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REGULARITY OF SEMIGROUPS IN TERMS OF HYBRID IDEALS AND HYBRID BI-IDEALS

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Abstract. In this paper, we establish some equivalent conditions for a semigroup to be regular and intra-regular, in terms of hybrid ideals and hybrid bi-ideals. We also characterize the left and right simple and the completely regular semigroups utilizing hybrid ideals and hybrid bi-ideals. We show that a semigroup $\mathcal{S}$ is left simple if and only if it is hybrid left simple. We also prove that if a semigroup $\mathcal{S}$ is intra-regular, then for each hybrid ideal $\tilde{j}_\mu$ of $\mathcal{S}$, we have $\tilde{j}_\mu(r_1r_2) = \tilde{j}_\mu(r_2r_1)$ for all $r_1, r_2 \in \mathcal{S}$.

1. Introduction

The fuzzy set theory is applicable to many mathematical branches which was introduced by Zadeh [11]. The fuzzification of algebraic structures and the introduction of the notion of fuzzy subgroups was inspired by Rosenfeld [10]. The definition given by Rosenfeld was a epochal turning point for pure mathematicians. Inspired by these studies, many authors have pursued the field of fuzzy algebraic structure in many different areas such as groups, rings, modules, vector spaces and so on (see [4, 5, 8]). Molodtsov [9] introduced the soft set theory as a new mathematical tool to deal with uncertainties. Many researchers have rigorously pursued and have studied extensively the fundamentals of soft set theory in recent years.

The notion of hybrid structure was introduced by Jun, Song and Muhiuddin [3] as a parallel circuit of fuzzy and soft sets. The notion of hybrid structure was introduced into a set of parameters on a initial universe set and it was applied to BCK/BCI algebras and linear spaces. In the year 2017, S. Anis, M. Khan and Y. B. Jun [1] gave

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the notions of hybrid sub-semigroups and hybrid left (resp. right) ideals in semigroups and obtained various properties. Several equivalent conditions for a semi-group hybrid bi-ideal to be a semi-group ideal were obtained by B. Elavarasan and Y. B. Jun [2] in their study of the notion of semi-group hybrid bi-ideals and they have also studied some of its important properties.

In this paper, we obtain some equivalent characterizations of a regular and intra-regular semigroup in terms of their hybrid ideals and hybrid bi-ideals. We present necessary definitions to be used in the sequel in Section 2 from the already available literature. Section 3 is devoted to major results of the characterization of the regular and intra-regular semigroup through the properties of their hybrid and hybrid bi-ideals.

2. Preliminaries

In this section, we collect some basic notions and results on semigroup and hybrid structures.

Let $\mathcal{S}$ be a semigroup. Let V and W be subsets of $\mathcal{S}$. Then the multiplication of V and W is defined as $VW = \{vw : v \in V \text{ and } w \in W\}$. A non-empty subset W of $\mathcal{S}$ is called a subsemigroup of $\mathcal{S}$ if $W^2 \subseteq W$. A subsemigroup W of $\mathcal{S}$ is called a left (resp. right) ideal of $\mathcal{S}$ if $\mathcal{S}W \subseteq W$ (resp. $W\mathcal{S} \subseteq W$). If W is both a left and right ideal of $\mathcal{S}$, then W is called a two-sided ideal or ideal of $\mathcal{S}$. Clearly for any $r \in \mathcal{S}$, $L(r) = \{r \cup \mathcal{S}r\}$ (resp. $R(r) = \{r \cup r\mathcal{S}\}$, $I(r) = \{r \cup r\mathcal{S} \cup \mathcal{S}r \cup \mathcal{S}r\mathcal{S}\}$ is a left ideal (resp. right ideal, ideal) of $\mathcal{S}$ generated by r in $\mathcal{S}$. A semigroup $\mathcal{S}$ is called regular if for each $d \in \mathcal{S}$, $d = dtd$ for some t in $\mathcal{S}$. A semigroup $\mathcal{S}$ is called intra-regular if for each $d \in \mathcal{S}$, $d = sd^2t$ for some $s, t \in \mathcal{S}$ [6]. A subsemigroup W of $\mathcal{S}$ is called a bi-ideal of $\mathcal{S}$ if $W\mathcal{S}W \subseteq W$. It is clear that for any $d \in \mathcal{S}$, $B(d) = \{d, d^2, d\mathcal{S}d\}$ is a bi-ideal of $\mathcal{S}$ generated by d in $\mathcal{S}$.

Throughout this paper, we denote $\mathcal{S}$ is a semigroup, J is a unit interval and $\mathscr{P}(U)$ is the power set of an initial universal set U.

Definition 2.1 ([1])**.** A hybrid structure in $\mathcal{S}$ over U is a mapping $\tilde{j}_\alpha := (\tilde{j}, \alpha) : \mathcal{S} \to \mathscr{P}(U) \times J,\ d \mapsto (\tilde{j}(d), \alpha(d))$, where $\tilde{j} : \mathcal{S} \to \mathscr{P}(U)$ and $\alpha : \mathcal{S} \to J$ are mappings.

For any semigroup $\mathcal{S}$, $HY(\mathcal{S})$ denote the set of all hybrid structures in $\mathcal{S}$ over U. We define an order $\ll$ in $HY(\mathcal{S})$ as follows.

For all $\tilde{j}_\nu, \tilde{k}_\mu \in HY(\mathcal{S})$, $\tilde{j}_\nu \ll \tilde{k}_\mu$ if and only if $\tilde{j} \mathrel{\tilde{\subseteq}} \tilde{k}$, $\nu \succeq \mu$, where $\tilde{j} \mathrel{\tilde{\subseteq}} \tilde{k}$ means that $\tilde{j}(d) \subseteq \tilde{k}(d)$ and $\nu \succeq \mu$ means that $\nu(d) \geq \mu(d)$ for all $d \in \mathcal{S}$.

For any $r_1, r_2 \in \mathcal{S}$, $\tilde{j}_\nu(r_1) = \tilde{k}_\mu(r_2)$ if and only if $\tilde{j}_\nu(r_1) \ll \tilde{k}_\mu(r_2)$ and $\tilde{k}_\mu(r_2) \ll \tilde{j}_\nu(r_1)$, where $\tilde{j}_\nu(r_1) \ll \tilde{k}_\mu(r_2)$ means that $\tilde{j}(r_1) \subseteq \tilde{k}(r_2)$ and $\nu(r_1) \geq \mu(r_2)$. Also, $\tilde{j}_\nu = \tilde{k}_\mu$ if and only if $\tilde{j}_\nu \ll \tilde{k}_\mu$ and $\tilde{k}_\mu \ll \tilde{j}_\nu$. Note that $(HY(\mathcal{S}), \ll)$ is a poset.

Remark 2.1. For a family of real numbers $\{a_i : i \in \alpha\}$, we define

$$\bigvee\{a_i : i \in \alpha\} := \begin{cases} \max\{a_i : i \in \alpha\}, & \text{if } \alpha \text{ is finite,} \\ \sup\{a_i : i \in \alpha\}, & \text{otherwise,} \end{cases}$$

and

$$\bigwedge\{a_i : i \in \alpha\} := \begin{cases} \min\{a_i : i \in \alpha\}, & \text{if } \alpha \text{ is finite,} \\ \inf\{a_i : i \in \alpha\}, & \text{otherwise.} \end{cases}$$

For real numbers r and k, we also use $r \vee k$ and $r \wedge k$ instead of $\bigvee\{r, k\}$ and $\bigwedge\{r, k\}$, respectively.

Definition 2.2 ([2]). A hybrid subsemigroup $\tilde{j}_\alpha$ in $\mathcal{S}$ over U is called a hybrid bi-ideal if for all $m, y, n \in \mathcal{S}$

(i) $\tilde{j}(myn) \supseteq \tilde{j}(m) \cap \tilde{j}(n)$;
(ii) $\alpha(myn) \leq \alpha(m) \vee \alpha(n)$.

Definition 2.3 ([1]). Let $\tilde{j}_\alpha \in HY(\mathcal{S})$. Then $\tilde{j}_\alpha$ is called a hybrid left (resp. right) ideal if for all $m, n \in \mathcal{S}$

(i) $\tilde{j}(mn) \supseteq \tilde{j}(n)$ (resp. $\tilde{j}(mn) \supseteq \tilde{j}(m)$);
(ii) $\alpha(mn) \leq \alpha(n)$ (resp. $\alpha(mn) \leq \alpha(m)$).

If $\tilde{j}_\alpha$ is both a hybrid left and hybrid right ideal of $\mathcal{S}$, then it is called a hybrid ideal of $\mathcal{S}$.

Definition 2.4. A semigroup $\mathcal{S}$ is called hybrid left (resp. right) duo over U if every hybrid left (resp. right) ideal of $\mathcal{S}$ over U is a hybrid ideal of $\mathcal{S}$ over U. $\mathcal{S}$ is called hybrid duo over U if it is both hybrid left and hybrid right duo over U.

Definition 2.5 ([1]). Let $\phi \neq B \subseteq \mathcal{S}$ and $\tilde{j}_\nu \in HY(\mathcal{S})$. Then the characteristic hybrid structure is denoted by $\chi_B(\tilde{j}_\nu) = (\chi_B(\tilde{j}), \chi_B(\nu))$, where $\chi_B(\tilde{j}) : \mathcal{S} \to \mathscr{P}(U)$,

$$d \mapsto \begin{cases} U, & \text{if } d \in B, \\ \phi, & \text{otherwise,} \end{cases}$$

and $\chi_B(\nu) : \mathcal{S} \to J$,

$$d \mapsto \begin{cases} 0, & \text{if } d \in B, \\ 1, & \text{otherwise.} \end{cases}$$

Definition 2.6 ([1]). Let $\tilde{j}_\alpha, \tilde{k}_\beta \in HY(\mathcal{S})$.

(i) The hybrid product of $\tilde{j}_\alpha$ and $\tilde{k}_\beta$ in $\mathcal{S}$ is defined as $\tilde{j}_\alpha \odot \tilde{k}_\beta = (\tilde{j}\tilde{\circ}\tilde{k}, \alpha\tilde{\circ}\beta)$ in $\mathcal{S}$ over U, where

$$(\tilde{j}\tilde{\circ}\tilde{k})(d) = \begin{cases} \bigcup\limits_{d=r_1r_2} \{\tilde{j}(r_1) \cap \tilde{k}(r_2)\}, & \text{if exist } r_1, r_2 \in \mathcal{S} \text{ such that } d = r_1r_2, \\ \phi, & \text{otherwise,} \end{cases}$$

and

$$(\alpha\tilde{\circ}\beta)(d) = \begin{cases} \bigwedge\limits_{d=r_1r_2} \{\alpha(r_1) \vee \beta(r_2)\} & \text{if exist } r_1, r_2 \in \mathcal{S} \text{ such that } d = r_1r_2, \\ 1 & \text{otherwise,} \end{cases}$$

for all $d \in \mathcal{S}$.

(ii) The hybrid intersection of $\tilde{j}_\alpha$ and $\tilde{k}_\beta$, denoted by $\tilde{j}_\alpha \Cap \tilde{k}_\beta$, is defined to be a hybrid structure $\tilde{j}_\alpha \Cap \tilde{k}_\beta : \mathcal{S} \to \mathscr{P}(U) \times I$, $d \mapsto ((\tilde{j}\tilde{\cap}\tilde{k})(r_1), (\alpha \vee \beta)(r_1))$, where $\tilde{j}\tilde{\cap}\tilde{k} : \mathcal{S} \longrightarrow \mathscr{P}(U)$, $r_1 \mapsto \tilde{j}(r_1) \cap \tilde{k}(r_1)$ and $\alpha \vee \beta : \mathcal{S} \to I$, $r_1 \mapsto \alpha(r_1) \vee \beta(r_1)$.

3. Hybrid Structures in Regular and Intra-Regular Semigroups

Lemma 3.1. ([1, Lemma 3.11]). *For subsets M and N of $\mathcal{S}$, we have*
(i) $\chi_M(\tilde{j}_\nu) \Cap \chi_N(\tilde{j}_\nu) = \chi_{M\cap N}(\tilde{j}_\nu)$;
(ii) $\chi_M(\tilde{j}_\nu) \odot \chi_N(\tilde{j}_\nu) = \chi_{MN}(\tilde{j}_\nu)$.

Theorem 3.1. ([1, Proposition 3.21]). *If $\tilde{j}_\nu$ and $\tilde{k}_\mu$ are hybrid right ideal and hybrid left ideal of $\mathcal{S}$ over U, respectively, then $\tilde{j}_\nu \odot \tilde{k}_\mu \ll \tilde{j}_\nu \Cap \tilde{k}_\mu$.*

We now obtain theorems on the characterizations of a regular and intra-regular semigroup in terms of different hybrid ideals of semigroups.

Theorem 3.2. *For any $\mathcal{S}$, the following assertions are equivalent:*
(i) $\tilde{j}_\nu \Cap \tilde{k}_\mu \ll \tilde{j}_\nu \odot \tilde{k}_\mu$ *for every hybrid bi-ideals $\tilde{j}_\nu$ and $\tilde{k}_\mu$ of $\mathcal{S}$;*
(ii) $\mathcal{S}$ is regular and intra-regular.

Proof. $(i) \Rightarrow (ii)$ If $\tilde{j}_\nu$ is a hybrid right ideal and $\tilde{k}_\mu$ is a hybrid left ideal of $\mathcal{S}$, then $\tilde{j}_\nu \Cap \tilde{k}_\mu \ll \tilde{j}_\nu \odot \tilde{k}_\mu$ and by Theorem 3.1, we can get $\tilde{j}_\nu \odot \tilde{k}_\mu \ll \tilde{j}_\nu \Cap \tilde{k}_\mu$. So, for any hybrid right ideal $\tilde{j}_\nu$ and hybrid left ideal $\tilde{k}_\mu$ of $\mathcal{S}$, we have $\tilde{j}_\nu \Cap \tilde{k}_\mu = \tilde{j}_\nu \odot \tilde{k}_\mu$.

Let E be a right ideal and D be a left ideal of $\mathcal{S}$ and $t \in E \cap D$. Then $\chi_E(\tilde{j}_\nu)(t) \Cap \chi_D(\tilde{j}_\nu)(t) = \chi_E(\tilde{j}_\nu)(t) \odot \chi_D(\tilde{j}_\nu)(t)$. By Lemma 3.1 (i), we have $\chi_{E\cap D}(\tilde{j}_\nu) = \chi_{ED}(\tilde{j}_\nu)$. Since $t \in E \cap D$ and $\chi_{E\cap D}(\tilde{j})(t) = U$, we have $\chi_{ED}(\tilde{j})(t) = U$, which implies $t \in ED$. Thus, $E \cap D \subseteq ED \subseteq E \cap D$ and hence $E \cap D = ED$. Therefore, $\mathcal{S}$ is regular. Also for $t \in \mathcal{S}$, we have $\chi_{B(t)}(\tilde{j}_\nu) \Cap \chi_{B(t)}(\tilde{j}_\nu) \ll \chi_{B(t)}(\tilde{j}_\nu) \odot \chi_{B(t)}(\tilde{j}_\nu)$. Again by Lemma 3.1 (i), we have $\chi_{B(t)}(\tilde{j}_\nu) \ll \chi_{B(t)B(t)}(\tilde{j}_\nu)$. Since $\chi_{B(t)}(\tilde{j})(t) = U$, we have $\chi_{B(t)B(t)}(\tilde{j})(t) = U$ which implies $t \in B(t)B(t)$, so $\mathcal{S}$ is intra-regular.

$(ii) \Rightarrow (i)$ Let $\tilde{j}_\nu$ and $\tilde{k}_\mu$ be hybrid bi-ideals of $\mathcal{S}$. Suppose $r_1 \in \mathcal{S}$. Then for some $x, y, z \in \mathcal{S}$, $r_1 = r_1xr_1 = r_1xr_1xr_1$ and $r_1 = yr_1^2z$, which imply $r_1 = r_1xyr_1^2zxr_1$. Now

$$\tilde{j}\tilde{\circ}\tilde{k} = \bigcup_{r_1=uv} \{\tilde{j}(u) \cap \tilde{k}(v)\} \supseteq \tilde{j}(r_1xyr_1) \cap \tilde{k}(r_1zxr_1) \supseteq \{\tilde{j}(r_1) \cap \tilde{j}(r_1)\} \cap \{\tilde{k}(r_1) \cap \tilde{k}(r_1)\}$$
$$= \tilde{j}(r_1) \cap \tilde{k}(r_1) = (\tilde{j} \cap \tilde{k})(r_1).$$

Also,

$$(\nu\tilde{\circ}\mu)(r_1) = \bigwedge_{r_1=uv} \{\nu(u) \vee \mu(v)\} \leq \nu(r_1xyr_1) \vee \mu(r_1zxr_1)$$
$$= \{\nu(r_1) \vee \nu(r_1)\} \vee \{\mu(r_1) \vee \mu(r_1)\} = \nu(r_1) \vee \mu(r_1) = (\nu \cap \mu)(r_1).$$

Therefore, $\tilde{j}_\nu \Cap \tilde{k}_\mu \ll \tilde{j}_\nu \odot \tilde{k}_\mu$. □

Theorem 3.3. *For any $\mathcal{S}$, the below assertions are equivalent:*
(i) $\tilde{j}_\nu \Cap \tilde{k}_\mu \ll (\tilde{j}_\nu \odot \tilde{k}_\mu) \Cap (\tilde{k}_\mu \odot \tilde{j}_\nu)$ *for every hybrid bi-ideals $\tilde{j}_\nu$ and $\tilde{k}_\mu$ of $\mathcal{S}$;*
(ii) $\mathcal{S}$ is intra-regular and regular.

Proof. $(i) \Rightarrow (ii)$ Let $\tilde{j}_\nu$ and $\tilde{k}_\mu$ be hybrid bi-ideals of $\mathcal{S}$. Then, by Theorem 3.2, we can have $\tilde{j}_\nu \Cap \tilde{k}_\mu \ll \tilde{j}_\nu \odot \tilde{k}_\mu$ and $\tilde{j}_\nu \Cap \tilde{k}_\mu = \tilde{k}_\mu \Cap \tilde{j}_\nu \ll \tilde{k}_\mu \odot \tilde{j}_\nu$, which imply $\tilde{j}_\nu \Cap \tilde{k}_\mu \ll (\tilde{j}_\nu \odot \tilde{k}_\mu) \Cap (\tilde{k}_\mu \odot \tilde{j}_\nu)$.

$(ii) \Rightarrow (i)$ By (ii), $\tilde{j}_\nu \Cap \tilde{k}_\mu \ll \tilde{j}_\nu \odot \tilde{k}_\mu$ for every hybrid bi-ideals $\tilde{j}_\nu$ and $\tilde{k}_\mu$ of $\mathcal{S}$. By Theorem 3.2, we get $\mathcal{S}$ is regular and intra-regular. □

Theorem 3.4. *For any $\mathcal{S}$, the following assertions are equivalent:*

(i) *$\tilde{i}_\nu \Cap \tilde{k}_\mu \ll \tilde{i}_\nu \odot \tilde{k}_\mu \odot \tilde{i}_\nu$ for every hybrid bi-ideals $\tilde{i}_\nu$ and $\tilde{k}_\mu$ of $\mathcal{S}$;*

(ii) *$\mathcal{S}$ is intra-regular and regular.*

Proof. $(i) \Rightarrow (ii)$ Let $w \in \mathcal{S}$. Then by assumption and by Lemma 3.1, we can get $\chi_{B(w)}(\tilde{i}_\nu) \ll \chi_{B(w)}(\tilde{i}_\nu) \odot \chi_{B(w)}(\tilde{i}_\nu) \odot \chi_{B(w)}(\tilde{i}_\nu) = \chi_{B(w)B(w)B(w)}(\tilde{i}_\nu)$. Since $\chi_{B(w)}(\tilde{i})(w) = U$, $\chi_{B(w)B(w)B(w)}(\tilde{i})(w) = U$ implies $w \in B(w)B(w)B(w)$. Therefore, $\mathcal{S}$ is regular and intra-regular.

$(ii) \Rightarrow (i)$ Let $\tilde{j}_\nu$ and $\tilde{k}_\mu$ be hybrid bi-ideals of $\mathcal{S}$ and let $w \in \mathcal{S}$. Then there exist $r_1, r_2, r_3 \in \mathcal{S}$ such that $w = wr_1w = wr_1wr_1w$ and $w = r_2w^2r_3$, which imply $w = wr_1r_2w^2r_3r_1r_2w^2r_3r_1w = (wr_1r_2w)(wr_3r_1r_2w)(wr_3r_1w)$. Then

$$
\begin{aligned}
(\tilde{j}\tilde{\circ}\tilde{k}\tilde{\circ}\tilde{j})(w) = & \bigcup_{w=w_1w_2} \{\tilde{j}(w_1) \cap (\tilde{k}\tilde{\circ}\tilde{j})(w_2)\} \\
\supseteq & \tilde{j}(wr_1r_2w) \cap (\tilde{k}\tilde{\circ}\tilde{j})((wr_3r_1r_2w)(wr_3r_1w)) \\
\supseteq & \{\tilde{j}(w) \cap \tilde{j}(w)\} \cap \{\tilde{k}(wr_3r_1r_2w) \cap \tilde{j}(wr_3r_1w)\} \\
\supseteq & \tilde{j}(w) \cap \tilde{k}(w) \cap \tilde{j}(w) = \tilde{j}(w) \cap \tilde{k}(w) = (\tilde{j} \cap \tilde{k})(w).
\end{aligned}
$$

Also,

$$
\begin{aligned}
(\nu\tilde{\circ}\mu\tilde{\circ}\nu)(w) = & \bigwedge_{w=w_1w_2} \{\nu(w_1) \vee (\mu\tilde{\circ}\nu)(w_2)\} \\
\leq & \nu(wr_1r_2w) \vee (\mu\tilde{\circ}\nu)((wr_3r_1r_2w)(wr_3r_1w)) \\
\leq & \{\nu(w) \vee \nu(w)\} \vee \{\mu(wr_3r_1r_2w) \vee \nu(wr_3r_1w)\} \\
\leq & \bigvee\{\nu(w), \mu(w), \nu(w)\} \\
= & \nu(w) \vee \mu(w) = (\nu \cap \mu)(w).
\end{aligned}
$$

Therefore, $\tilde{j}_\nu \Cap \tilde{k}_\mu \ll \tilde{j}_\nu \odot \tilde{k}_\mu \odot \tilde{j}_\nu$. □

Theorem 3.5. ([1, Theorem 3.5]). *For $\varnothing \neq M \subset \mathcal{S}$, the following assertions are equivalent:*

(i) *$\chi_M(\tilde{i}_\nu)$ in $\mathcal{S}$ is hybrid right (resp. left) ideal;*

(ii) *M is a right (resp. left) ideal of $\mathcal{S}$.*

Theorem 3.6. ([2, Theorem 2.12]). *Let $\mathcal{S}$ be a regular right duo (resp. left duo, duo) semigroup. Then the following assertions are equivalent:*

(i) *$\tilde{i}_\nu$ is a hybrid left ideal (resp. right ideal) of $\mathcal{S}$;*

(ii) *$\tilde{i}_\nu$ is a hybrid bi-ideal of $\mathcal{S}$.*

Theorem 3.7. *For any* $\mathcal{S}$, *the following assertions are equivalent:*
(i) *if* $\tilde{k}_\mu$ *is a hybrid ideal of* $\mathcal{S}$, *then* $\tilde{k}_\mu(r) = \tilde{k}_\mu(r^2)$ *for all* $r \in \mathcal{S}$*;*
(ii) $\mathcal{S}$ *is intra-regular.*

Proof. (i) $\Rightarrow$ (ii) Let $r \in \mathcal{S}$. Then $I(r^2)$ is an ideal of $\mathcal{S}$ and by Theorem 3.5, $\chi_{I(r^2)}(\tilde{k}_\mu)(r) = \chi_{I(r^2)}(\tilde{k}_\mu)(r^2)$. Since $r^2 \in I(r^2)$, we have $\chi_{I(r^2)}(\tilde{k})(r^2) = U$ implies that $\chi_{I(r^2)}(\tilde{k}_\mu)(r) = U$. Thus, $r \in I(r^2)$ and hence $\mathcal{S}$ is intra-regular.

(ii) $\Rightarrow$ (i) Let $r \in \mathcal{S}$. Then there exist $x, y \in \mathcal{S}$ such that $r = xr^2y$. Let $\tilde{k}_\mu$ be a hybrid ideal of $\mathcal{S}$. Then $\tilde{k}_\mu(r) \ll \tilde{k}_\mu(r^2)$. Also,

$$\tilde{k}(r) = \tilde{k}(xr^2y) = \tilde{k}((xrr)y) \supseteq \tilde{k}(xrr) \supseteq \tilde{k}(rr) = \tilde{k}(r^2)$$

and $\mu(r) = \mu(xr^2y) = \mu((xrr)y) \leq \mu(xrr) \leq \mu(rr) = \mu(r^2)$. Thus, $\tilde{k}_\mu(r^2) \ll \tilde{k}_\mu(r)$ and hence $\tilde{k}_\mu(r) = \tilde{k}_\mu(r^2)$. □

As a consequence of above two theorems, we have the following result.

Corollary 3.1. *If* $\mathcal{S}$ *is a regular duo, then the following assertions are equivalent:*
(i) *if* $\tilde{k}_\nu$ *is a hybrid bi-ideal in* $\mathcal{S}$, *then* $\tilde{k}_\nu(r) = \tilde{k}_\nu(r^2)$ *for all* $r \in \mathcal{S}$*;*
(ii) $\mathcal{S}$ *is intra-regular.*

Theorem 3.8. *If* $\mathcal{S}$ *is an intra-regular semigroup, then for every hybrid ideal* $\tilde{k}_\nu$ *in* $\mathcal{S}$, *we have* $\tilde{k}_\nu(r_1r_2) = \tilde{k}_\nu(r_2r_1)$ *for all* $r_1, r_2 \in \mathcal{S}$.

Proof. Let $\tilde{k}_\nu$ be a hybrid ideal in $\mathcal{S}$. Then by Theorem 3.7, we get $\tilde{k}_\nu(r_1r_2) = \tilde{k}_\nu((r_1r_2)^2)$ and $\tilde{k}_\nu(r_2r_1) = \tilde{k}_\nu((r_2r_1)^2)$. Now

$$\tilde{k}(r_1r_2) = \tilde{k}(r_1r_2r_1r_2) \supseteq \tilde{k}(r_1r_2r_1) \supseteq \tilde{k}(r_2r_1)$$

and $\tilde{k}(r_2r_1) = \tilde{k}(r_2r_1r_2r_1) \supseteq \tilde{k}(r_2r_1r_2) \supseteq \tilde{k}(r_1r_2)$. So, $\tilde{k}(r_1r_2) = \tilde{k}(r_2r_1)$. Also,

$$\nu(r_1r_2) = \nu((r_1r_2)^2) = \nu(r_1r_2r_1r_2) \leq \nu(r_2r_1)$$

and $\nu(r_2r_1) = \nu((r_2r_1)^2) = \nu(r_2r_1r_2r_1) \leq \nu(r_1r_2r_1) \leq \nu(r_1r_2)$, so $\nu(r_1r_2) = \nu(r_2r_1)$. Therefore, $\tilde{k}_\nu(ab) = \tilde{k}_\nu(r_2r_1)$. □

A semigroup $\mathcal{S}$ is called completely regular if for each $r \in \mathcal{S}$, there exists $t \in \mathcal{S}$ such that $r = rtr$ and $rt = tr$. $\mathcal{S}$ is called left (resp. right) regular if for each $r \in \mathcal{S}$ there exists $t \in \mathcal{S}$ such that $r = tr^2$ (resp. $r = r^2t$) [7].

Theorem 3.9. *For any* $\mathcal{S}$, *the following statements are equivalent:*
(i) *for every hybrid left ideal* $\tilde{k}_\mu$ (*resp. right ideal*) *of* $\mathcal{S}$, *we have* $\tilde{k}_\mu(r) = \tilde{k}_\mu(r^2)$ *for all* $r \in \mathcal{S}$*;*
(ii) $\mathcal{S}$ *is left regular* (*resp. right regular*)*.*

Proof. (i) $\Rightarrow$ (ii) Let $\tilde{k}_\mu$ be a hybrid left ideal in $\mathcal{S}$. Then for any $r \in \mathcal{S}$, we have $\chi_{L(r^2)}(\tilde{k}_\mu)(r) = \chi_{L(r^2)}(\tilde{k}_\mu)(r^2)$ by Theorem 3.5. Since $r^2 \in L(r^2)$, we have $\chi_{L(r^2)}(\tilde{k})(r^2) = U$ implies that $\chi_{L(r^2)}(\tilde{k}_\mu)(r) = U$. Thus, $r \in L(r^2)$ and hence $\mathcal{S}$ is left-regular.

$(ii) \Rightarrow (i)$ Let $s \in \mathcal{S}$. Then there exists $x \in \mathcal{S}$ such that $s = xs^2$. Let $\tilde{k}_\mu$ be a hybrid left ideal $\mathcal{S}$. Then $\tilde{k}(s) = \tilde{k}(xs^2) \supseteq \tilde{k}(s^2) \supseteq \tilde{k}(s)$ and $\mu(s) = \mu(xs^2) \leq \mu(s^2) \leq \mu(s)$. So, $\tilde{k}_\mu(s) = \tilde{k}_\mu(s^2)$. □

Corollary 3.2. *If $\mathcal{S}$ is regular left duo (resp. right duo), then the below assertions are equivalent:*

(i) if $\tilde{k}_\mu$ is a hybrid bi-ideal of $\mathcal{S}$, then $\tilde{k}_\mu(r) = \tilde{k}_\mu(r^2)$ for all $r \in \mathcal{S}$;

(ii) $\mathcal{S}$ is right regular.

Proof. It is evident from Theorem 3.6 and Theorem 3.9. □

Theorem 3.10. ([2, Theorem 2.9]). *Let $\phi \neq X \subseteq \mathcal{S}$. Then X is bi-ideal if and only if $\chi_X(\tilde{i}_\lambda)$ is hybrid bi-ideal.*

The equivalent conditions for complete regularity of a semigroup are given below.

Theorem 3.11. *For any $\mathcal{S}$, the following assertions are equivalent:*

(i) for each hybrid bi-ideal $\tilde{j}_\mu$ in $\mathcal{S}$, we have $\tilde{j}_\mu(r) = \tilde{j}_\mu(r^2)$ for all $r \in \mathcal{S}$;

(ii) $\mathcal{S}$ is completely regular;

(iii) for every hybrid left ideal $\tilde{j}_\mu$ and hybrid right ideal $\tilde{k}_\lambda$ of $\mathcal{S}$, we have $\tilde{j}_\mu(r) = \tilde{j}_\mu(r^2)$ and $\tilde{k}_\lambda(r) = \tilde{k}_\lambda(r^2)$ for all $r \in \mathcal{S}$.

Proof. $(i) \Rightarrow (ii)$ Let $\tilde{j}_\mu$ be a hybrid bi-ideal of $\mathcal{S}$. Then for any $r \in \mathcal{S}$, by Theorem 3.10 we have $\chi_{B(r^2)}(\tilde{j}_\mu)(r) = \chi_{B(r^2)}(\tilde{j}_\mu)(r^2)$. Since $r^2 \in B(r^2)$, we have $\chi_{B(r^2)}(\tilde{j})(r^2) = U$ implies that $\chi_{B(r^2)}(\tilde{j}_\mu)(r) = U$. Thus, $r \in B(r^2)$ and hence $\mathcal{S}$ is completely regular.

$(ii) \Rightarrow (i)$ Let $r \in \mathcal{S}$. Then there exists $q \in \mathcal{S}$ such that $r = r^2qr^2$. Let $\tilde{j}_\mu$ be a hybrid bi-ideal in $\mathcal{S}$. Then $\tilde{j}(r) = \tilde{j}(r^2qr^2) = \tilde{j}(r(rqr)r) \supseteq \tilde{j}(r) \cap \tilde{j}(r) = \tilde{j}(r)$ and $\mu(r) = \mu(r^2qr^2) = \mu(r(rqr)r) \leq \mu(r) \vee \mu(r) = \mu(r)$. Therefore, $\tilde{j}_\mu(r) = \tilde{j}_\mu(r^2)$.

$(ii) \Leftrightarrow (iii)$ It is evident from Theorem 3.9. □

A semigroup $\mathcal{S}$ is said to be right (resp. left) simple if $\mathcal{S}$ has no proper right (resp. left) ideals of $\mathcal{S}$. A semigroup $\mathcal{S}$ is said to be simple if $\mathcal{S}$ has no proper ideals.

A hybrid structure $\tilde{j}_\nu$ in $\mathcal{S}$ is called a constant function if $\tilde{j} : \mathcal{S} \to \mathscr{P}(U)$ and $\nu : \mathcal{S} \to J$ are constant mappings.

A semigroup $\mathcal{S}$ is called hybrid right (resp. left) simple if every hybrid right (resp. left) ideal of $\mathcal{S}$ over U is a constant function, and hybrid simple if every hybrid ideal of $\mathcal{S}$ is a constant function.

Theorem 3.12. *For any $\mathcal{S}$, the following assertions are equivalent:*

(i) $\mathcal{S}$ is hybrid left simple (resp. hybrid right simple, hybrid simple);

(ii) $\mathcal{S}$ is left simple (resp. right simple, simple).

Proof. $(i) \Rightarrow (ii)$ Suppose $\mathcal{S}$ is hybrid left simple and let D be a left ideal of $\mathcal{S}$. Then $\chi_D(\tilde{k}_\mu)$ is hybrid left ideal. Since $D \neq \phi$, the constant value is U. So, every element of $\mathcal{S}$ is in D. Thus, $\mathcal{S} = D$ and hence $\mathcal{S}$ is left simple.

$(ii) \Rightarrow (i)$ Suppose $\mathcal{S}$ is hybrid left simple. Then for any $w_1, w_2 \in \mathcal{S}$, we have $\mathcal{S}w_1 = \mathcal{S} = \mathcal{S}w_2$ implies $w_2 = xw_1$ and $w_1 = yw_2$ for some $x, y \in \mathcal{S}$. Let $\tilde{k}_\mu$ be a hybrid left ideal of $\mathcal{S}$. Then $\tilde{k}(w_1) = \tilde{k}(yw_2) \supseteq \tilde{k}(w_2) = \tilde{k}(xw_1) \supseteq \tilde{k}(w_1)$ implies $\tilde{k}(w_1) = \tilde{k}(w_2)$. Also, $\mu(w_1) = \mu(yw_2) \leq \mu(w_2) = \mu(xw_1) \leq \mu(w_1)$ implies $\mu(w_1) = \mu(w_2)$. Thus, $\tilde{k}_\mu$ is a constant function and hence $\mathcal{S}$ is hybrid left simple. □

Theorem 3.13. *Let $\mathcal{S}$ be a hybrid left (resp. right) simple semigroup. Then every hybrid bi-ideal is hybrid right ideal (resp. hybrid left ideal).*

Proof. Let $\tilde{k}_\mu$ be any hybrid bi-ideal and $u, v \in \mathcal{S}$. Then $\mathcal{S}u = \mathcal{S}$ and there exists $k \in \mathcal{S}$ such that $v = ku$, so $uv = uku$. Now $\tilde{k}(uv) = \tilde{k}(uku) \supseteq \tilde{k}(u) \cap \tilde{k}(u) = \tilde{k}(u)$ and $\mu(uv) = \mu(uku) \leq \mu(u)$. Therefore, $\tilde{k}_\mu$ is hybrid right ideal. □

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