

k -TYPE BI-NULL CARTAN SLANT HELICES IN \mathbb{R}_2^6

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ABSTRACT. In the present paper, we give the notion of k -type bi-null Cartan slant helices in \mathbb{R}_2^6 , where $k \in \{1, 2, 3, 4, 5, 6\}$. We give the necessary and sufficient conditions for bi-null Cartan curves to be k -type slant helices in terms of their curvature functions.

1. INTRODUCTION

The notion of a slant helix is introduced by Izumiya and Takeuchi [4]. A curve γ with non-zero curvature is called a slant helix in Euclidean 3-space \mathbb{R}^3 if the principal normal line of γ makes a constant angle with a fixed vector in \mathbb{R}^3 . Some characterizations of such curves were presented in [1, 5, 6, 9].

Further, k -type slant helices emerged and attracted attention of researchers. Ergüt et al. ([3]) studied k -slant helices in Minkowski 3-space \mathbb{R}_1^3 . The curves of such type were studied in Minkowski space-time \mathbb{R}_1^4 by some researchers in [2, 7].

Bi-null Cartan curves in \mathbb{R}_2^n are defined in [8]. Some characterizations of bi-null Cartan curves in terms of their curvature functions in \mathbb{R}_2^n for $n \geq 6$ are also given in [8]. The necessary and the sufficient conditions for bi-null curves to be k -type slant helices in semi-Euclidean spaces \mathbb{R}_3^6 and \mathbb{R}_2^5 are given in [10, 11].

On the other hand, the third named author of this paper gave the notion of bi-null Cartan curves in semi-Euclidean spaces \mathbb{R}_2^n of index 2, together with the unique Cartan frame and the Cartan curvatures. He discussed some properties of bi-null Cartan curves in terms of the Cartan curvatures in the case where $n \geq 6$ ([8]). In \mathbb{R}_2^5 and \mathbb{R}_3^6 , we define k -type bi-null slant helices and we give the necessary and sufficient

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conditions for bi-null curves to be k -type slant helices in terms of their curvature functions ([10, 11]).

In this paper, we give the notion of k -type bi-null Cartan slant helices in \mathbb{R}_2^6 , where $k \in \{1, 2, 3, 4, 5, 6\}$. We give the necessary and sufficient conditions for bi-null Cartan curves to be k -type slant helices in terms of their curvature functions.

2. PRILIMINARIES

In this section, following [8], we recall the Frenet equations for bi-null Cartan curves in \mathbb{R}_2^6 . Let \mathbb{R}_2^6 be the 6-dimensional semi-Euclidean space of index 2 with standard coordinate system $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ and metric

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 - dx_5^2 - dx_6^2.$$

We denote by $\langle \cdot, \cdot \rangle$ the inner product on \mathbb{R}_2^6 . Recall that a vector $v \in \mathbb{R}_2^6 \setminus \{0\}$ can be spacelike if $\langle v, v \rangle > 0$, timelike if $\langle v, v \rangle < 0$ and null (lightlike) if $\langle v, v \rangle = 0$. In particular, the vector $v = 0$ is said to be spacelike. The norm of a vector v is given by $\|v\| = \sqrt{|\langle v, v \rangle|}$. Two vectors v and w are said to be orthogonal, if $\langle v, w \rangle = 0$.

We say that a curve $\gamma(t)$ in \mathbb{R}_2^6 is a bi-null curve if $\text{span}\{\gamma'(t), \gamma''(t)\}$ is an isotropic 2-plane for all t . That is,

$$\langle \gamma'(t), \gamma'(t) \rangle = \langle \gamma'(t), \gamma''(t) \rangle = \langle \gamma''(t), \gamma''(t) \rangle = 0,$$

and $\{\gamma'(t), \gamma''(t)\}$ are linearly independent. The condition is independent of the choice of the parameter.

We say that a bi-null curve $\gamma(t)$ in \mathbb{R}_2^6 is parametrized by the bi-null arc if $\langle \gamma^{(3)}(t), \gamma^{(3)}(t) \rangle = 1$. If a bi-null curve $\gamma(t)$ in \mathbb{R}_2^6 satisfies $\langle \gamma^{(3)}(t), \gamma^{(3)}(t) \rangle \neq 0$, then by the anti-isometry of \mathbb{R}_2^6 , we may assume that $\langle \gamma^{(3)}(t), \gamma^{(3)}(t) \rangle > 0$ and we can see that

$$u(t) = \int_{t_0}^t \langle \gamma^{(3)}(t), \gamma^{(3)}(t) \rangle^{1/6} dt$$

becomes the bi-null arc parameter.

Let us say that a bi-null curve $\gamma(t)$ in \mathbb{R}_2^6 with $\langle \gamma^{(3)}(t), \gamma^{(3)}(t) \rangle > 0$ is a bi-null Cartan curve if $\{\gamma'(t), \gamma''(t), \gamma^{(3)}(t), \gamma^{(4)}(t), \gamma^{(5)}(t)\}$ are linearly independent for any t .

For a bi-null Cartan curve $\gamma(t)$ in \mathbb{R}_2^6 with bi-null arc parameter t , there exists a unique pseudo-orthonormal frame $\{L_1, L_2, N_1, N_2, W_1, W_2\}$ such that

$$\begin{aligned}
 \gamma' &= L_1, & L_1' &= L_2, & L_2' &= W_1, \\
 W_1' &= -k_0 L_2 - N_2, \\
 N_2' &= -k_1 L_1 - N_1 + k_0 W_1, \\
 N_1' &= k_1 L_2 + k_2 W_2, \\
 W_2' &= -k_2 L_1,
 \end{aligned}
 \tag{2.1}$$

where N_1, N_2 are null, $\langle L_1, N_1 \rangle = \langle L_2, N_2 \rangle = 1$, $\{L_1, N_1\}$, $\{L_2, N_2\}$ and $\{W_1, W_2\}$ are mutually orthogonal, $\{W_1, W_2\}$ are orthonormal of signature $(+, +)$, and $\{L_1, L_2, N_1, N_2, W_1, W_2\}$ is positively oriented.

We say that the pseudo-orthonormal frame $\{L_1, L_2, N_1, N_2, W_1, W_2\}$ is the Cartan frame and the functions $\{k_0, k_1, k_2\}$ are the Cartan curvatures of γ .

3. k -TYPE BI-NULL CARTAN SLANT HELICES

Let $\gamma(t)$ be a bi-null Cartan curve in \mathbb{R}_2^6 with bi-null arc parameter t and Cartan frame $\{L_1, L_2, N_1, N_2, W_1, W_2\}$. Let us set $V_1 = L_1, V_2 = L_2, V_3 = N_1, V_4 = N_2, V_5 = W_1$ and $V_6 = W_2$. Then we give the following definition.

Definition 3.1. A bi-null Cartan curve γ in \mathbb{R}_2^6 with Cartan frame $\{V_1, V_2, V_3, V_4, V_5, V_6\}$ is called a k -type bi-null Cartan slant helix if there exists a non-zero fixed vector $U \in \mathbb{R}_2^6$ such that the following holds

$$\langle V_k, U \rangle = \text{constant}, \quad \text{where } k \in \{1, 2, 3, 4, 5, 6\}.$$

Firstly, we consider 1-type bi-null Cartan slant helices in \mathbb{R}_2^6 .

Theorem 3.1. Let $\gamma(t)$ be a bi-null Cartan curve in \mathbb{R}_2^6 parametrized by bi-null arc t with k_0, k_1 and non-zero k_2 . Then $\gamma(t)$ is a 1-type bi-null Cartan slant helix if and only if k_1 and k_2 satisfy the following

$$(3.1) \quad k_2 - \left(\frac{k_1'}{k_2}\right)' = 0.$$

Proof. Assume that $\gamma(t)$ is a 1-type bi-null Cartan slant helix parametrized by bi-null arc t with k_0, k_1 and non-zero k_2 . Then there exists a non-zero fixed vector $U \in \mathbb{R}_2^6$ such that

$$(3.2) \quad \langle L_1, U \rangle = c, \quad c \in \mathbb{R}.$$

Taking derivative of the equation (3.2) with respect to t and using equations (2.1), we get

$$(3.3) \quad \langle L_2, U \rangle = 0, \quad \langle W_1, U \rangle = 0, \quad \langle N_2, U \rangle = 0.$$

Then using (3.3), we can write U as follows

$$(3.4) \quad U = \lambda_1 L_1 + c N_1 + \lambda_2 W_2,$$

where λ_1 and λ_2 are differentiable functions. Taking derivative of the equation (3.4) with respect to t and using equations (2.1), we have

$$0 = (\lambda_1' - \lambda_2 k_2) L_1 + (\lambda_1 + c k_1) L_2 + (c k_2 + \lambda_2') W_2,$$

which implies that

$$(3.5) \quad \begin{cases} \lambda_1' - \lambda_2 k_2 = 0, \\ \lambda_1 + c k_1 = 0, \\ c k_2 + \lambda_2' = 0. \end{cases}$$

From (3.5), we find that $c \neq 0$ and

$$\lambda_1 = -ck_1, \quad \lambda_2 = -c\frac{k'_1}{k_2}, \quad k_2 - \left(\frac{k'_1}{k_2}\right)' = 0.$$

Conversely, assume that k_1 and k_2 satisfy

$$k_2 - \left(\frac{k'_1}{k_2}\right)' = 0.$$

For $c \neq 0$, choosing the vector U as

$$(3.6) \quad U = -ck_1L_1 + cN_1 - (ck'_1/k_2)W_2,$$

we get $U' = 0$ and $\langle L_1, U \rangle = c$ (constant). Thus, $\gamma(t)$ is a 1-type bi-null Cartan slant helix. \square

Example 3.1. The following curvature functions satisfy (3.1):

- (i) $k_1 = t^2/2, k_2 = 1;$
- (ii) $k_1 = 2t^2, k_2 = -2.$

Corollary 3.1. *The axis of a 1-type bi-null Cartan slant helix $\gamma(t)$ in \mathbb{R}_2^6 with k_0, k_1 and non-zero k_2 , is given by*

$$(3.7) \quad U = -ck_1L_1 + cN_1 - (ck'_1/k_2)W_2,$$

where $c \in \mathbb{R}/\{0\}$.

Corollary 3.2. *There exists no 1-type bi-null Cartan slant helix $\gamma(t)$ in \mathbb{R}_2^6 with k_0, k_1 and non-zero k_2 , whose axis U satisfies $\langle L_1, U \rangle = 0$.*

Corollary 3.3. *Let $\gamma(t)$ be a bi-null Cartan curve in \mathbb{R}_2^6 parametrized by bi-null arc t with k_0, k_1 and non-zero k_2 . Assume that $k'_1 \neq 0$. Then $\gamma(t)$ is a 1-type bi-null Cartan slant helix if and only if k_1 and k_2 satisfy that*

$$(3.8) \quad -2k_1 + \left(\frac{k'_1}{k_2}\right)^2 = \text{constant}.$$

Proof. Assume that $\gamma(t)$ is a 1-type bi-null Cartan slant helix in \mathbb{R}_2^6 parametrized by bi-null arc t with k_0, k_1 and non-zero k_2 . Using Corollary 3.1 and that U is constant, we obtain

$$-2k_1 + \left(\frac{k'_1}{k_2}\right)^2 = \text{constant}.$$

Conversely, assume that the relation (3.8) holds. Then taking derivative of the equation (3.8) with respect to t , we get

$$k_2 - \left(\frac{k'_1}{k_2}\right)' = 0.$$

Thus from Theorem 3.1, we find that $\gamma(t)$ is a 1-type bi-null Cartan slant helix. \square

Corollary 3.4. *Let $\gamma(t)$ be a 1-type bi-null Cartan slant helix in \mathbb{R}_2^6 parametrized by bi-null arc t with k_0, k_1 and non-zero k_2 . Then we get*

$$k_1 = \frac{1}{2} \left(\int k_2(t) dt + a \right)^2 + b,$$

where $a, b \in \mathbb{R}$.

Secondly, we consider 2-type bi-null Cartan slant helices in \mathbb{R}_2^6 .

Theorem 3.2. *Let $\gamma(t)$ be a bi-null Cartan curve in \mathbb{R}_2^6 parametrized by bi-null arc t with k_0, k_1 and non-zero k_2 . Then $\gamma(t)$ is a 2-type bi-null Cartan slant helix if and only if k_0, k_1 and k_2 satisfy*

$$(3.9) \quad k_2(ct + a) + \left[\frac{1}{k_2} (ck_0'' - 2ck_1 - k_1'(ct + a)) \right]' = 0,$$

where $a, c \in \mathbb{R}$ and $(a, c) \neq (0, 0)$.

Proof. Assume that $\gamma(t)$ is a 2-type bi-null Cartan slant helix parametrized by bi-null arc t with k_0, k_1 and non-zero k_2 . Then there exists a non-zero fixed vector $U \in \mathbb{R}_2^6$ such that

$$(3.10) \quad \langle L_2, U \rangle = c, \quad c \in \mathbb{R}.$$

Taking derivative of the equation (3.10) with respect to t and using equations (2.1), we get $\langle W_1, U \rangle = 0$. Then we can write the vector U as follows

$$(3.11) \quad U = \lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 N_1 + cN_2 + \lambda_4 W_2,$$

where λ_i is a differentiable function ($i \in \{1, 2, 3, 4\}$). Differentiating the equation (3.11) with respect to t and using equations (2.1), we get

$$0 = (\lambda_1' - ck_1 - \lambda_4 k_2) L_1 + (\lambda_1 + \lambda_2' + \lambda_3 k_1) L_2 + (\lambda_3' - c) N_1 + (\lambda_2 + ck_0) W_1 + (\lambda_3 k_2 + \lambda_4') W_2,$$

which implies that

$$(3.12) \quad \begin{cases} \lambda_1' - ck_1 - \lambda_4 k_2 = 0, \\ \lambda_1 + \lambda_2' + \lambda_3 k_1 = 0, \\ \lambda_3' - c = 0, \\ \lambda_2 + ck_0 = 0, \\ \lambda_3 k_2 + \lambda_4' = 0. \end{cases}$$

Solving (3.12), we get

$$\lambda_1 = ck_0' - k_1(ct + a), \quad \lambda_2 = -ck_0, \quad \lambda_3 = ct + a, \\ \lambda_4 = \frac{1}{k_2} (ck_0'' - 2ck_1 - k_1'(ct + a))$$

and

$$k_2(ct + a) + \left[\frac{1}{k_2} (ck_0'' - 2ck_1 - k_1'(ct + a)) \right]' = 0,$$

where $a, c \in \mathbb{R}$ and $(a, c) \neq (0, 0)$.

Conversely, assume that the following relation holds

$$k_2(ct + a) + \left[\frac{1}{k_2} (ck_0'' - 2ck_1 - k_1'(ct + a)) \right]' = 0,$$

where $a, c \in \mathbb{R}$ and $(a, c) \neq (0, 0)$. Then choosing the vector U as follows

$$U = (ck_0' - k_1(ct + a))L_1 - ck_0L_2 + (ct + a)N_1 + cN_2 + \frac{1}{k_2} (ck_0'' - 2ck_1 - k_1'(ct + a))W_2,$$

we get $U' = 0$ and $\langle L_2, U \rangle = c$ (constant). Thus, $\gamma(t)$ is a 2-type bi-null Cartan slant helix. □

Example 3.2. The following curvature functions satisfy (3.9):

- (i) $c = 0, a = 1, k_0 = t^2, k_1 = t^2/2, k_2 = 1;$
- (ii) $c = 1, a = 0, k_0 = t, k_1 = t^2/8, k_2 = 1.$

Corollary 3.5. *The axis of a 2-type bi-null Cartan slant helix $\gamma(t)$ in \mathbb{R}_2^6 with k_0, k_1 and $k_2 \neq 0$, is given by*

$$U = (ck_0' - k_1(ct + a))L_1 - ck_0L_2 + (ct + a)N_1 + cN_2 + \frac{1}{k_2} (ck_0'' - 2ck_1 - k_1'(ct + a))W_2,$$

where $a, c \in \mathbb{R}$ and $(a, c) \neq (0, 0)$.

Corollary 3.6. *Let $\gamma(t)$ be a bi-null Cartan curve in \mathbb{R}_2^6 parametrized by bi-null arc t with k_0, k_1 and $k_2 \neq 0$. If $\gamma(t)$ is a 1-type bi-null Cartan slant helix then $\gamma(t)$ is a 2-type bi-null Cartan slant helix whose principal normal L_2 is orthogonal to the axis U of the helix.*

Thirdly, we consider 3-type bi-null Cartan slant helices in \mathbb{R}_2^6 .

Theorem 3.3. *Let $\gamma(t)$ be a bi-null Cartan curve in \mathbb{R}_2^6 parametrized by bi-null arc t with k_0, k_1 and non-zero k_2 . Then $\gamma(t)$ is a 3-type bi-null Cartan slant helix if and only if k_0, k_1 and k_2 satisfy*

$$(3.13) \quad \lambda_2^{(4)} + 2k_0\lambda_2'' + k_0'\lambda_2' - \lambda_2k_1 = c$$

and

$$(3.14) \quad k_2\lambda_2 - \left(\frac{k_1\lambda_2'}{k_2} \right)' = 0,$$

where $c \in \mathbb{R}$ and λ_2 is a differentiable function which is not identically zero.

Proof. Assume that $\gamma(t)$ is a 3-type bi-null Cartan slant helix parametrized by bi-null arc t with k_0, k_1 and non-zero k_2 . Then there exists a non-zero fixed vector $U \in \mathbb{R}_2^6$ such that

$$(3.15) \quad \langle N_1, U \rangle = c, \quad c \in \mathbb{R}.$$

Then we can write U as follows

$$(3.16) \quad U = cL_1 + \lambda_1L_2 + \lambda_2N_1 + \lambda_3N_2 + \lambda_4W_1 + \lambda_5W_2,$$

where λ_i is a differentiable function. Taking derivative of the equation (3.16) with respect to t and using equations (2.1), we get

$$0 = (-k_1\lambda_3 - \lambda_5k_2)L_1 + (c + \lambda'_1 + \lambda_2k_1 - \lambda_4k_0)L_2 + (\lambda'_2 - \lambda_3)N_1 + (\lambda'_3 - \lambda_4)N_2 + (\lambda_1 + k_0\lambda_3 + \lambda'_4)W_1 + (k_2\lambda_2 + \lambda'_5)W_2,$$

which implies that

$$(3.17) \quad \begin{cases} -k_1\lambda_3 - \lambda_5k_2 = 0, \\ c + \lambda'_1 + \lambda_2k_1 - \lambda_4k_0 = 0, \\ \lambda'_2 - \lambda_3 = 0, \\ \lambda'_3 - \lambda_4 = 0, \\ \lambda_1 + k_0\lambda_3 + \lambda'_4 = 0, \\ k_2\lambda_2 + \lambda'_5 = 0. \end{cases}$$

By (3.17), λ_2 cannot be identically zero. Solving (3.17), we get

$$\lambda_2^{(4)} + 2k_0\lambda_2'' + k_0'\lambda_2' - \lambda_2k_1 = c$$

and

$$k_2\lambda_2 - \left(\frac{k_1\lambda_2'}{k_2}\right)' = 0,$$

where $c \in \mathbb{R}$.

Conversely, assume that the following relation holds

$$\lambda_2^{(4)} + 2k_0\lambda_2'' + k_0'\lambda_2' - \lambda_2k_1 = c$$

and

$$k_2\lambda_2 - \left(\frac{k_1\lambda_2'}{k_2}\right)' = 0,$$

where $c \in \mathbb{R}$ and λ_2 is a differentiable function which is not identically zero. Then choosing the vector U as follows

$$U = cL_1 - (k_0\lambda_2' + \lambda_2^{(3)})L_2 + \lambda_2N_1 + \lambda_2'N_2 + \lambda_2''W_1 - \frac{k_1}{k_2}\lambda_2'W_2,$$

we get $U' = 0$ and $\langle N_1, U \rangle = c$ (constant). Thus, $\gamma(t)$ is a 3-type bi-null Cartan slant helix. □

Example 3.3. The following curvature functions satisfy the equations (3.13) and (3.14):

- (i) $\lambda_2 = t, k_0 = -t + (t^4/8), k_1 = t^2/2, k_2 = 1, c = -1;$
- (ii) $\lambda_2 = t^2, k_0 = t^6/128, k_1 = t^4/8, k_2 = t, c = 0.$

Corollary 3.7. *The axis of a 3-type bi-null Cartan slant helix $\gamma(t)$ in \mathbb{R}_2^6 with k_0, k_1 and $k_2 \neq 0$, is given by*

$$U = cL_1 - (k_0\lambda_2' + \lambda_2^{(3)})L_2 + \lambda_2N_1 + \lambda_2'N_2 + \lambda_2''W_1 - \frac{k_1}{k_2}\lambda_2'W_2,$$

where $c \in \mathbb{R}$ and λ_2 is a differentiable function that is not identically zero.

Let us consider 4-type bi-null Cartan slant helices in \mathbb{R}_2^6 . In the following, we omit the proofs.

Theorem 3.4. *Let $\gamma(t)$ be a bi-null Cartan curve in \mathbb{R}_2^6 parametrized by bi-null arc t with k_0, k_1 and non-zero k_2 . Then $\gamma(t)$ is a 4-type bi-null Cartan slant helix if and only if k_0, k_1 and k_2 satisfy*

$$(3.18) \quad k_2\lambda_2 + \left[\frac{1}{k_2} \left(k_0\lambda_2^{(3)} + k_0'\lambda_2'' - k_1'\lambda_2 - 2k_1\lambda_2' \right) \right]' = 0$$

and

$$(3.19) \quad c + k_0\lambda_2' + \lambda_2^{(3)} = 0,$$

where $c \in \mathbb{R}$ and λ_2 is a differentiable function which is not identically zero.

Example 3.4. The following curvature functions satisfy the equations (3.18) and (3.19):

- (i) $\lambda_2 = t, k_0 = -c, k_1 = t^2/8, k_2 = 1;$
- (ii) $\lambda_2 = 1, k_0 = t, k_1 = t^2/2, k_2 = 1, c = 0.$

Corollary 3.8. *The axis of a 4-type bi-null Cartan slant helix $\gamma(t)$ in \mathbb{R}_2^6 with k_0, k_1 and $k_2 \neq 0$, is given by*

$$U = (k_0\lambda_2'' - k_1\lambda_2) L_1 + cL_2 + \lambda_2 N_1 + \lambda_2' N_2 + \lambda_2'' W_1 + \frac{1}{k_2} \left(k_0\lambda_2^{(3)} + k_0'\lambda_2'' - k_1'\lambda_2 - 2k_1\lambda_2' \right) W_2,$$

where $c \in \mathbb{R}$ and λ_2 is a differentiable function that is not identically zero.

Let us consider 5-type bi-null Cartan slant helices in \mathbb{R}_2^6 . In the following, we omit the proofs.

Theorem 3.5. *Let $\gamma(t)$ be a bi-null Cartan curve in \mathbb{R}_2^6 parametrized by bi-null arc t with k_0, k_1 and non-zero k_2 . Then $\gamma(t)$ is a 5-type bi-null Cartan slant helix if and only if k_0, k_1 and k_2 satisfy*

$$\lambda' + k_2 \left(c \frac{t^2}{2} + a_1 t + a_2 \right) = 0,$$

where

$$\lambda = \frac{1}{k_2} \left(3ck_0' + (k_0'' - 2k_1)(ct + a_1) - k_1' \left(\frac{ct^2}{2} + a_1 t + a_2 \right) \right),$$

$c, a_1, a_2 \in \mathbb{R}$ and $(c, a_1, a_2) \neq (0, 0, 0)$.

Example 3.5. The following curvature functions satisfy the above equations:

- (i) $c = a_1 = 0, a_2 = 1, k_0 = t^2, k_1 = -t^2/2, k_2 = 1;$
- (ii) $c = a_2 = 0, a_1 = 1, k_0 = 1, k_1 = (t^2/8) + (1/2t^2), k_2 = 1.$

Corollary 3.9. *The axis of a 5-type bi-null Cartan slant helix $\gamma(t)$ in \mathbb{R}_2^6 , with k_0, k_1 and $k_2 \neq 0$, is given by*

$$\begin{aligned}
 U = & \left(2ck_0 + k'_0(ct + a_1) - k_1 \left(\frac{ct^2}{2} + a_1t + a_2 \right) \right) L_1 - k_0(ct + a_1) L_2 \\
 & + \left(\frac{ct^2}{2} + a_1t + a_2 \right) N_1 + (ct + a_1) N_2 + cW_1 \\
 & + \frac{1}{k_2} \left(3ck'_0 + (k''_0 - 2k_1)(ct + a_1) - k'_1 \left(\frac{ct^2}{2} + a_1t + a_2 \right) \right) W_2,
 \end{aligned}$$

where $c, a_1, a_2 \in \mathbb{R}$ and $(c, a_1, a_2) \neq (0, 0, 0)$.

Corollary 3.10. *Let $\gamma(t)$ be a bi-null Cartan curve in \mathbb{R}_2^6 parametrized by bi-null arc t with k_0, k_1 and $k_2 \neq 0$. If $\gamma(t)$ is a 2-type bi-null Cartan slant helix then $\gamma(t)$ is a 5-type bi-null Cartan slant helix such that W_1 is orthogonal to the axis U of the helix.*

Lastly, we consider 6-type bi-null Cartan slant helices in \mathbb{R}_2^6 .

Theorem 3.6. *Let $\gamma(t)$ be a bi-null Cartan curve in \mathbb{R}_2^6 parametrized by bi-null arc t with k_0, k_1, k_2 . If $\gamma(t)$ is a 6-type bi-null Cartan slant helix, then it lies in some hyperplane of index 2.*

Proof. Assume that $\gamma(t)$ is a 6-type bi-null Cartan slant helix parametrized by bi-null arc t with k_0, k_1, k_2 . Then there exists a non-zero fixed vector $U \in \mathbb{R}_2^6$ such that

$$(3.20) \quad \langle W_2, U \rangle = c, \quad c \in \mathbb{R}.$$

Taking derivative of the equation (3.20) with respect to t and using equations (2.1), we get

$$(3.21) \quad k_2 \langle L_1, U \rangle = 0.$$

Assume that $k_2 \neq 0$. Then we have

$$\langle L_1, U \rangle = \langle L_2, U \rangle = \langle W_1, U \rangle = \langle N_2, U \rangle = \langle N_1, U \rangle = \langle W_2, U \rangle = 0,$$

and U is zero, which is a contradiction. Thus, $k_2 = 0$ which means that $\gamma(t)$ lies in some hyperplane of index 2. □

Remark 3.1. Any bi-null Cartan curve in \mathbb{R}_2^6 with $k_2 = 0$ is 6-type bi-null Cartan helix, since W_2 is a constant vector, so it trivially makes constant angle with any fixed direction.

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