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INFINITELY MANY SOLUTIONS TO A FOURTH-ORDER IMPULSIVE DIFFERENTIAL EQUATION WITH TWO CONTROL PARAMETERS

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Abstract. In this article, we give some new criteria to guarantee the infinitely many solutions for a fourth-order impulsive boundary value problem. Our main tool to ensure the existence of infinitely many solutions is the classical Ricceri's Variational Principle.

1. Introduction.

In this paper, we consider the following boundary value problem for a fourth-order impulsive differential equation:

$$\begin{cases} u^{(4)}(t) + Au''(t) + Bu(t) = \lambda f(t,u(t)) + \mu g(t,u(t)), & t \neq t_j, t \in [0,1], \\ \triangle u''(t_j) = I_{1j}(u'(t_j)), \quad -\triangle u'''(t_j) = I_{2j}(u(t_j)), & j = 1,2,\dots,m, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{1.1}$$

where A, B are two real constants, $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are two L^2-Carathéodory functions, $I_{1j}, I_{2j} \in C(\mathbb{R},\mathbb{R})$, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = 1$, the operator Δ is defined as $\Delta U(t_j) = U(t_j^+) - U(t_j^-)$, where $U(t_j^+)$ $\left(U(t_j^-)\right)$ denotes the right-hand (left-hand) limit of U at t_j and $\lambda > 0$ and $\mu \geq 0$ are referred to as control parameters.

In recent years, a great deal of work has been done in the study of the existence of solutions for impulsive boundary value problems (IBVPs for short).

Some classical tools have been used to study such problems in the literatures. These techniques include the coincidence degree theory of Mawhin, the method of upper and

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lower solutions with monotone iterative technique, and some fixed point theorems in cones.

On the other hand, in the last few years, many researchers have used variational methods to study the existence of solutions for IBVPs. We refer the interested readers to [1–4, 6–8].

Motivated by the paper [1], in the present paper, by employing the classical Ricceri's Variational Principle, we obtain a sequence of solutions to problem (1.1) which is unbounded. Note that when $\mu = 0$ system (1.1) reduces to the one studied in [8]. Our results extend those ones in [8].

The remaining part of this paper is organized as follows. Some fundamental facts will be given in Section 2 and the main result of this paper will be presented in Section 3.

2. Preliminaries

Our main tool to ensure the existence of infinitely many solutions for the problem (1.1) is the classical Ricceri's Variational Principle ([5, Theorem 2.5]) that we now recall here.

Theorem 2.1. *Let X be a reflexive real Banach space. Let $\phi, \psi : X \to \mathbb{R}$ be two Gateaux differentiable functionals such that ϕ is sequentially weakly lower semicontinuous, strongly continuous and coercive and ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \phi$, let us put*

$$\varphi(r) = \inf_{u\in\phi^{-1}(]-\infty,r[)} \frac{\sup_{v\in\phi^{-1}(]-\infty,r])} \psi(v) - \psi(u)}{r - \phi(u)}.$$

If $\gamma := \liminf_{r\to+\infty} \varphi(r) < +\infty$, then for each $\lambda \in]0, \frac{1}{\gamma}[$, only one of the following statements holds to the functional $I_\lambda := \phi - \lambda\psi$:

(A1) *I_λ possesses a global minimum;*

(A2) *there is a sequence (u_n) of critical points (local minima) of I_λ such that*

$$\lim_{n\to+\infty} \phi(u_n) = +\infty.$$

Here and in the sequel, we suppose that A and B satisfy the following condition:

$$A \le 0 \le B. \tag{2.1}$$

Define

$$\begin{aligned} H_0^1([0,1]) :=&\{u \in L^2([0,1]) : u' \in L^2([0,1]), u(0) = u(1) = 0\},\\ H^2([0,1]) :=&\{u \in L^2([0,1]) : u', u'' \in L^2([0,1])\}. \end{aligned}$$

Take $X := H^2([0,1]) \bigcap H_0^1([0,1])$ and define

$$\|u\|_X = \left(\int_0^1 |u''(t)|^2 - A|u'(t)|^2 + B|u(t)|^2 dt \right)^{\frac{1}{2}}. \tag{2.2}$$

Since A, B satisfy (2.1), it is straightforward to verify that (2.2) defines a norm for the Sobolev space X and this norm is equivalent to the usual norm defined as follows:

$$\|u\| := \|u''\|_{L^2([0,1])}.$$

It follows from (2.1) that $\|u\| \leq \|u\|_X$. For the norm in $C^1([0,1])$

$$\|u\|_\infty = \max\left\{\max_{t\in[0,1]} |u(t)|, \max_{t\in[0,1]} |u'(t)|\right\},$$

we have the following relation.

Lemma 2.1 ([8]). *Let $M_1 := 1 + \frac{1}{\pi}$. Then $\|u\|_\infty \leq M_1\|u\|_X$ for all $u \in X$.*

Definition 2.1. By a weak solution of the problem (1.1), we mean any $u \in X$ such that

$$\int_0^1 [u''(t)v''(t) - Au'(t)v'(t) + Bu(t)v(t)]dt + \sum_{j=1}^m I_{2j}(u(t_j))v(t_j) + \sum_{j=1}^m I_{1j}(u'(t_j))v'(t_j)$$
$$=\lambda \int_0^1 f(t,u(t))v(t)dt + \mu \int_0^1 g(t,u(t))v(t)dt$$

holds for every $v \in X$.

Put

$$F(t,x) := \int_0^x f(t,\xi)d\xi, \quad G(t,x) := \int_0^x g(t,\xi)d\xi,$$

for all $(t,x) \in [0,1] \times \mathbb{R}$.

3. Main Results

In this section, we present our main results. To this end, we need the following assumptions.

$(H1)$ Assume that there exist two positive constants k_1 and k_2 such that for each $u \in X$

$$0 \leq \sum_{j=1}^m \int_0^{u'(t_j)} I_{1j}(s)ds \leq k_1 \max_{j\in\{1,2,\ldots,m\}} |u'(t_j)|^2$$

and

$$0 \leq \sum_{j=1}^m \int_0^{u(t_j)} I_{2j}(s)ds \leq k_2 \max_{j\in\{1,2,\ldots,m\}} |u(t_j)|^2.$$

Also put $k_3 := 2.048(\frac{3}{8} - \frac{9}{10.4^4}A + \frac{79}{14.4^8}B)$ and $k_4 := k_2 + k_3$. These constants will be used in the hypotheses of Theorem 3.1.

$(H2)$ Assume that $\{t_1, t_2, \ldots, t_m\} \subseteq [\frac{1}{4}, \frac{3}{4}]$.

$(H3)$ Assume that $F(t,u) \geq 0$ for $(t,u) \in \Big([0, \frac{1}{4}] \bigcup [\frac{3}{4}, 1]\Big) \times \mathbb{R}$.

Theorem 3.1. *Suppose that* $(H1)$, $(H2)$ *and* $(H3)$ *are satisfied. Also* $(H4)$

$$M_1^2 \liminf_{\xi\to+\infty} \frac{\int_0^1 \sup_{|x|\leq\xi} F(t,x)dt}{\xi^2} < \frac{1}{k_4} \limsup_{\xi\to+\infty} \frac{\int_{\frac{1}{4}}^{\frac{3}{4}} F(t,\xi)dt}{\xi^2},$$

and $\lambda \in]\lambda_1, \lambda_2[$, *where*

$$\lambda_1 := \frac{k_4}{\limsup\limits_{\xi\to+\infty} \frac{\int_{\frac{1}{4}}^{\frac{3}{4}} F(t,\xi)dt}{\xi^2}}, \quad \lambda_2 := \frac{1}{M_1^2 \liminf\limits_{\xi\to+\infty} \frac{\int_0^1 \sup_{|x|\leq\xi} F(t,x)dt}{\xi^2}}.$$

If G *is a nonnegative function satisfying the condition*

$$g_\infty := \lim_{\xi\to+\infty} \frac{\int_0^1 \sup_{|x|\leq\xi} G(t,x)dt}{\left(\frac{\xi^2}{M_1^2}\right)} < +\infty, \tag{3.1}$$

then for every $\mu \in [0, \mu_{g,\lambda}[$, *where*

$$\mu_{g,\lambda} := \frac{1}{g_\infty}\left(1 - \lambda M_1^2 \liminf_{\xi\to+\infty} \frac{\int_0^1 \sup_{|x|\leq\xi} F(t,x)dt}{\xi^2}\right),$$

the problem (1.1) *has an unbounded sequence of weak solutions in* X.

Proof. Fix $\lambda \in]\lambda_1, \lambda_2[$ and let g be a function satisfying the condition (3.1). Since $\lambda < \lambda_2$, one has $\mu_{g,\lambda} > 0$. Fix $\mu \in [0, \mu_{g,\lambda}[$ and put

$$v_1 := \lambda_1, \quad v_2 := \frac{\lambda_2}{1 + (\frac{\mu}{\lambda})\lambda_2 g_\infty}.$$

If $g_\infty = 0$, clearly, $v_1 = \lambda_1$, $v_2 = \lambda_2$ and $\lambda \in]v_1, v_2[$. If $g_\infty \neq 0$, since $\mu < \mu_{g,\lambda}$, we obtain $\frac{\lambda}{\lambda_2} + \mu g_\infty < 1$, and so $v_2 > \lambda$. Hence, since $\lambda > \lambda_1 = v_1$, one has $\lambda \in]v_1, v_2[$. Take $X = H^2\big([0,1]\big) \cap H_0^1\big([0,1]\big)$ and define in X the functional I_λ for each $u \in X$, as follows

$$I_\lambda(u) := \phi(u) - \lambda\psi(u),$$

where

$$\phi(u) = \frac{1}{2}\|u\|_X^2 + \sum_{j=1}^m \int_0^{u'(t_j)} I_{1j}(s)ds + \sum_{j=1}^m \int_0^{u(t_j)} I_{2j}(s)ds,$$

$$\psi(u) = \int_0^1 F(t,u(t))dt + \frac{\mu}{\lambda}\int_0^1 G(t,u(t))dt.$$

It is not hard to show that every critical point of I_λ is a weak solution of system (1.1). So, our goal is to apply Theorem 2.1 to ϕ and ψ. In the first step, it is well known that $\phi, \psi : X \to \mathbb{R}$ are two Gateaux differentiable functionals such that ϕ is sequentially weakly lower semicontinuous, strongly continuous and coercive.

Moreover, ψ is sequentially weakly upper semicontinuous. Now, we wish to prove that $\gamma = \liminf_{r\to+\infty} \varphi(r) < \infty$, where

$$\varphi(r) = \inf_{\phi(u)<r} \frac{\sup_{\phi(v)\le r} \psi(v) - \psi(u)}{r - \phi(u)}.$$

Let

$$Q(t,x) := F(t,x) + \frac{\mu}{\lambda} G(t,x), \quad (t,x) \in [0,1] \times \mathbb{R}.$$

Let (ξ_n) be a real sequence such that $\xi_n > 0$ for all $n \in \mathbb{N}$ and $\xi_n \to +\infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{\int_0^1 \sup_{|x|\le\xi_n} Q(t,x)dt}{\xi_n^2} = \liminf_{\xi\to+\infty} \frac{\int_0^1 \sup_{|x|\le\xi} Q(t,x)dt}{\xi^2}.$$

Put $r_n = \dfrac{\xi_n^2}{2M_1^2}$ for all $n \in \mathbb{N}$. Then for every $u \in X$, with $\phi(u) < r_n$, we have

$$\|u\|_\infty^2 \le M_1^2 \|u\|_X^2 \le 2M_1^2 \phi(u) < 2M_1^2 r_n = \xi_n^2,$$

thus

$$\phi^{-1}(]-\infty, r_n[) \subseteq \{u \in X : \|u\|_\infty \le \xi_n\}.$$

Hence, taking into account that $\phi(0) = \psi(0) = 0$ for every n large enough, one has

$$\begin{aligned}
\varphi(r_n) &= \inf_{\phi(u)<r_n} \frac{\sup_{\phi(v)\le r_n} \psi(v) - \psi(u)}{r_n - \phi(u)} \le \frac{\sup_{\phi(v)\le r_n} \psi(v) - \psi(0)}{r_n - \phi(0)} \\
&= \frac{1}{r_n} \sup_{\phi(v)\le r_n} \psi(v) = \frac{2M_1^2}{\xi_n^2} \sup_{\phi(v)\le r_n} \psi(v) \\
&\le \frac{2M_1^2}{\xi_n^2} \int_0^1 \sup_{|x|\le\xi_n} Q(t,x)dt \\
&\le \frac{\int_0^1 \sup_{|x|\le\xi_n} F(t,x)dt}{\frac{\xi_n^2}{2M_1^2}} + \frac{\mu}{\lambda} \cdot \frac{\int_0^1 \sup_{|x|\le\xi_n} G(t,x)dt}{\frac{\xi_n^2}{2M_1^2}}.
\end{aligned}$$

Therefore, it follows from $(H4)$ and condition (3.1) that

$$\gamma \le \liminf_{n\to+\infty} \varphi(r_n) \le \lim_{n\to+\infty} \frac{\int_0^1 \sup_{|x|\le\xi_n} F(t,x) + (\frac{\mu}{\lambda})G(t,x)dt}{\frac{\xi_n^2}{2M_1^2}} < +\infty. \tag{3.2}$$

Here, we can observe that $]\lambda_1, \lambda_2[\subseteq]0, \frac{1}{\gamma}[$. Hence, for the fixed $\lambda \in]\lambda_1, \lambda_2[$, the inequality (3.2) assures that Theorem 2.1 can be used and either I_λ has a global minimum or there exists a sequence (u_n) of weak solutions of the problem (1.1) such that $\lim_{n\to\infty} \|u_n\|_X = +\infty$.

The other step is to verity that the functional I_λ has no global minimum. Since

$$\frac{k_4}{\lambda} < \limsup_{\xi\to+\infty} \frac{\int_{\frac{1}{4}}^{\frac{3}{4}} F(t,\xi)dt}{\xi^2},$$

we can consider a real sequence (γ_n) and a positive constant τ such that $\gamma_n \to +\infty$ as $n \to \infty$ and

$$\frac{k_4}{\lambda} < \tau < \frac{\int_{\frac{1}{4}}^{\frac{3}{4}} F(t,\gamma_n)dt}{\gamma_n^2}, \tag{3.3}$$

for each $n \in \mathbb{N}$ large enough. Thus, if we consider a sequence (w_n) in X defined by setting

$$w_n(t) = \begin{cases} 64\gamma_n\left(t^3 - \frac{3}{4}t^2 + \frac{3}{16}t\right), & t \in \left[0, \frac{1}{4}\right[, \\ \gamma_n, & t \in \left[\frac{1}{4}, \frac{3}{4}\right], \\ 64\gamma_n\left(-t^3 + \frac{9}{4}t^2 - \frac{27}{16}t + \frac{7}{16}\right), & t \in \left]\frac{3}{4}, 1\right], \end{cases}$$

then, taking $(H1)$ and $(H2)$ into account, we conclude

$$\phi(w_n) = 2.048\left(\frac{3}{8} - \frac{9}{10.4^4}A + \frac{79}{14.4^8}B\right)\gamma_n^2 + \sum_{j=1}^{m}\int_0^{\gamma_n} I_{2j}(s)ds \le k_3\gamma_n^2 + k_2\gamma_n^2 = k_4\gamma_n^2.$$

On the other hand, since G is nonnegative, we observe

$$\psi(w_n) \ge \int_{\frac{1}{4}}^{\frac{3}{4}} F(t,\gamma_n)dt.$$

So, from (3.3), we conclude

$$I_\lambda(w_n) = \phi(w_n) - \lambda\psi(w_n) \le k_4\gamma_n^2 - \lambda\int_{\frac{1}{4}}^{\frac{3}{4}} F(t,\gamma_n)dt \le \gamma_n^2(k_4 - \lambda\tau),$$

for every $n \in \mathbb{N}$ large enough. Note that $k_4 - \lambda\tau < 0$. Hence, the functional I_λ is unbounded from below, and it follows that I_λ has no global minimum and we have the conclusion. $\square$

We have the following corollary as a special case of Theorem 3.1.

Corollary 3.1. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a nonnegative function and let $F(x) = \int_0^x f(\xi)d\xi$ for all $x \in \mathbb{R}$. Also,*

$$\liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} = 0 \quad \textit{and} \quad \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} = +\infty.$$

Then for every continuous function $g : \mathbb{R} \to \mathbb{R}$ whose $G(x) = \int_0^x g(\xi)d\xi$ for every $x \in \mathbb{R}$, is a nonnegative function satisfying the condition

$$g_* := \lim_{\xi\to+\infty} \frac{\sup_{|x|\le\xi} G(x)}{\frac{\xi^2}{M_1^2}} < +\infty,$$

and for every $\mu \in [0, \mu_*[$, where $\mu_* := \frac{1}{g_*}\left(1 - M_1^2 \liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^2}\right)$, the problem

$$\begin{cases} u^{(4)}(t) + Au''(t) + Bu(t) = f(u(t)) + \mu g(u(t)), & t \in [0,1], \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases}$$

has an unbounded sequence of weak solutions.

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