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ON THE COMPOSITION OF CONDITIONAL EXPECTATION AND MULTIPLICATION OPERATORS

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Abstract. In this paper, first we provide some necessary and sufficient conditions for quasi- normality and quasi- hyponormality of weighted conditional type operators. And then the spectrum, residual spectrum, point spectrum and spectral radius of weighted conditional type operators are computed. As an application, we give an equivalent conditions for weighted conditional type operators to be quasinilpotent. Also, some examples are provided to illustrate concrete applications of the main results.

1. Introduction and Preliminaries

This paper is about an important operator in statistics and analysis, that is called conditional expectation. Theory of conditional type operators is one of important arguments in the connection of operator theory and measure theory. By the projection theorem, the conditional expectation $E(X)$ is the best mean square predictor of X in range E, also in analysis it is proved that lots of operators are of the form E and of the form of combinations of E and multiplications operators. Conditional expectations have been studied in an operator theoretic setting, by, for example in [8], S.-T. C. Moy characterized all operators on L^p of the form $f \to E(fg)$ for g in L^q with $E(|g|)$ bounded, P. G. Dodds, C. B. Huijsmans and B. De Pagter [1], extended these characterizations to the setting of function ideals and vector lattices and J. Herron presented some assertions about the operator EM_u on L^p spaces in [6]. Also, some results about multiplication conditional expectation operators can be found in [5, 7]. In [3] we investigated some classic properties of multiplication conditional expectation operators M_wEM_u on L^p spaces.

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Let (X, Σ, μ) be a σ-finite measure space. For any sub-algebra $\mathcal{A} \subseteq \Sigma$, the L^p-space $L^p(X, \mathcal{A}, \mu_{|\mathcal{A}})$ is abbreviated by $L^p(\mathcal{A})$, and its norm is denoted by $\|.\|_p$, where $1 \le p < \infty$. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. The support of a measurable function f is defined as $S(f) = \{x \in X : f(x) \neq 0\}$. We denote the vector space of all equivalence classes of almost everywhere finite valued measurable functions on X by $L^0(\Sigma)$.

The notion of conditional expectation plays an important role throughout the paper and so we recall the definition and some elementary properties for the reader's covenience. Let (X, Σ, μ) be a σ-finite measure space and $\mathcal{A} \subseteq \Sigma$ be a σ-finite subalgebra of Σ. For all non-negative Σ-measurable functions like f as well as for all $f \in L^p(\Sigma)$, we denote by $E^{\mathcal{A}}f$, the (μ-a.e.) unique $\mathcal{A}$-measurable function with the property that

$$\int_A f d\mu = \int_A E^{\mathcal{A}} f d\mu, \quad \text{for all } A \in \mathcal{A}.$$

The existence of $E^{\mathcal{A}}(f)$ is a consequence of the Radon-Nikodym theorem. The function $E^{\mathcal{A}}(f)$ is called conditional expectation of f with respect to $\mathcal{A}$. As an operator on $L^p(\Sigma)$, $E^{\mathcal{A}}$ is idempotent and $E^{\mathcal{A}}(L^p(\Sigma)) = L^p(\mathcal{A})$. This operator will play a major role in our work. Let $f \in L^0(\Sigma)$, then f is said to be conditionable with respect to E if $f \in \mathcal{D}(E) := \{g \in L^0(\Sigma) : E(|g|) \in L^0(\mathcal{A})\}$. Throughout this paper we take u and w in $\mathcal{D}(E)$. If there is no possibility of confusion, we write $E(f)$ in place of $E^{\mathcal{A}}(f)$. A detailed discussion about this operator may be found in [10]. We list here some useful properties of conditional expectation operator:

- if g is $\mathcal{A}$-measurable, then $E(fg) = E(f)g$;
- $|E(f)|^p \le E(|f|^p)$ for all $f \in L^p(\Sigma)$;
- if $f \ge 0$, then $E(f) \ge 0$; if $f > 0$, then $E(f) > 0$;
- $|E(fg)| \le (E(|f|^p))^{\frac{1}{p}} (E(|g|^{p'}))^{\frac{1}{p'}}$, where $p^{-1} + p'^{-1} = 1$ (Hölder inequality);
- for each $f \ge 0$, $S(f) \subseteq S(E(f))$.

For a measurable function $w \in L^0(\Sigma)$, the operator $M_w : L^0(\Sigma) \to L^0(\Sigma)$ with $M_w(f) = w.f$, for every $f \in L^0(\Sigma)$, is called Multiplication operator. Now we give a definition for weighted conditional type operators on L^p-spaces.

Definition 1.1. Let (X, Σ, μ) be a σ-finite measure space and let $\mathcal{A}$ be a σ-sub-algebra of Σ such that $(X, \mathcal{A}, \mu_{\mathcal{A}})$ is also σ-finite. Let E be the conditional expectation operator on $L^p(\Sigma)$ relative to $\mathcal{A}$. If $w, u \in L^0(\Sigma)$ such that uf is conditionable and $wE(uf) \in L^p(\Sigma)$, for all $f \in L^p(\Sigma)$, then the corresponding weighted conditional type operator is the linear transformation $T : L^p(\Sigma) \to L^p(\Sigma)$ defined by $f \to wE(uf)$.

In this paper we will be concerned with characterizing weighted conditional expectation type operators on $L^p(\Sigma)$, provide necessary and sufficient conditions for quasi-normality and quasi-hyponormality, computing the spectrum, residual spectrum, point spectrum and spectral radius. The results of [1] state that our results are valid for a large class of linear operators.

2. Quasi-Normality and Quasi-Hyponormality

In this section first we reminisce some properties of weighted conditional type operators that we proved in [3]. Also, we give some examples of conditional expectation operator. I recall that throughout this paper we assume that (X, Σ, μ) and $(X, \mathcal{A}, \mu_{\mathcal{A}})$ are σ-finite measure spaces.

Let $T = M_w E M_u$ be a bounded operator on $L^2(\Sigma)$ and let $p \in (0, \infty)$. Then

$$(T^*T)^p = M_{\bar{u}(E(|u|^2))^{p-1}\chi_S(E(|w|^2))^p} E M_u, \tag{2.1}$$

$$(TT^*)^p = M_{w(E(|w|^2))^{p-1}\chi_G(E(|u|^2))^p} E M_{\bar{w}}, \tag{2.2}$$

where $S = S(E(|u|^2))$ and $G = S(E(|w|^2))$.

Let $\mathcal{H}$ be a Hilbert spaces and $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. The operator $T \in \mathcal{B}(\mathcal{H})$ is called quasi-normal if the equation $T(T^*T) = (T^*T)T$ holds, in which T^* is the adjoint of T. Also, for $M > 0$, the operator $T \in \mathcal{B}(\mathcal{H})$ is called M-quasi-hyponormal if $M^2 {T^*}^2 T^2 - (T^*T)^2 \geq 0$. Specially T is called quasi-hyponormal if T is 1-quasi-hyponormal. This definition comes from [11]. There is another concept with almost the same names but different definition in [4]. There is a small difference between them as you see, M-quasi-hyponormal (capital letter M) and k-quasi-hyponormal (small letter k).

In the next theorem we give some necessary and sufficient conditions for quasi-normality of weighted conditional type operators.

Theorem 2.1. *Let $T = M_w E M_u \in \mathcal{B}(L^2(\Sigma))$. Then we have the foolowing.*

(a) If $\bar{u}E(uw) = wE(|u|^2)$ on G, then T is quasi-normal.

(b) If T is quasi-normal, then $E(|w|^2)E(uw)|E(u)|^2 = E(|u|^2)E(|w|^2)E(u)E(w)$.

Proof. (a) By using (2.1) and (2.2) easily we can obtain that

$$T(T^*T) = M_{wE(|u|^2)E(|w|^2)} E M_u, \quad (T^*T)T = M_{E(|w|^2)E(uw)\bar{u}} E M_u.$$

So, for every $f \in L^2(\Sigma)$ we have

$$\begin{aligned}
&\langle T(T^*T)f - (T^*T)Tf, f\rangle \\
&= \int_X (E(|u|^2)E(|w|^2)wE(uf)\bar{f} - E(|w|^2)E(uw)\bar{u}E(uf)\bar{f})d\mu.
\end{aligned}$$

This implies that, if $\bar{u}E(uw) = wE(|u|^2)$ on G, then T is quasi-normal.

(b) If T is quasi-normal then, for all $f \in L^2(\Sigma)$ we have

$$\begin{aligned}
\langle T(T^*T)f - (T^*T)Tf, f\rangle &= \int_X (E(|u|^2)E(|w|^2)wE(uf)\bar{f} \\
&\quad - E(|w|^2)E(uw)\bar{u}E(uf)\bar{f})d\mu \\
&= \int_X (E(|u|^2)E(|w|^2)E(uf)E(w\bar{f}) \\
&\quad - E(|w|^2)E(uw)|E(uf)|^2)d\mu \\
&= 0.
\end{aligned}$$

Let $A \in \mathcal{A}$, with $0 < \mu(A) < \infty$. By replacing f to χ_A, we have

$$\int_A (E(|u|^2)E(|w|^2)E(u)E(w) - E(|w|^2)E(uw)|E(u)|^2)d\mu = 0.$$

Since $A \in \mathcal{A}$ is arbitrary, then

$$E(|u|^2)E(|w|^2)E(u)E(w) - E(|w|^2)E(uw)|E(u)|^2 = 0. \qquad \square$$

Now we provide some necessary and sufficient conditions for M-quasi-hyponormality of weighted conditional type operators.

Theorem 2.2. *Let $T = M_w E M_u \in \mathcal{B}(L^2(\Sigma))$. Then we have the following.*

(a) If $M^2|E(uw)|^2E(|w|^2) - E(|u|^2)(E(|w|^2))^2 \geq 0$, then T is M-quasi-hyponormal.

(b) If T is M-quasi-hyponormal, then $M^2|E(uw)|^2E(|w|^2) - E(|u|^2)(E(|w|^2))^2 \geq 0$ on $S(E(u))$.

Proof. (a) Direct computation shows that for all $f \in L^2(\Sigma)$

$$M^2T^{*^2}T^2(f) - (T^*T)^2(f) = M^2\bar{u}|E(uw)|^2E(|w|^2)E(uf) - \bar{u}(E(|w|^2))^2E(|u|^2)E(uf).$$

If $M^2|E(uw)|^2E(|w|^2) - E(|u|^2)(E(|w|^2))^2 \geq 0$, then for all $f \in L^2(\Sigma)$

$$\begin{aligned}
&\langle M^2T^{*^2}T^2(f) - (T^*T)^2(f), f\rangle \\
&= \int_X (M^2\bar{u}|E(uw)|^2E(|w|^2)E(uf)\bar{f} - \bar{u}(E(|w|^2))^2E(|u|^2)E(uf)\bar{f})d\mu \\
&= \int_X (M^2|E(uw)|^2E(|w|^2)|E(uf)|^2 - (E(|w|^2))^2E(|u|^2)|E(uf)|^2)d\mu \geq 0,
\end{aligned}$$

so T is M-quasi-hyponormal.

(b) If T is M-quasi-hyponormal, then for all $f \in L^2(\Sigma)$ we have

$$\int_X (M^2|E(uw)|^2E(|w|^2)|E(uf)|^2 - (E(|w|^2))^2E(|u|^2)|E(uf)|^2)d\mu \geq 0.$$

Let $A \in \mathcal{A}$, with $0 < \mu(A) < \infty$. By replacing f to χ_A, we have

$$\int_A (M^2|E(uw)|^2E(|w|^2)|E(u)|^2 - (E(|w|^2))^2E(|u|^2)|E(u)|^2)d\mu \geq 0.$$

Since $A \in \mathcal{A}$ is arbitrary, then

$$M^2|E(uw)|^2E(|w|^2) - E(|u|^2)(E(|w|^2))^2 \geq 0$$

on $S(E(u))$. $\square$

Hence, we get the next corollary.

Corollary 2.1. *Let $T = EM_u$ and $S(E(u)) = X$. Then*

(a) T is M-quasi hyponormal if and only if $M^2|E(u)|^2 - E(|u|^2) \geq 0$;

(b) T is quasi hyponormal if and only if $u \in L^0(\mathcal{A})$;

(c) T is quasi-normal if and only if $u \in L^0(\mathcal{A})$.

Now we provide an example for Theorem 2.2 in which we have a weighted conditional type operator is M-quasi-hyponormal, but the inequality in (a) does not hold.

Example 2.1. Let $X = [-1, 1]$, $d\mu = \frac{1}{2}dx$ and $\mathcal{A} = \langle\{(-a, a) : 0 \leq a \leq 1\}\rangle$ (σ-algebra generated by symmetric intervals). Then

$$E^{\mathcal{A}}(f)(x) = \frac{f(x) + f(-x)}{2}, \quad x \in X,$$

whenever $E^{\mathcal{A}}(f)$ is defined. Let $w(x) = 1$, $u(x) = x + 1$ for $x \in X$. Then easily we get that $E(u)(x) = 1$ and $E(|u|^2)(x) = x^2 + 1$. Hence, we have $|E(u)(x)|^2 < E(|u|^2)(x)$ for all $x \in X \setminus \{0\}$. Also, for $f \in L^p(X)$ and $M \geq \sqrt{2}$ we have

$$\int_X (M^2|E(uf)|^2 - E(|u|^2)|E(uf)|^2)d\mu = \int_X (M^2 - E(|u|^2))|E(uf)|^2 d\mu \geq 0,$$

this implies that the operator $M_w EM_u$ is M-quasi-hyponormal.

3. The Spectrum

In this section we shall denote by $\sigma(T)$, $\sigma_p(T)$, $\sigma_{jp}(T)$, $\sigma_r(T)$, $r(T)$ the spectrum of T, the point spectrum of T, the joint point spectrum of T, the residual spectrum, the spectral radius of T, respectively for $T \in \mathcal{B}(X)$ in which X is a Banach space. The spectrum of an operator T is the set

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}.$$

A complex number $\lambda \in \mathbb{C}$ is said to be in the point spectrum $\sigma_p(T)$ of the operator T, if there is a unit vector x satisfying $(T - \lambda)x = 0$. If in addition, $(T^* - \bar{\lambda})x = 0$, then λ is said to be in the joint point spectrum $\sigma_{jp}(T)$ of T. The residual spectrum of T is equal to

$$\{\lambda \in \mathbb{C} : (T - \lambda I)^{-1} \text{ exists and } \overline{R(T)} \subsetneq X\}.$$

Also, the spectral radius of T is defined by $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$. For more information one can see [4].

If A is a unital algebra and $a, b \in A$, then it is well known that $\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$, $\sigma_p(ab) \setminus \{0\} = \sigma_p(ba) \setminus \{0\}$, $\sigma_{jp}(ab) \setminus \{0\} = \sigma_{jp}(ba) \setminus \{0\}$ and $\sigma_r(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$. J. Herron showed that if $EM_u : L^p(\Sigma) \to L^p(\Sigma)$ is bounded, then $\sigma(EM_u) = \operatorname{ess\,range}(E(u)) \cup \{0\}$[6]. By means of the above mentioned properties of weighted conditional expectation type operators we have the following theorems.

Theorem 3.1. *Let $\mathcal{A} \subsetneq \Sigma$ and $EM_u : L^p(\Sigma) \to L^p(\mathcal{A})$ be bounded, for $1 \leq p \leq \infty$. Then*

- $\sigma_p(EM_u) \setminus \{0\} = \{\lambda \in \mathbb{C} \setminus \{0\} : \mu(\{x \in X | E(u)(x) = \lambda\}) > 0\}$;
- $\rho(EM_u) \setminus \{0\} = \{\lambda \in \mathbb{C} \setminus \{0\} : (\exists \epsilon > 0)$ *such that* $|\lambda - E(u)| \geq \epsilon$ *a.e*$\}$,
- $\sigma(EM_u) \setminus \{0\} = \{\lambda \in \mathbb{C} \setminus \{0\} : (\nexists \epsilon > 0)$ *such that* $|\lambda - E(u)| \geq \epsilon$ *a.e*$\}$;
- $\sigma_r(EM_u) \setminus \{0\} = \begin{cases} \emptyset, & 1 \leq p < \infty, \\ B_\infty, & p = \infty, \end{cases}$

  in which $B_\infty = \{\lambda \in \mathbb{C} : (\nexists \epsilon > 0)$ *such that* $|\lambda - E(u)| \geq \epsilon$ *a.e*$\} \setminus \sigma_p(EM_u)$.

Proof. Point spectrum. Let $A_\lambda = \{x \in X : E(u)(x) = \lambda\}$, for $\lambda \in \mathbb{C}$. Suppose that $\mu(A_\lambda) > 0$. Since $\mathcal{A}$ is σ-finite, there exists an $\mathcal{A}$-measurable subset B of A_λ such that $0 < \mu(B) < \infty$ and $f = \chi_B \in L^p(\mathcal{A}) \subseteq L^p(\Sigma)$. Now

$$EM_u(f) - \lambda f = E(u)\chi_B - \lambda\chi_B = 0.$$

This implies that $\lambda \in \sigma_p(EM_u)$.

If there exists $f \in L^p(\Sigma)$ such that $f\chi_C \neq 0$ μ-a.e, for $C \in \Sigma$ of positive measure and $E(uf) = \lambda f$ for $\lambda \in \mathbb{C}$, which means that f is $\mathcal{A}$-measurable. Therefore $E(uf) = E(u)f = \lambda f$ and $(E(u) - \lambda)f = 0$. This implies that $C \subseteq A_\lambda$ and so $\mu(A_\lambda) > 0$.

Resolvent set. Let $\lambda \in \mathbb{C} \setminus \{0\}$ such that $|\lambda - E(u)| > \epsilon$, a.e, for some $\epsilon > 0$. We show that $EM_u - \lambda I$ is invertible. If $\lambda f - E(uf) = 0$, then f is $\mathcal{A}$-measurable and so, $(\lambda - E(u))f = 0$. This implies that $f = 0$ a.e, therefore $\lambda I - EM_u$ is injective. Now we show that $EM_u - \lambda I$ is surjective. Let $g \in L^p(\Sigma)$. We can write

$$g = g - E(g) + E(g), \quad g_2 = g - E(g), \quad g_1 = E(g).$$

Then $g_1 \in L^p(\mathcal{A})$ and $g_2 \in L^p(\Sigma)$, $E(g_2) = 0$. Let

$$f_1 = \frac{\lambda g_1 + E(u(g_2))}{\lambda(E(u) - \lambda)}, \quad f_2 = -\frac{g_2}{\lambda}.$$

Since $|E(u) - \lambda| \geq \varepsilon$ a.e for some $\varepsilon > 0$, then $\|\frac{1}{E(u)-\lambda}\|_\infty \leq \frac{1}{\varepsilon}$. So, $f_2 \in L^p(\mathcal{A})$, $f_1 \in L^p(\Sigma)$ and $f := f_1 + f_2 \in L^p(\Sigma)$. Now, we show that $T(f) - \lambda f = g$. We have

$$E(uf) = E(u\left(\frac{\lambda g_1 + E(ug_2) - g_2(E(u) - \lambda)}{\lambda(E(u) - \lambda)}\right) = \frac{g_1 E(u) + E(ug_2)}{E(u) - \lambda}.$$

So,

$$\begin{aligned}(EM_u - \lambda I)f &= \frac{g_1 E(u) + E(ug_2)}{E(u) - \lambda} - \lambda\frac{\lambda g_1 + E(ug_2) - g_2(E(u) - \lambda)}{\lambda(E(u) - \lambda)} \\ &= \frac{g_1(E(u) - \lambda) + g_2(E(u) - \lambda)}{E(u) - \lambda} \\ &= g_1 + g_2 = g.\end{aligned}$$

This implies that $EM_u - \lambda I$ is invertible and so $\lambda \in \rho(EM_u)$. Hence

$$\rho(EM_u) \supseteq \{\lambda \in \mathbb{C} \setminus \{0\} : (\exists\epsilon > 0) \text{ such that } |\lambda - E(u)| \geq \epsilon \text{ a.e}\}.$$

Conversely, let $\lambda \in \rho(EM_u)$, then $\lambda I - EM_u$ has an inverse operator. Define the linear transformation L on $L^p(\Sigma)$ as follows

$$Lf = \frac{E(uf) - f(E(u) - \lambda)}{\lambda(E(u) - \lambda)}, \quad f \in L^p(\Sigma).$$

If there exists $\epsilon > 0$ such that $|\lambda - E(u)| > \epsilon$ a.e then $\|\frac{1}{E(u)-\lambda}\|_\infty \leq \frac{1}{\varepsilon}$. So,

$$\|Sf\|_p \leq \left\|\frac{E(uf)}{\lambda(E(u) - \lambda)}\right\|_p + \left\|\frac{f}{\lambda}\right\|_p \leq \left(\frac{\|EM_u\|}{\lambda\varepsilon} + \frac{1}{\lambda}\right)\|f\|_p.$$

Thus, L is bounded on $L^p(\Sigma)$. If L is bounded on $L^p(\Sigma)$, then for $f \in L^p(\mathcal{A}) \subseteq L^p(\Sigma)$, $Lf = \alpha f = M_\alpha f$, where $\alpha = \frac{1}{Eu-\lambda}$. Thus, multiplication operator M_α is bounded on $L^p(\mathcal{A})$. This implies that $\alpha \in L^\infty(\mathcal{A})$ and so there exists some $\varepsilon > 0$ such that $E(u) - \lambda = \frac{1}{\alpha} \geq \varepsilon$ a.e. Also, we have $L \circ (EM_u - \lambda I) = I$. Indeed for each $f \in L^p(\Sigma)$ we have

$$\begin{aligned} L \circ (EM_u - \lambda I)(f) &= L(E(uf) - \lambda f) \\ &= \frac{E[u(E(uf) - \lambda f)] - [E(uf) - \lambda f](E(u) - \lambda)}{\lambda(E(u) - \lambda)} \\ &= \frac{\lambda f(E(u) - \lambda)}{\lambda(E(u) - \lambda)} \\ &= f. \end{aligned}$$

Thus, $(EM_u - \lambda I)^{-1} = L$ and so L have to be bounded. Hence there exists some $\varepsilon > 0$ such that $E(u) - \lambda = \frac{1}{\alpha} \geq \varepsilon$ a.e.

Residual spectrum. Let $\lambda \in \mathbb{C} \setminus (\sigma_p(EM_u) \cup \rho(EM_u))$. So, $\mu(\{x \in X : E(u)(x) = \lambda\}) = 0$, but on the other hand $\mu(\{x \in X : |E(u)(x) - \lambda| > \epsilon\}) > 0$ for every $\epsilon > 0$. We wish to determine if the range of $\lambda I - EM_u$, i.e., the domain of $(\lambda I - EM_u)^{-1}$, is dense. Set, for each $n \in \mathbb{N}$,

$$E_n = \left\{ x \in X : |E(u)(x) - \lambda| \geq \frac{1}{n} \right\}.$$

The range of $\lambda I - EM_u$ contains $\{\chi_{E_n} f : f \in L^p(\mathcal{A}), n \in \mathbb{N}\}$, because, for every $f \in L^p(\mathcal{A})$, $(\lambda I - EM_u)(\frac{1}{\lambda - E(u)} \chi_{E_n} f) = \chi_{E_n} f$. Furthermore, $\chi_{E_n} f$ converges pointwise almost everywhere to f as $n \to \infty$. So, if $1 \leq p < \infty$, the Lebesgue dominated convergence theorem implies that $\chi_{E_n} f$ converges in $L^p(X, \mathcal{A}, \mu)$ to f as $n \to \infty$. Thus the range of $\lambda I - EM_u$ is dense in $L^p(X, \mathcal{A}, \mu)$ if $1 \leq p < \infty$. On the other hand, if $p = \infty$, the constant function 1 is not in the closure of the range of $\lambda I - EM_u$ because, for every $0 \neq f \in L^p(X, \mathcal{A}, \mu)$, there is $A \in \mathcal{A}$ such that $\mu(A) > 0$ and $|\lambda - E(u)| \leq \frac{1}{2\|f\|_\infty}$ on A and hence $|1 - (\lambda - E(u))f)| \geq \frac{1}{2}$ on A. Thus, the proof is completed. □

Let $\mathcal{H}$ be the infinite dimensional complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded operators on $\mathcal{H}$. Every operator T on a Hilbert space $\mathcal{H}$ can be decomposed into $T = U|T|$ with a partial isometry U, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is determined uniquely by the kernel condition $\mathcal{N}(U) = \mathcal{N}(|T|)$. Then this decomposition is called the polar decomposition. The unique polar decomposition of bounded operator $T = M_w EM_u$, were given in [3], is $U|T|$, where

$$|T|(f) = \left(\frac{E(|w|^2)}{E(|u|^2)} \right)^{\frac{1}{2}} \chi_S \bar{u} E(uf)$$

and

$$U(f)=\left(\frac{\chi_{S\cap G}}{E(|w|^2)E(|u|^2)}\right)^{\frac{1}{2}}wE(uf),$$

for all $f\in L^2(\Sigma)$. Also, the Aluthge transformation of $T=M_wEM_u$ is

$$\widehat{T}(f)=\frac{\chi_S E(uw)}{E(|u|^2)}\bar{u}E(uf),\quad f\in L^2(\Sigma).$$

Now, by using Theorem 2.1 we get the next theorem for M_wEM_u.

Theorem 3.2. *Let $T=M_wEM_u:L^p(\Sigma)\to L^p(\Sigma)$, for $1\le p\le\infty$ and*

$$E(|u|^p),E(|w|^p)\in L^\infty(\mathcal{A}).$$

Then we have the following.

(a) $\sigma(M_wEM_u)\setminus\{0\}=\{\lambda\in\mathbb{C}:(\nexists\epsilon>0)$ such that $|\lambda-E(uw)|\ge\epsilon$ a.e$\}\setminus\{0\}$.

(b) If $S\cap G=X$ and $p=2$, then

$$\sigma(M_wEM_u)=\{\lambda\in\mathbb{C}:(\nexists\epsilon>0)\text{ such that }|\lambda-E(uw)|\ge\epsilon\text{ a.e}\},$$

where $S=S(E(|u|^2))$ and $G=S(E(|w|^2))$.

(c) $\sigma_p(M_wEM_u)\setminus\{0\}=\{\lambda\in\mathbb{C}\setminus\{0\}:\mu(A_{\lambda,w})>0\}$, where $A_{\lambda,w}=\{x\in X:E(uw)(x)=\lambda\}$.

(d) $\rho(M_wEM_u)\setminus\{0\}=\{\lambda\in\mathbb{C}:(\exists\epsilon>0)$ such that $|\lambda-E(uw)|\ge\epsilon$ a.e$\}\setminus\{0\}$.

(e) $\sigma_r(M_wEM_u)\setminus\{0\}=\begin{cases}\emptyset, & 1\le p<\infty,\\ ER(E(uw))\setminus\{0\}\cup\sigma_p(M_wEM_u), & p=\infty,\end{cases}$ in which $ER(E(uw))=\{\lambda\in\mathbb{C}:(\nexists\epsilon>0)$ such that $|\lambda-E(uw)|\ge\epsilon$ a.e$\}$.

Proof. (a) Since

$$\sigma(M_wEM_u)\setminus\{0\}=\sigma(EM_uM_w)\setminus\{0\}=\sigma(EM_{uw})\setminus\{0\}=\operatorname{ess\,range}(E(uw))\setminus\{0\},$$

then we have

$$\sigma(M_wEM_u)\setminus\{0\}=\operatorname{ess\,range}(E(uw))\setminus\{0\}.$$

(b) We know that $\sigma(EM_{uw})=\operatorname{ess\,range}(E(uw))$. So, we have to prove that $0\notin\sigma(EM_{uw})$ if and only if $0\notin\sigma(M_wEM_u)$ by (a).

Let $0\notin\sigma(EM_{uw})$. Then EM_{uw} is surjective and so $\mathcal{A}=\Sigma$. Thus, $E=I$. So, $0\notin\sigma(M_wEM_u)$.

Conversely, we know that the polar decomposition of $M_wEM_u=U|M_wEM_u|$ is as follow

$$|M_wEM_u|(f)=\left(\frac{E(|w|^2)}{E(|u|^2)}\right)^{\frac{1}{2}}\chi_S\bar{u}E(uf)$$

and

$$U(f)=\left(\frac{\chi_{S\cap G}}{E(|w|^2)E(|u|^2)}\right)^{\frac{1}{2}}wE(uf),$$

for all $f\in L^2(\Sigma)$.

If $0 \notin \sigma(M_wEM_u)$, then $|M_wEM_u|$ is invertible and U is unitary [page 73, [4]]. Therefore, $U^*U = UU^* = I$. The equation $UU^* = I$ implies that $w \in L^0(\mathcal{A}_{S\cap G})$, where $\mathcal{A}_{S\cap G} = \{A \cap S \cap G : A \in \mathcal{A}\}$. Since $S \cap G = X$, then $w \in L^0(\mathcal{A})$. Hence, $0 \notin \sigma(M_wEM_u) = \sigma(EM_{uw})$.

(c) By Theorem 3.1 we have

$$\sigma_p(M_wEM_u) \setminus \{0\} = \sigma_p(EM_uM_w) \setminus \{0\} = \sigma_p(EM_{uw}) \setminus \{0\}.$$

So,

$$\sigma_p(M_wEM_u) \setminus \{0\} = \{\lambda \in \mathbb{C} \setminus \{0\} : \mu(A_{\lambda,w}) > 0\}.$$

The proof of (d) and (e) are as the same as (c). □

If the converse of conditional-type Hölder inequality is satisfied for w and u, then we get that the joint point spectrum and point spectrum of M_wEM_u are equal. Hence we have the next remark.

Remark 3.1. If $M_wEM_u : L^2(\Sigma) \to L^2(\Sigma)$ and $|E(uw)|^2 \geq E(|u|^2)E(|w|^2)$, then the following hold.

(a) $\sigma_{jp}(M_wEM_u) \setminus \{0\} = \{\lambda \in \mathbb{C} \setminus \{0\} : \mu(A_{\lambda,w}) > 0\}$.

(b) If $S \cap G = X$, then

$$\sigma_{jp}(M_wEM_u) = \operatorname{ess\,range}(E(uw)) = \{\lambda \in \mathbb{C} : \mu(A_{\lambda,w}) > 0\}.$$

By Theorem 3.2 we have the following corollary.

Corollary 3.1. *For* $T = M_wEM_u : L^p(\Sigma) \to L^p(\Sigma)$ *and* $1 \leq p \leq \infty$ *we have* $r(T) = \|E(uw)\|_\infty$.

Recall that an operator T is quasinilpotent if $\sigma(T) = \{0\}$. In light of Theorem 3.2, we have the following proposition.

Proposition 3.1. *If* $T = M_wEM_u : L^p(\Sigma) \to L^p(\Sigma)$, $1 \leq p \leq \infty$, *and* $\frac{1}{p} + \frac{1}{p'} = 1$, *then the following are equivalent:*

(a) M_wEM_u *is quasinilpotent;*

(b) $E(uw) \equiv 0$*;*

(c) $M_{wE(uw)}EM_u \equiv 0$.

Proof. ($a \Leftrightarrow b$) Since $\sigma(M_wEM_u) \setminus \{0\} = \operatorname{ess\,range}(E(uw)) \setminus \{0\}$, it follows that $\sigma(M_wEM_u) = 0$ if and only if $E(uw) = 0$ almost every where.

($b \Leftrightarrow c$) As we investigated the norm of weighted conditional type operators in [3], we have

$$\|M_{wE(uw)}EM_u\| = \|E(uw)(E(|w|^p))^{\frac{1}{p}}(E(|u|^{p'}))^{\frac{1}{p'}}\|_\infty$$

and

$$|E(uw)| \leq (E(|w|^p))^{\frac{1}{p}}(E(|u|^{p'}))^{\frac{1}{p'}},$$

it follows that $E(uw) = 0$, a.e, if and only if $E(uw)(E(|w|^p))^{\frac{1}{p}}(E(|u|^{p'}))^{\frac{1}{p'}} = 0$, a.e, if and only if $\|M_{wE(uw)}EM_u\| = 0$. □

Now, we give some example of conditional expectation.

Example 3.1. (a) Let $X=\mathbb{N}\cup\{0\}$, $\mathcal{G}=2^{\mathbb{N}}$ and let $\mu(\{x\})=\frac{e^{-\theta}\theta^x}{x!}$, for each $x\in X$ and $\theta\geq 0$. Elementary calculations show that μ is a probability measure on $\mathcal{G}$. Let $\mathcal{A}$ be the σ-algebra generated by the partition $B=\{\emptyset, X, \{0\}, X_1=\{1,3,5,7,9,\dots\}, X_2=\{2,4,6,8,\dots\},\}$ of $\mathbb{N}$. Note that, $\mathcal{A}$ is a sub-σ-algebra of Σ and each of element of $\mathcal{A}$ is an $\mathcal{A}$-atom. Thus, the conditional expectation of any $f\in\mathcal{D}(E)$ relative to $\mathcal{A}$ is constant on $\mathcal{A}$-atoms. Hence, there exist scalars a_1, a_2, a_3 such that

$$E(f)=a_1\chi_0+a_2\chi_{X_1}+a_3\chi_{X_2}.$$

So,

$$E(f)(0)=a_1,\quad E(f)(2n-1)=a_2,\quad E(f)(2n)=a_3,$$

for all $n\in\mathbb{N}$. By definition of conditional expectation with respect to $\mathcal{A}$, we have

$$f(0)\mu(\{0\})=\int_{\{0\}}fd\mu=\int_{\{0\}}E(f)d\mu=a_1\mu(\{0\}),$$

so $a_1=f(0)$. Also,

$$\sum_{n\in\mathbb{N}}f(2n-1)\frac{e^{-\theta}\theta^{2n-1}}{(2n-1)!}=\int_{X_1}fd\mu=\int_{X_1}E(f)d\mu=a_2\mu(X_2)=a_2\sum_{n\in\mathbb{N}}\frac{e^{-\theta}\theta^{2n-1}}{(2n-1)!}.$$

So,

$$a_2=\frac{\sum_{n\in\mathbb{N}}f(2n-1)\frac{e^{-\theta}\theta^{2n-1}}{(2n-1)!}}{\sum_{n\in\mathbb{N}}\frac{e^{-\theta}\theta^{2n-1}}{(2n-1)!}}.$$

By the same method we have

$$a_3=\frac{\sum_{n\in\mathbb{N}}f(2n)\frac{e^{-\theta}\theta^{2n}}{(2n)!}}{\sum_{n\in\mathbb{N}}\frac{e^{-\theta}\theta^{2n}}{(2n)!}}.$$

If we set $f(x)=x$, then $E(f)$ is a special function as follows:

$$E(f)=\theta\coth(\theta)\chi_{X_1}+\frac{\cosh(\theta)-1}{\cosh(\theta)}\chi_{X_2}.$$

Also, if u and w are real functions on X such that M_wEM_u is bounded on l^p, then by Theorem 3.2 we have

$$\begin{aligned}&\sigma(M_wEM_u)\\&=\left\{u(0)w(0),\frac{\sum_{n\in\mathbb{N}}u(2n-1)w(2n-1)\frac{e^{-\theta}\theta^{2n-1}}{(2n-1)!}}{\sum_{n\in\mathbb{N}}\frac{e^{-\theta}\theta^{2n-1}}{(2n-1)!}},\frac{\sum_{n\in\mathbb{N}}u(2n)w(2n)\frac{e^{-\theta}\theta^{2n}}{(2n)!}}{\sum_{n\in\mathbb{N}}\frac{e^{-\theta}\theta^{2n}}{(2n)!}}\right\}.\end{aligned}$$

(b) Let $X=\mathbb{N}$, $\mathcal{G}=2^{\mathbb{N}}$ and let $\mu(\{x\})=pq^{x-1}$ for each $x\in X$, $0\leq p\leq 1$ and $q=1-p$. Elementary calculations show that μ is a probability measure on $\mathcal{G}$. Let $\mathcal{A}$ be the σ-algebra generated by the partition $B=\{X_1=\{3n:n\geq 1\}, X_1^c\}$ of X. So, for every $f\in\mathcal{D}(E^{\mathcal{A}})$

$$E(f)=\alpha_1\chi_{X_1}+\alpha_2\chi_{X_1^c},$$

and direct computations show that

$$\alpha_1 = \frac{\sum_{n\geq 1} f(3n)pq^{3n-1}}{\sum_{n\geq 1} pq^{3n-1}}$$

and

$$\alpha_2 = \frac{\sum_{n\geq 1} f(n)pq^{n-1} - \sum_{n\geq 1} f(3n)pq^{3n-1}}{\sum_{n\geq 1} pq^{n-1} - \sum_{n\geq 1} pq^{3n-1}}.$$

For example, if we set $f(x) = x$, then $E(f)$ is a special function as follows

$$\alpha_1 = \frac{3}{1-q^3}, \quad \alpha_2 = \frac{1+q^6-3q^4+4q^3-3q^2}{(1-q^2)(1-q^3)}.$$

So, if u and w are real functions on X such that M_wEM_u is bounded on l^p, then by Theorem 3.2 we have

$$\begin{aligned}&\sigma(M_wEM_u)\\ &=\left\{\frac{\sum_{n\geq 1} u(3n)w(2n)pq^{3n-1}}{\sum_{n\geq 1} pq^{3n-1}}, \frac{\sum_{n\geq 1} u(n)w(n)pq^{n-1} - \sum_{n\geq 1} u(3n)w(3n)pq^{3n-1}}{\sum_{n\geq 1} pq^{n-1} - \sum_{n\geq 1} pq^{3n-1}}\right\}.\end{aligned}$$

(c) Let $X = [0,1)$, Σ is σ-algebra of Lebesgue measurable subsets of X, μ is the Lebesgue measure on X. Let $s : [0,1) \to [0,1)$ be defined by $s(x) = x + \frac{1}{4}(\text{mod } 1)$. Let $\mathcal{B} = \{E \in \Sigma : s(E) = E\}$. In this case

$$E^{\mathcal{B}}(f)(x) = \frac{f(x) + f(s(x)) + f(s^2(x)) + f(s^3(x))}{4},$$

where s^j denotes the jth iteration of s. Also, $|f| \leq 3E^{\mathcal{B}}(|f|)$ a.e. Hence, the operator EM_u is bounded on $L^p([0,1))$ if and only if $u \in L^\infty([0,1))$.

(d) Let $X = [0,a] \times [0,a]$ for $a > 0$, $d\mu = dxdy$, Σ the Lebesgue subsets of X and let $\mathcal{A} = \{A \times [0,a] : A \text{ is a Lebesgue set in } [0,a]\}$. Then, for each $f \in \mathcal{D}(E)$, $(Ef)(x,y) = \int_0^a f(x,t)dt$, which is independent of the second coordinate. For example, if we set $a = 1$, $w(x,y) = 1$ and $u(x,y) = e^{(x+y)}$, then $E(u)(x,y) = e^x - e^{x+1}$ and M_wEM_u is bounded. Therefore, by Theorem 3.2 $\sigma(M_wEM_u) = [e - e^2, 1 - e]$.

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