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NONLINEAR SEQUENTIAL CAPUTO AND CAPUTO-HADAMARD FRACTIONAL DIFFERENTIAL EQUATIONS WITH DIRICHLET BOUNDARY CONDITIONS IN BANACH SPACES

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ABSTRACT. This paper is devoted to the existence of solutions for certain classes of nonlinear sequential Caputo and Caputo-Hadamard fractional differential equations with Dirichlet boundary conditions in Banach spaces. Moreover, our analysis is based on Darbo's fixed point theorem in conjunction with the technique of Hausdorff measure of noncompactness. An example is also presented to illustrate the effectiveness of the main results.

1. INTRODUCTION

Fractional calculus and fractional differential equations describe various phenomena in diverse areas of natural science such as physics, aerodynamics, biology, control theory, and chemistry; see for instance [26,29,31,39–41,43]. On the other hand, there are several definitions of fractional integrals and derivatives in the literature, but the most popular definitions are in the sense of the Riemann-Liouville and Caputo. However, there is another kind of fractional derivatives that appears in the literature due to Hadamard [23], which is known as Hadamard derivative and differs from the preceding ones in the sense that its definition involves logarithmic function of arbitrary exponent. Another significant aspect of Hadamard derivative is that its expression can be viewed as a generalization operator $\left(t\frac{\mathrm{d}}{\mathrm{dt}}\right)^n$ [6,23], whilst the Riemann-Liouville derivative is regarded as an extension of the classical differential operator $\left(\frac{\mathrm{d}}{\mathrm{dt}}\right)^n$. For some developments on the existence results of the Hadamard fractional differential equations, we

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can refer to [6,16,18,28,44]. In recent times, another derivative was proposed by modifying the Hadamard derivative with the Caputo one, known as Caputo-Hadamard derivative [24]. It is obtained from the Hadamard derivative by changing the order of its differentiation and integration. In addition, the main difference between the Caputo-Hadamard fractional derivative and the Hadamard fractional derivative is that the Caputo-Hadamard derivative of a constant is zero; another aspect is that the Cauchy problems for Caputo-Hadamard fractional differential equations contain initial conditions which can be physically interpretable, similarly to the case with Caputo fractional derivatives. From these points of view, it is imperative to study Caputo-Hadamard fractional calculus. To the best of our knowledge, few results can be found in the literature concerning boundary value problems for Caputo-Hadamard fractional differential equations [7,8,17,27]. Moreover, it has been noticed that most of the above-mentioned work on the topic is based on the technique of nonlinear analysis such as Banach fixed point theorem, Schauder's fixed point theorem and Leray-Schauder nonlinear alternative, etc. But if compactness and Lipschitz condition are not satisfied these results cannot be used. Measure of noncompactness comes handy in such situations. For instance, the celebrated Darbo fixed point theorem and Mönch fixed point theorem are used by several authors with the end goal to establish existence results for nonlinear integral equations (see [1,2,4,13,15,21,45] and references therein).

In 2018, Tariboon et al. in [44], discussed the existence and uniqueness of solutions for two sequential Caputo-Hadamard and Hadamard-Caputo fractional differential equations subject to separated boundary conditions as

$$\begin{cases} {}^{C}\mathfrak{D}^{p}\left[{}^{H}\mathfrak{D}^{q}u(t)\right] = f(t,u(t)), & t\in(a,b),\\ a_1u(a)+b_1{}^{H}\mathfrak{D}^{q}u(a)=0, & \\ a_2u(b)+b_2{}^{H}\mathfrak{D}^{q}u(b)=0, & \end{cases}$$

and

$$\begin{cases} {}^{H}\mathfrak{D}^{q}\left[{}^{C}\mathfrak{D}^{p}u(t)\right] = f(t,u(t)), & t\in(a,b),\\ a_1u(a)+b_1{}^{C}\mathfrak{D}^{p}u(a)=0, & \\ a_2u(b)+b_2{}^{C}\mathfrak{D}^{p}u(b)=0, & \end{cases}$$

where ${}^{C}\mathfrak{D}^{p}$ and ${}^{H}\mathfrak{D}^{q}$ are the Caputo and Hadamard fractional derivatives of orders p and q, respectively, $0<p,q\leq 1$, $f:[a,b]\times\mathbb{R}\to\mathbb{R}$ is a continuous function, $a>0$ and $a_i,b_i\in\mathbb{R}$, $i=1,2$.

Very recently, in [30], the authors considered the infinite system of second-order differential equations of the type

$$\begin{cases} t\frac{\mathrm{d}^2v_j}{\mathrm{dt}^2}+\frac{\mathrm{d}v_j}{\mathrm{dt}} = f_i(t,v(t)), & t\in J:=[1,T],\\ v_j(1)=v_j(T)=0, & \end{cases} \tag{1.1}$$

where $v(t)=\{v_j(t)\}_{j=1}^{\infty}$, in Banach sequence space ℓ^p, $p\geq 1$. The authors obtained the existence of solutions by using the Hausdorff measure of noncompactness and Darbo type fixed point theorem. Additionally, for more interesting details about infinite systems of differential equations or integral equations in some Banach sequence spaces we suggest some works [5, 32–38]. Moreover, the reader is advised to see the recent book [12] where several applications of the measure of noncompactness can be found.

No contributions exist, as far as we know, concerning nonlinear sequential Caputo and Caputo-Hadamard fractional differential equations in Banach spaces. As a result, the goal of this paper is to enrich this academic area. So, in this paper, we mainly study the following boundary value problem of the form.

$$\begin{cases} {}^{C}\mathfrak{D}^{q}\left[{}^{C}_{H}\mathfrak{D}^{p}u(t)\right] = f(t, u(t)), \quad 0 < p, q \leq 1,\, t \in J := [a, b], \\ u(a) = u(b) = \theta, \end{cases} \tag{1.2}$$

where ${}^{C}_{H}\mathfrak{D}^{p}u(t)$ and ${}^{C}\mathfrak{D}^{q}$ are the Caputo Hadamard and Caputo fractional derivatives of orders p and q, respectively, $0 < p, q \leq 1$, $f : [a, b] \times E \to E$ is a given function satisfying some assumptions that will be specified later, E is a Banach space with norm $\|\cdot\|$, and θ refers to the null vector in the space E.

The main motivation for the elaboration of this paper comes from the above highlighted articles on the existence of solutions of fractional differential equations. In addition, as in the Banach space (in general in any infinite-dimensional linear space) a closed and bounded set is not necessarily compact set, mere continuity of the function f does not guarantee the existence of a solution of differential equations. The arguments are based on Darbo's fixed point theorem combined with the technique of measures of noncompactness to establish the existence of solution for (1.2). Obviously, BVP (1.2) is more general than the problems discussed in some recent literature (such as [30, 44]). Firstly, our results are not only new in the given configuration but also correspond to some new situations associated with the specific values of the parameters involved in the given problem. For example, if we take $a = p = q = 1$, $b = T$ and $E = \ell^p$, then the BVP (1.2) corresponds to the infinite system represented in (1.1). Secondly, the required conditions to prove the existence of solutions for the system (1.1) depend strongly on the chosen Banach space of sequences. This is because the formula for the Hausdorff MNC is, of course, different from one space to another. However, our conditions do not depend on the chosen Banach space.

Here is a brief outline of the paper. The next section provides the definitions and preliminary results that we will need to prove our main results. Then, we present the existence results in Section 3. In Section 4, we give an example to illustrate the obtained results. The last section concludes this paper.

2. Preliminaries

We start this section by introducing some necessary definitions and basic results required for further developments.

Let $C(J,E)$ be the Banach space of all continuous functions u from J into E with the supremum (uniform) norm

$$\|u\|_\infty = \sup_{t\in J}\|u(t)\|.$$

By $L^1(J)$ we denote the space of Bochner-integrable functions $u : J \to E$, with the norm

$$\|u\|_1 = \int_a^b \|u(t)\|\mathrm{dt}.$$

Next, we define the Hausdorff measure of noncompactness and give some of its important properties.

Definition 2.1 ([11]). Let E be a Banach space and B a bounded subsets of E . Then Hausdorff measure of non-compactness of B is defined by

$$\chi(B) = \inf\Big\{\varepsilon > 0 : B \text{ has a finite cover by closed balls of radius } \varepsilon\Big\}.$$

To discuss the problem in this paper, we need the following lemmas.

Lemma 2.1. *Let $A, B \subset E$ be bounded. Then Hausdorff measure of non-compactness has the following properties:*

(1) $A \subset B \Rightarrow \chi(A) \le \chi(B)$;
(2) $\chi(A) = 0 \Leftrightarrow A$ *is relatively compact;*
(3) $\chi(A\cup B) = \max\{\chi(A), \chi(B)\}$;
(4) $\chi(A) = \chi(\overline{A}) = \chi(\mathrm{conv}(A))$, *where* $\overline{A}$ *and* $convA$ *represent the closure and the convex hull of* A, *respectively;*
(5) $\chi(A+B) \le \chi(A) + \chi(B)$, *where* $A+B = \{x+y : x\in A, y\in B\}$;
(6) $\chi(\lambda A) \le |\lambda|\chi(A)$ *for any* $\lambda\in\mathbb{R}$,

For more details and the proof of these properties see [11].

Lemma 2.2 ([11]). *If $W \subseteq C(J,E)$ is bounded and equicontinuous, then $\chi(W(t))$ is continuous on J and*

$$\chi(W) = \sup_{t\in J}\chi(W(t)).$$

We call $B \subset L^1(J,E)$ uniformly integrable if there exists $\eta \in L^1(J,\mathbb{R}^+)$ such that

$$\|u(s)\| \le \eta(s), \quad \text{for all } u\in B \text{ and a.e. } s\in J.$$

Lemma 2.3 ([25]). *If $\{u_n\}_{n=1}^\infty \subset L^1(J,E)$ is uniformly integrable, then $\chi(\{u_n\}_{n=1}^\infty)$ is measurable, and*

$$\chi\left(\left\{\int_0^t u_n(s)\mathrm{ds}\right\}_{n=1}^\infty\right) \le 2\int_0^t \chi(\{u_n(s)\}_{n=1}^\infty)\mathrm{ds}.$$

Lemma 2.4 ([19]). *If W is bounded, then for each ε, there is a sequence $\{u_n\}_{n=1}^\infty \subset W$, such that*

$$\chi(W) \le 2\chi(\{u_n\}_{n=1}^\infty) + \varepsilon.$$

Definition 2.2 ([46])**.** A function $f : [a, b] \times E \to E$ is said to satisfy the Carathéodory conditions, if the following hold

- $f(t, u)$ is measurable with respect to t for $u \in E$;
- $f(t, u)$ is continuous with respect to $u \in E$ for $t \in J$.

Definition 2.3 ([10])**.** The mapping $\mathfrak{T} : \Omega \subset E \to E$ is said to be a χ-contraction, if there exists a positive constant $k < 1$ such that

$$\chi(\mathfrak{T}(W)) \leq k\chi(W),$$

for every bounded subset W of Ω.

A useful fixed point result for our goals is the following, proved in [11, 20].

Theorem 2.1 (Darbo and Sadovskii)**.** *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $\mathfrak{T} : \Omega \to \Omega$ be a continuous operator. If $\mathfrak{T}$ is a χ-contraction, then $\mathfrak{T}$ has at least one fixed point.*

Let us recall some preliminary concepts of fractional calculus related to our work.

Definition 2.4 ([29])**.** The Riemann-Liouville fractional integral of order $p > 0$ of a function $u \in L^1([a, b])$ is defined by

$$^{RL}\mathfrak{J}^p u(t) = \frac{1}{\Gamma(p)} \int_a^t (t-s)^{p-1} u(s)\mathrm{ds}, \quad t > a,\, p > 0,$$

where $\Gamma(\cdot)$ is the (Euler's) Gamma function $\Gamma(p) = \int_0^{+\infty} e^{-t} t^{p-1} \mathrm{dt}$, $p > 0$. Moreover, for $p = 0$, we set $^{RL}\mathfrak{J}^p u := u$.

Lemma 2.5 ([29])**.** *The following basic properties of the Riemann-Liouville integrals hold.*

(a) *The integral operator $^{RL}\mathfrak{J}^p$ is linear.*

(b) *The semigroup property of the fractional integration operator $^{RL}\mathfrak{J}^p$ is given by the following result*

$$^{RL}\mathfrak{J}^p(^{RL}\mathfrak{J}^q u(t)) = {}^{RL}\mathfrak{J}^{p+q} u(t), \quad p, q > 0,$$

holds at every point if $u \in C([a, b])$ and holds almost everywhere if $u \in L^1([a, b])$.

(c) *Commutativity*

$$^{RL}\mathfrak{J}^p(^{RL}\mathfrak{J}^q u(t)) = {}^{RL}\mathfrak{J}^q(^{RL}\mathfrak{J}^p u(t)), \quad p, q > 0.$$

(d) *The fractional integration operator $^{RL}\mathfrak{J}^p$ is bounded in $L^p[a, b]$, $1 \leq p \leq \infty$,*

$$\|^{RL}\mathfrak{J}^p u\|_{L^p} \leq \frac{1}{\Gamma(p+1)} \|u\|_{L^p}.$$

Example 2.1. The Riemann-Liouville fractional integral of the power function $(t - a)^q$, $p > 0$, $q > -1$

$$^{RL}\mathfrak{J}^p (t-a)^q = \frac{\Gamma(q+1)}{\Gamma(p+q+1)} (t-a)^{p+q}.$$

Definition 2.5 ([29,40])**.** The Caputo fractional derivative ${}^{C}\mathfrak{D}^{p}$ of order p of a function $u \in AC^n([a,b])$ is represented by

$$ {}^{C}\mathfrak{D}^{p}u(t) = \begin{cases} \frac{1}{\Gamma(n-p)} \int_a^t (t-s)^{n-p-1} u^{(n)}(s)\mathrm{ds}, & \text{if } p \notin \mathbb{N}, \\ u^{(n)}(t), & \text{if } p \in \mathbb{N}, \end{cases} $$

where $u^{(n)}(t) = \frac{\mathrm{d}^n u(t)}{\mathrm{d}t^n}$, $p > 0$, $n = [p] + 1$ and $[p]$ denotes the integer part of the real number p.

Example 2.2. The Caputo fractional derivative of order $n-1 < p < n$ for $(t-a)^q$ is given by

$$ {}^{C}\mathfrak{D}^{p}(t-a)^q = \begin{cases} \frac{\Gamma(q+1)}{\Gamma(q-p+1)}(t-a)^{q-p}, & q \in \mathbb{N} \text{ and } q \geq n \text{ or } q \notin \mathbb{N} \text{ and } q > n-1, \\ 0, & q \in \{0, \dots, n-1\}. \end{cases} \tag{2.1} $$

Lemma 2.6 ([29,40])**.** *Let $p > 0$ and $n = [p] + 1$, then the differential equation*

$$ {}^{C}\mathfrak{D}^{p}u(t) = 0 $$

has solutions

$$ u(t) = \sum_{j=0}^{n-1} c_j (t-a)^j, \quad c_j \in \mathbb{R},\ j = 0, \dots, n-1. $$

Lemma 2.7 ([29,40])**.** *Let $p > q > 0$ and $u \in L^1([a,b])$. Then we have:*

(1) *the Caputo fractional derivative is linear;*
(2) *the Caputo fractional derivative obeys the following property:*

$$ {}^{RL}\mathfrak{J}^{p}\,{}^{C}\mathfrak{D}^{p}u(t) = u(t) + \sum_{j=0}^{n-1} c_j (t-a)^j, $$

for some $c_j \in \mathbb{R}$, $j = 0, 1, 2, \dots, n-1$, where $n = [p] + 1$;
(3) ${}^{C}\mathfrak{D}^{p}\,{}^{RL}\mathfrak{J}^{p}u(t) = u(t)$*;*
(4) ${}^{C}\mathfrak{D}^{q}\,{}^{RL}\mathfrak{J}^{p}u(t) = {}^{RL}\mathfrak{J}^{p-q}u(t)$*.*

Definition 2.6 ([29])**.** The Hadamard fractional integral of order $p > 0$ for a function $u \in L^1(J)$ is defined as

$$ \left({}^{H}\mathfrak{J}^{p}u\right)(t) = \frac{1}{\Gamma(p)} \int_a^t \left(\log \frac{t}{s}\right)^{p-1} u(s) \frac{\mathrm{ds}}{s}, \quad p > 0. $$

Set

$$ \delta = t\frac{\mathrm{d}}{\mathrm{dt}}, \quad p > 0,\ n = [p] + 1, $$

where $[p]$ denotes the integer part of p. Define the space

$$ AC_{\delta}^{n}[a,b] := \left\{ u : [a,b] \to \mathbb{R} : \delta^{n-1}u(t) \in AC([a,b]) \right\}. $$

Definition 2.7 ([29]). The Hadamard fractional derivative of order $p > 0$ applied to the function $u \in AC^n_\delta[a, b]$ is defined as

$$\left({}^H\mathfrak{D}^p u\right)(t) = \delta^n \left({}^H\mathfrak{J}^{n-p} u\right)(t).$$

Definition 2.8 ([24, 29]). The Caputo-Hadamard fractional derivative of order $p > 0$ applied to the function $u \in AC^n_\delta[a, b]$ is defined as

$$\left({}^C_H\mathfrak{D}^p u\right)(t) = \left({}^H\mathfrak{J}^{n-p} \delta^n u\right)(t).$$

Lemmas of the following type are rather standard in the study of fractional differential equations.

Lemma 2.8 ([24, 29]). *Let $p > 0$, $r > 0$, $n = [p] + 1$ and $a > 0$, then the following relations hold :*

- $\left({}^H\mathfrak{J}^p \left(\log \frac{s}{a}\right)^{r-1}\right)(t) = \frac{\Gamma(r)}{\Gamma(p+r)} \left(\log \frac{t}{a}\right)^{p+r-1}$;
- $$\left({}^C_H\mathfrak{D}^p \left(\log \frac{s}{a}\right)^{r-1}\right)(t) = \begin{cases} \frac{\Gamma(r)}{\Gamma(r-p)} \left(\log \frac{t}{a}\right)^{r-p-1}, & r > n, \\ 0, & r \in \{0, \dots, n-1\}. \end{cases}$$

Lemma 2.9 ([22, 29]). *Let $p > q > 0$ and $u \in AC^n_\delta[a, b]$. Then we have:*

- ${}^H\mathfrak{J}^p\, {}^H\mathfrak{J}^q u(t) = {}^H\mathfrak{J}^{p+q} u(t)$;
- ${}^C_H\mathfrak{D}^p\, {}^H\mathfrak{J}^p u(t) = u(t)$;
- ${}^C_H\mathfrak{D}^q\, {}^H\mathfrak{J}^p u(t) = {}^H\mathfrak{J}^{p-q} u(t)$.

Lemma 2.10 ([24, 29]). *Let $p \geq 0$ and $n = [p] + 1$. If $u \in AC^n_\delta[a, b]$, then the Caputo-Hadamard fractional differential equation*

$$\left({}^C_H\mathfrak{D}^p u\right)(t) = 0,$$

has a solution

$$u(t) = \sum_{j=0}^{n-1} c_j \left(\log \frac{t}{a}\right)^j,$$

and the following formula holds:

$${}^H\mathfrak{J}^p \left({}^C_H\mathfrak{D}^p u(t)\right) = u(t) + \sum_{j=0}^{n-1} c_j \left(\log \frac{t}{a}\right)^j,$$

where $c_j \in \mathbb{R}$, $j = 0, 1, 2, \dots, n-1$.

Remark 2.1. Note that for an abstract function $u : J \to E$, the integrals which appear in the previous definitions are taken in Bochner's sense (see, for instance, [42]).

3. Main Results

Let us recall the definition and lemma of a solution for problem (1.2).

First of all, we define what we mean by a solution for the boundary value problem (1.2).

Definition 3.1. A function $u \in C(J, E)$ is said to be a solution of (1.2) if u satisfies the equation ${}^{C}\mathfrak{D}^{q}\left[{}^{C}_{H}\mathfrak{D}^{p}u(t)\right] = f(t, u(t))$ a.e. on J and the condition $u(a) = u(b) = \theta$.

For the existence of solutions for the problem (1.2) we need the following lemma.

Lemma 3.1. *For a given $h \in C(J, \mathbb{R})$, the unique solution of the linear fractional boundary value problem*

$$\begin{cases} {}^{C}\mathfrak{D}^{q}\left[{}^{C}_{H}\mathfrak{D}^{p}u(t)\right] = h(t), \quad 0 < p, q \leq 1,\ t \in J := [a, b], \\ u(a) = u(b) = 0, \end{cases} \tag{3.1}$$

is given by

$$\begin{aligned} u(t) =& {}^{H}\mathfrak{J}^{p}\left({}^{RL}\mathfrak{J}^{q}h\right)(t) - \left(\frac{\log(t/a)}{\log(b/a)}\right)^{p} {}^{H}\mathfrak{J}^{p}\left({}^{RL}\mathfrak{J}^{q}h\right)(b) \\ =& \frac{1}{\Gamma(p)\Gamma(q)}\int_{a}^{t}\int_{a}^{s}\left(\log\frac{t}{s}\right)^{p-1}(s-\tau)^{q-1}h(\tau)\mathrm{d}\tau\frac{\mathrm{ds}}{\mathrm{s}} \\ & - \left(\frac{\log(t/a)}{\log(b/a)}\right)^{p}\frac{1}{\Gamma(p)\Gamma(q)}\int_{a}^{b}\int_{a}^{s}\left(\log\frac{b}{s}\right)^{p-1}(s-\tau)^{q-1}h(\tau)\mathrm{d}\tau\frac{\mathrm{ds}}{\mathrm{s}}. \end{aligned} \tag{3.2}$$

Proof. Taking the Riemann-Liouville fractional integral of order q to the first equation of (3.1), we get

$${}^{C}_{H}\mathfrak{D}^{p}u(t) = {}^{RL}\mathfrak{J}^{q}h(t) + k_0, \quad k_0 \in \mathbb{R}. \tag{3.3}$$

Again taking the Hadamard fractional integral of order p to the above equation, we obtain

$$u(t) = {}^{H}\mathfrak{J}^{p}\left({}^{RL}\mathfrak{J}^{q}h\right)(t) + k_0\frac{(\log(t/a))^{p}}{\Gamma(p+1)} + k_1, \quad k_0 \in \mathbb{R}. \tag{3.4}$$

Substituting $t = a$ in (3.4) and applying the first boundary condition of (3.1), it follows that $k_1 = 0$. For $t = b$ in (3.4) and using the second boundary condition of (3.1), it yields

$$u(b) = 0 = {}^{H}\mathfrak{J}^{p}\left({}^{RL}\mathfrak{J}^{q}h\right)(b) + k_0\frac{(\log(b/a))^{p}}{\Gamma(p+1)}. \tag{3.5}$$

By solving (3.5), we find that

$$k_0 = -\frac{\Gamma(p+1)}{(\log(b/a))^{p}}{}^{H}\mathfrak{J}^{p}\left({}^{RL}\mathfrak{J}^{q}h\right)(b). \tag{3.6}$$

Substituting the values of k_0 and k_1 into (3.4), we get the integral equation (3.2). The converse follows by the direct computation which completes the proof. $\square$

Now, we shall present our main result concerning the existence of solutions of problem (1.2). Let us introduce the following hypotheses.
(H1) The function $f : [a, b] \times E \longrightarrow E$ satisfies Carathéodory conditions.
(H2) There exists function $\psi \in L^\infty(J, \mathbb{R}_+)$ such that

$$\|f(t, u(t))\| \le \psi(t)(1 + \|u\|), \quad \text{for all } u \in C(J, E).$$

(H3) For each bounded set $W \subset E$ and each $t \in J$, the following inequality holds

$$\chi(f(t, W)) \le \psi(t)\chi(W).$$

For computational convenience we put

$$\mathcal{M}_\psi = \frac{2\|\psi\| \, (b-a)^q \left(\log \frac{b}{a}\right)^p}{\Gamma(p+1)\Gamma(q+1)}. \tag{3.7}$$

Now, we shall prove the following theorem concerning the existence of solutions of problem (1.2)

Theorem 3.1. *Assume that the hypotheses* (H1)-(H3) *are satisfied. If*

$$4\mathcal{M}_\psi < 1, \tag{3.8}$$

then the problem (1.2) *has at least one solution defined on* J.

Proof. Consider the operator $\mathcal{N} : C(J, E) \to C(J, E)$ defined by:

$$\begin{aligned} \mathcal{N}u(t) =& \frac{1}{\Gamma(p)\Gamma(q)} \int_a^t \int_a^s \left(\log \frac{t}{s}\right)^{p-1} (s-\tau)^{q-1} f(\tau, u(\tau)) \mathrm{d}\tau \frac{\mathrm{ds}}{\mathrm{s}} \\ & - \left(\frac{\log(t/a)}{\log(b/a)}\right)^p \frac{1}{\Gamma(p)\Gamma(q)} \int_a^b \int_a^s \left(\log \frac{b}{s}\right)^{p-1} (s-\tau)^{q-1} f(\tau, u(\tau)) \mathrm{d}\tau \frac{\mathrm{ds}}{\mathrm{s}}. \end{aligned} \tag{3.9}$$

It is obvious that $\mathcal{N}$ is well defined due to (H1) and (H2). Then, fractional integral equation (3.2) can be written as the following operator equation

$$u = \mathcal{N}u. \tag{3.10}$$

Thus, the existence of a solution for (1.2) is equivalent to the existence of a fixed point for operator $\mathcal{N}$ which satisfies operator equation (3.10). Define a bounded closed convex set

$$B_R = \{w \in C(J, E) : \|w\|_\infty \le R\},$$

with $R > 0$, such that

$$R \ge \frac{\mathcal{M}_\psi}{1 - \mathcal{M}_\psi}.$$

In order to satisfy the hypotheses of the Darbo fixed point theorem, we split the proof into four steps.

Step 1. The operator $\mathcal{N}$ maps the set B_R into itself. By the assumption (H2), we have

$$\|\mathcal{N}u(t)\| \le \frac{1}{\Gamma(p)\Gamma(q)} \int_a^t \int_a^s \left(\log \frac{t}{s}\right)^{p-1} (s-\tau)^{q-1} \|f(\tau, u(\tau))\| \mathrm{d}\tau \frac{\mathrm{ds}}{\mathrm{s}}$$

$$
\begin{aligned}
&+\left(\frac{\log(t/a)}{\log(b/a)}\right)^{p}\frac{1}{\Gamma(p)\Gamma(q)}\int_a^b\int_a^s\left(\log\frac{b}{s}\right)^{p-1}(s-\tau)^{q-1}\|f(\tau,u(\tau))\|\mathrm{d}\tau\frac{\mathrm{ds}}{\mathrm{s}}\\
\leq&\frac{1}{\Gamma(p)\Gamma(q)}\int_a^t\int_a^s\left(\log\frac{t}{s}\right)^{p-1}(s-\tau)^{q-1}\psi(\tau)(1+\|u(\tau)\|)\mathrm{d}\tau\frac{\mathrm{ds}}{\mathrm{s}}\\
&+\left(\frac{\log(t/a)}{\log(b/a)}\right)^{p}\frac{1}{\Gamma(p)\Gamma(q)}\int_a^b\int_a^s\left(\log\frac{b}{s}\right)^{p-1}(s-\tau)^{q-1}\psi(\tau)(1+\|u(\tau)\|)\mathrm{d}\tau\frac{\mathrm{ds}}{\mathrm{s}}\\
\leq&\frac{\|\psi\|(1+\|u\|)}{\Gamma(p)\Gamma(q)}\int_a^t\int_a^s\left(\log\frac{t}{s}\right)^{p-1}(s-\tau)^{q-1}\mathrm{d}\tau\frac{\mathrm{ds}}{\mathrm{s}}\\
&+\frac{\|\psi\|(1+\|u\|)}{\Gamma(p)\Gamma(q)}\int_a^b\int_a^s\left(\log\frac{b}{s}\right)^{p-1}(s-\tau)^{q-1}\mathrm{d}\tau\frac{\mathrm{ds}}{\mathrm{s}}.
\end{aligned}
$$

Also, note that

$$
\frac{1}{\Gamma(p)\Gamma(q)}\int_a^t\int_a^s\left(\log\frac{t}{s}\right)^{p-1}(s-\tau)^{q-1}\mathrm{d}\tau\frac{\mathrm{ds}}{\mathrm{s}}\leq\frac{(b-a)^q\left(\log\frac{b}{a}\right)^p}{\Gamma(p+1)\Gamma(q+1)},
$$

where we have used the fact that $(s-a)^q\leq(b-a)^q$ for $0<q\leq 1$. Using the above arguments, we have

$$
\|\mathcal{N}u(t)\|\leq\|\psi\|(1+\|u\|)\frac{2\,(b-a)^q\left(\log\frac{b}{a}\right)^p}{\Gamma(p+1)\Gamma(q+1)}\leq(1+R)\mathcal{M}_\psi\leq R.
$$

Thus, $\|\mathcal{N}u\|\leq R$. This proves that $\mathcal{N}$ transforms the ball B_R into itself.

Step 2. The operator $\mathcal{N}$ is continuous. Suppose that $\{u_n\}$ is a sequence such that $u_n\to u$ in B_R as $n\to\infty$. It is easy to see that $f(s,u_n(s))\to f(s,u(s))$ as $n\to+\infty$, due to the Carathéodory continuity of f. On the other hand taking (H2) into consideration we get

$$
\begin{aligned}
&\|\mathcal{N}u_n(t)-\mathcal{N}u(t)\|\\
\leq&\frac{1}{\Gamma(p)\Gamma(q)}\int_a^t\int_a^s\left(\log\frac{t}{s}\right)^{p-1}(s-\tau)^{q-1}\|f(\tau,u_n(\tau))-f(\tau,u(\tau))\|\mathrm{d}\tau\frac{\mathrm{ds}}{\mathrm{s}}\\
&+\left(\frac{\log(t/a)}{\log(b/a)}\right)^{p}\frac{1}{\Gamma(p)\Gamma(q)}\int_a^b\int_a^s\left(\log\frac{b}{s}\right)^{p-1}(s-\tau)^{q-1}\|f(\tau,u_n(\tau))-f(\tau,u(\tau))\|\mathrm{d}\tau\frac{\mathrm{ds}}{\mathrm{s}}\\
\leq&\frac{2\,(b-a)^q\left(\log\frac{b}{a}\right)^p}{\Gamma(p+1)\Gamma(q+1)}\|f(\cdot,u_n(\cdot))-f(\cdot,u(\cdot))\|.
\end{aligned}
$$

By using the Lebesgue dominated convergence theorem, we know that

$$
\|\mathcal{N}u_n(t)-\mathcal{N}u(t)\|\to 0\quad\text{as}\quad n\to+\infty,
$$

for any $t\in J$. Therefore, we get that

$$
\|\mathcal{N}u_n-\mathcal{N}u\|\to 0\quad\text{as}\quad n\to+\infty,
$$

which implies the continuity of the operator $\mathcal{N}$.

Step 3. The operator $\mathcal{N}$ is equicontinuous. For any $a < t_1 < t_2 < b$ and $u \in B_R$, we get

$$\begin{aligned}
&\|\mathcal{N}(u)(t_2) - \mathcal{N}(u)(t_1)\| \\
&\leq \frac{1}{\Gamma(p)\Gamma(q)} \int_a^{t_1} \int_a^s \left[\left(\log \frac{t_1}{s}\right)^{p-1} - \left(\log \frac{t_2}{s}\right)^{p-1} \right] (s-\tau)^{q-1} \|f(\tau, u(\tau))\| \mathrm{d}\tau \frac{\mathrm{ds}}{\mathrm{s}} \\
&+ \frac{1}{\Gamma(p)\Gamma(q)} \int_{t_1}^{t_2} \int_a^s \left(\log \frac{t_2}{s}\right)^{p-1} (s-\tau)^{q-1} \|f(\tau, u(\tau))\| \mathrm{d}\tau \frac{\mathrm{ds}}{\mathrm{s}} \\
&+ \frac{(\log(t_2/a))^p - (\log(t_1/a))^p}{\Gamma(p)\Gamma(q)\,(\log(b/a))^p} \int_a^b \int_a^s \left(\log \frac{b}{s}\right)^{p-1} (s-\tau)^{q-1} \|f(\tau, u(\tau))\| \mathrm{d}\tau \frac{\mathrm{ds}}{\mathrm{s}} \\
\leq & \frac{\|\psi\|(1+r)\,(b-a)^q}{\Gamma(p)\Gamma(q+1)} \left[\int_a^{t_1} \left[\left(\log \frac{t_1}{s}\right)^{p-1} - \left(\log \frac{t_2}{s}\right)^{p-1} \right] \frac{\mathrm{ds}}{\mathrm{s}} + \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{p-1} \frac{\mathrm{ds}}{\mathrm{s}} \right] \\
&+ \frac{\|\psi\|(1+r)\,(b-a)^q}{\Gamma(p)\Gamma(q+1)} \frac{(\log(t_2/a))^p - (\log(t_1/a))^p}{\Gamma(p)\,(\log(b/a))^p} \int_a^b \left(\log \frac{b}{s}\right)^{p-1} \frac{\mathrm{ds}}{\mathrm{s}} \\
\leq & 2 \frac{\|\psi\|(1+r)\,(b-a)^q}{\Gamma(p+1)\Gamma(q+1)} \left(\log \frac{t_2}{t_1}\right)^p.
\end{aligned}$$

As $t_2 \to t_1$ the right-hand side of the above inequality tends to zero independently of $u \in B_R$. Hence, we conclude that $\mathcal{N}(B_R) \subseteq C(J, E)$ is bounded and equicontinuous.
Step 4: Our aim in this step is to show that $\mathcal{N}$ is χ-contraction on B_R. For every bounded subset $W \subset B_R$ and $\varepsilon > 0$ using Lemma 2.4 and the properties of χ, there exist sequences $\{u_k\}_{k=1}^{\infty} \subset W$ such that

$$\begin{aligned}
&\chi\left(\mathcal{N}W(t)\right) \\
\leq & 2\chi \Bigg\{ \frac{1}{\Gamma(p)\Gamma(q)} \int_a^t \int_a^s \left(\log \frac{t}{s}\right)^{p-1} (s-\tau)^{q-1} f(\tau, \{u_k(\tau)\}_{k=1}^{\infty}) \mathrm{d}\tau \frac{\mathrm{ds}}{\mathrm{s}} \\
&- \left(\frac{\log(t/a)}{\log(b/a)}\right)^p \frac{1}{\Gamma(p)\Gamma(q)} \int_a^b \int_a^s \left(\log \frac{b}{s}\right)^{p-1} (s-\tau)^{q-1} f(\tau, \{u_k(\tau)\}_{k=1}^{\infty}) \mathrm{d}\tau \frac{\mathrm{ds}}{\mathrm{s}} \Bigg\} + \varepsilon.
\end{aligned}$$

Next, by Lemma 2.3 and the properties of χ and (H3) we have

$$\begin{aligned}
&\chi\left(\mathcal{N}W(t)\right) \\
\leq & 4 \Bigg\{ \frac{1}{\Gamma(p)\Gamma(q)} \int_a^t \int_a^s \left(\log \frac{t}{s}\right)^{p-1} (s-\tau)^{q-1} \chi(f(\tau, \{u_k(\tau)\}_{k=1}^{\infty})) \mathrm{d}\tau \frac{\mathrm{ds}}{\mathrm{s}} \\
&+ \left(\frac{\log(t/a)}{\log(b/a)}\right)^p \frac{1}{\Gamma(p)\Gamma(q)} \int_a^b \int_a^s \left(\log \frac{b}{s}\right)^{p-1} (s-\tau)^{q-1} \chi(f(\tau, \{u_k(\tau)\}_{k=1}^{\infty})) \mathrm{d}\tau \frac{\mathrm{ds}}{\mathrm{s}} \Bigg\} + \varepsilon \\
\leq & 4 \Bigg\{ \frac{1}{\Gamma(p)\Gamma(q)} \int_a^t \int_a^s \left(\log \frac{t}{s}\right)^{p-1} (s-\tau)^{q-1} \psi(\tau) \chi(\{u_k(\tau)\}_{k=1}^{\infty}) \mathrm{d}\tau \frac{\mathrm{ds}}{\mathrm{s}}
\end{aligned}$$

$$+\left(\frac{\log(t/a)}{\log(b/a)}\right)^p \frac{1}{\Gamma(p)\Gamma(q)} \int_a^b \int_a^s \left(\log\frac{b}{s}\right)^{p-1} (s-\tau)^{q-1}\psi(\tau)\chi(\{u_k(\tau)\}_{k=1}^{\infty})\mathrm{d}\tau \frac{\mathrm{ds}}{\mathrm{s}}\Bigg\} + \varepsilon$$

$$\leq 4\frac{2\|\psi\|\,(b-a)^q\left(\log\frac{b}{a}\right)^p}{\Gamma(p+1)\Gamma(q+1)}\chi(B) + \varepsilon.$$

As the last inequality is true for every $\varepsilon > 0$ we infer

$$\chi(\mathcal{N}W) = \sup_{t\in J}\chi(\mathcal{N}W(t)) \leq 4\mathcal{M}_\psi\chi(B).$$

Using the condition (3.8), we claim that $\mathcal{N}$ is a χ-contraction on B_R. By Theorem 2.1, there is a fixed point u of $\mathcal{N}$ on B_R, which is a solution of (1.2). This completes the proof. □

4. An Example

In this section we give an example to illustrate the usefulness of our main result. Let

$$E = c_0 = \{u = (u_1, u_2, \ldots, u_n, \ldots) : u_n \to 0\ (n \to \infty)\},$$

be the Banach space of real sequences converging to zero, endowed its usual norm

$$\|u\|_\infty = \sup_{n\geq 1}|u_n|.$$

Example 4.1. Consider the following boundary value problem of a fractional differential posed in c_0:

$$\begin{cases} {}^{C}\mathcal{D}^{\frac{3}{4}}\left[{}^{C}_{H}\mathcal{D}^{\frac{7}{8}}u(t)\right] = f(t,u(t)), \quad 0 < p, q \leq 1,\ t \in J := \left[1, \frac{3}{2}\right], \\ u(1) = u\left(\frac{3}{2}\right) = 0. \end{cases} \tag{4.1}$$

Note that this problem is a particular case of BVP (1.2), where

$$p = \frac{7}{8} = q = \frac{3}{4}, \quad a = 1, \quad b = \frac{3}{2},$$

and $f : J \times c_0 \to c_0$ given by

$$f(t,u) = \left\{\frac{1}{(t^2+2)^2}\left(\frac{1}{n^2} + \sin|u_n|\right)\right\}_{n\geq 1}, \quad \text{for } t \in J,\ u = \{u_n\}_{n\geq 1} \in c_0.$$

It is clear that condition (H1) holds, and as

$$\|f(t,u)\| = \left\|\frac{1}{(t^2+2)^2}\left(\frac{1}{n^2} + \sin|u_n|\right)\right\| \leq \frac{1}{(t^2+2)^2}(1 + \|u\|) = \psi(t)(1 + \|u\|).$$

Therefore, assumption (H2) of Theorem 3.1 is satisfied, with $\psi(t) = \frac{1}{(t^2+2)^2}$, $t \in J$.

On the other hand, for any bounded set $W \subset c_0$, we have

$$\chi(f(t,W)) \leq \frac{1}{(t^2+2)^2}\chi(W), \quad \text{for each } t \in J.$$

Hence (H3) is satisfied.

We shall check that condition (3.8) is satisfied. Indeed, $4\mathcal{M}_\psi = 0.616 < 1$ and $(1+R)\mathcal{M}_\psi \le R$. Thus,

$$R \ge \frac{\mathcal{M}_\psi}{1-\mathcal{M}_\psi} = 1.6041.$$

Then R can be chosen as $R = 2 > 1.6041$. Consequently, Theorem 3.1 implies that problem (4.1) has at least one solution $u \in C(J, c_0)$.

5. Conclusions

We have proved the existence of solutions for certain classes of nonlinear sequential Caputo and Caputo-Hadamard fractional differential equations with Dirichlet boundary conditions in a given Banach space. The problem is issued by applying Darbo's fixed point theorem combined with the technique of Hausdorff measure of noncompactness. We also provide an example to make our results clear.

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