k} \right) \left(\frac{\sqrt{k^n |a_n| - lm - \sqrt{|a_0|}}}{\sqrt{k^n |a_n| - lm}} \right) lm + \frac{n}{2} \left(\frac{|\alpha| + k}{k} \right) lm, \quad \text{if } n > 2. \end{aligned}$$

Equivalently, we have for $|\alpha| \geq k$, $0 \leq l < 1$ and $|z| = 1$

$$\begin{aligned} \max_{|z|=1} |D_\alpha P(z)| &\geq \frac{n}{1 + k^n} \left\{ (|\alpha| - k) \max_{|z|=1} |P(z)| + (|\alpha| + 1/k^{n-1}) lm \right\} \\ &\quad + \frac{(|\alpha| - k)}{k^n (k^n + 1)} \left(\frac{\sqrt{k^n |a_n| - lm - \sqrt{|a_0|}}}{\sqrt{k^n |a_n| - lm}} \right) \left(k^n \max_{|z|=1} |P(z)| - lm \right) \\ &\quad + \frac{(|\alpha| - k) |a_{n-1}|}{k(1 + k^n)} \left(n + \frac{\sqrt{k^n |a_n| - lm - \sqrt{|a_0|}}}{\sqrt{k^n |a_n| - lm}} \right) \\ &\quad \times \left(\frac{k^n - 1}{n} - \frac{k^{n-2} - 1}{n - 2} \right) + (1 - 1/k^2) |na_0 + \alpha a_1|, \quad \text{if } n > 2. \end{aligned}$$

That proves the inequality (2.3) for $n > 2$. For the case $n = 2$, the result follows on similar lines by using inequality (3.3) of Lemma 3.4 and inequality (3.8) of Lemma 3.7 in the inequality (4.10). This completes the proof of Theorem 2.2. \square

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