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(F, G)-DERIVATIONS ON A LATTICE

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Abstract. In the present paper, we introduce the notion of (F, G)-derivation on a lattice as a generalization of the notion of $(\wedge, \vee)$-derivation. This newly notion is based on two arbitrary binary operations F and G instead of the meet $(\wedge)$ and the join $(\vee)$ operations. Also, we investigate properties of (F, G)-derivation on a lattice in details. Furthermore, we define and study the notion of principal (F, G)-derivations as a particular class of (F, G)-derivations. As applications, we provide two representations of a given lattice in terms of its principal (F, G)-derivations.

1. Introduction

Binary operations are among the oldest fundamental concepts in algebraic structures. Since their introduction, they have become the key notion in the consepts of groups, monoids, semigroups, rings, and in more algebraic structures studied in abstract algebra [6, 15]. Binary operations have become essential tools in lattice theory and its applications [8]. Several notions and properties, and the notion of the lattice itself can be interpreted in terms of binary operations on it [5, 21]. Furthermore, it is not surprising that binary operations with specific properties appear in various theoretical and application fields. For instance, aggregation functions (as binary operations with specific properties) on bounded lattices and their wide use in various fields of applied sciences, including, computer and information sciences, economics, and social sciences (see, e.g., [9, 11] and [12, 13, 16, 18]). Also, they play an important role (as generalization of the basic connectives between fuzzy sets) in theories of fuzzy sets and logic [3].

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The notion of derivation appeared first on the ring structures and it has many applications (see, e.g., [2]). Szász [23] has extended this notion to the lattice structures based on the meet and the join operations ($(\wedge, \vee)$-derivation, for short), i.e., a $(\wedge, \vee)$-derivation on given lattice L is a function d of L into itself satisfying the following two conditions: $d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y))$ and $d(x \vee y) = d(x) \vee d(y)$ for any $x, y \in L$. Ferrari [7] has investigated some properties of this notion and provided some interesting examples in particular classes of lattices. Xin et al. [27] have ameliorated the notion of derivation on a lattice by considering only the first condition, and they have shown that the second condition obviously holds for the isotone derivations on a distributive lattice. In the same paper, they have characterized the distributive and modular lattices in terms of their isotone derivations. Later on, Xin [26] has focused his attention to the structure of the set of fixed points of a derivation on a lattice and has shown some relationships between lattice ideals and this set of fixed points.

The notion of $(\wedge, \vee)$-derivation on a lattice is witnessing increased attention. It studies, among others, in partially ordered sets [1, 31], in distributive lattices [30], in semilattices [29], in bounded hyperlattices [24], in quantales and residuated lattices [10, 25] and in several kinds of algebras [14, 17, 19]. Furthermore, it used in the definition of congruences and ideals in a distributive lattice [20].

In this paper, we generalize the notion of $(\wedge, \vee)$-derivation on a lattice to the (F, G)-derivation, where F and G are arbitrary binary operations on that lattice. More precisely, we introduce the notion of derivation on a lattice L with respect to two arbitrary binary operations F and G on L instead of the meet $(\wedge)$ and the join $(\vee)$ operations of L. Also, we investigate their properties in details. Furthermore, we define the principal (F, G)-derivations as a particular class of (F, G)-derivations on a lattice, and we study their various properties. Specific attention is paid to the lattice structure of the poset of principal (F, G)-derivations on a lattice. As applications, we provide two representations of a given lattice in terms of its principal (F, G)-derivations. These representations are draw upon some properties of binary operations on a lattice we investigated in [28].

The remainder of the paper is structured as follows. In Section 2, we recall the necessary basic concepts and properties of lattices and binary operations on lattices. In Section 3, we introduce the notion of (F, G)-derivation on a lattice and investigate their properties. In Section 4, we define the principal (F, G)-derivations on a lattice and study their various properties. In Section 5, we provide two representations of a given lattice in terms of its principal (F, G)-derivations. Finally, we present some concluding remarks in Section 6.

2. Basic concepts

In this section, we recall the necessary basic concepts and properties of lattices and binary operations on lattices.

2.1. **Lattice.** An *order relation* $\leqslant$ on a set X is a binary relation on X that is *reflexive* (i.e., $x \leqslant x$, for any $x \in X$), *antisymmetric* (i.e., $x \leqslant y$ and $y \leqslant x$ imply $x = y$, for any $x, y \in X$) and *transitive* (i.e., $x \leqslant y$ and $y \leqslant z$ imply $x \leqslant z$, for any $x, y, z \in X$). A set X equipped with an order relation $\leqslant$ is called a *partially ordered set* (a *poset*, for short), denoted $(X, \leqslant)$. Let $(X, \leqslant)$ be a poset and A be a subset of X. An element $x_0 \in X$ is called a *lower bound* of A if $x_0 \leqslant x$, for any $x \in A$. x_0 is called the *greatest lower bound* (or the *infimum*) of A if x_0 is a lower bound and $m \leqslant x_0$, for any lower bound m of A. *Upper bound* and *least upper bound* (or *supremum*) are defined dually.

A poset $(L, \leqslant)$ is called a $\wedge$-semi-lattice if any two elements x and y have a greatest lower bound, denoted by $x \wedge y$ and called the meet (infimum) of x and y. Analogously, it is called a $\vee$-semi-lattice if any two elements x and y have a smallest upper bound, denoted by $x \vee y$ and called the join (supremum) of x and y. A poset $(L, \leqslant)$ is called a lattice if it is both a $\wedge$-semi-lattice and a $\vee$-semi-lattice. Usually, the notation $(L, \leqslant, \wedge, \vee)$ is used. A poset $(L, \leqslant)$ is called bounded if it has a smallest and a greatest element, respectively denoted by 0 and 1. Often, the notation $(L, \leqslant, \wedge, \vee, 0, 1)$ is used to describe a bounded lattice. A lattice $(L, \leqslant, \wedge, \vee)$ is called distributive if one of the following two equivalent conditions hold:

(a) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for any $x, y, z \in L$;
(a^δ) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for any $x, y, z \in L$.

Let $(X, \leqslant)$ and $(Y, \preceq)$ be two posets. A mapping φ from X into Y is called an *order isomorphism* if it is surjective and satisfies

$$x \leqslant y \quad \text{if and only if} \quad \varphi(x) \preceq \varphi(y), \quad \text{for any } x, y \in X.$$

Let $(L, \leqslant, \wedge, \vee)$ and $(M, \preceq, \frown, \smile)$ be two lattices. A mapping φ from L into M is called a *lattice homomorphism*, if it satisfies $\varphi(x \wedge y) = \varphi(x) \frown \varphi(y)$ and $\varphi(x \vee y) = \varphi(x) \smile \varphi(y)$, for any $x, y \in L$. A *lattice isomorphism* is a bijective lattice homomorphism.

The following proposition shows that an order isomorphism between two lattices is a lattice isomorphism.

Proposition 2.1 ([5]). *Let L, M be two lattices and φ be a mapping from L into M. The following statements are equivalent:*

(i) *φ is an order isomorphism;*
(ii) *φ is a lattice isomorphism.*

For more details on lattices, we refer to [5, 15, 21, 22].

2.2. **Binary operations on a lattice.** In this subsection, we recall properties of binary operations on a lattice.

Let F be a binary operation on a non-empty set X. An element $e \in X$ is called:

(i) a right- (resp. left-) neutral element of F, if $F(x, e) = x$ (resp. $F(e, x) = x$) for any $x \in X$;

(ii) neutral element of F, if it is right- and left-neutral element, i.e., $F(e,x) = F(x,e) = x$ for any $x \in X$.

An element $k \in X$ is called:

(i) a right- (resp. left-) absorbing element of F, if $F(x,k) = k$ (resp. $F(k,x) = k$) for any $x \in X$;

(ii) an absorbing element of F, if it is right- and left-absorbing element, i.e.,

$$F(x,k) = F(k,x) = k, \quad \text{for any } x \in X.$$

The following properties of binary operations on a lattice are interest in this paper.

Definition 2.1 ([28])**.** Let $(L, \leqslant, \wedge, \vee)$ be a lattice and F be a binary operation on L. F is called:

(i) *idempotent*, if $F(x,x) = x$ for any $x \in L$;

(ii) *conjunctive*, if $F(x,y) \leqslant x \wedge y$ for any $x, y \in L$;

(iii) *increasing with respect to the first variable*, if $x \leqslant y$, implies $F(x,z) \leqslant F(y,z)$ for any $x, y, z \in L$;

(iv) *increasing*, if $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$ implies $F(x_1, y_1) \leqslant F(x_2, y_2)$ for any $x_1, y_1, x_2, y_2 \in L$.

In what follows, the statement $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$, can be written shortly by using the coordinate-wise order $(x_1, y_1) \leqslant_{L\times L} (x_2, y_2)$.

Definition 2.2 ([4])**.** A *triangular norm* (*t-norm*, for short) T on a bounded lattice $(L, \leqslant, \wedge, \vee, 0, 1)$ is a commutative, associative and increasing binary operation on L, and it has the neutral element $1 \in L$. Dually, a *triangular conorm* (*t-conorm*, for short) S on L, is a commutative, associative and increasing binary operation on L, and it has the neutral element $0 \in L$.

3. (F,G)-Derivations on a Lattice

In this section, we introduce the notion of (F,G)-derivation on a lattice and investigate their properties. This newly notion is a natural generalization of the notion of derivation on a lattice given by Xin et al. [27] with respect to the meet and join operations.

3.1. **Definitions and examples.** The notion of derivation on a lattice was first introduced by Szász [23].

Definition 3.1 ([23])**.** Let $(L, \leqslant, \wedge, \vee)$ be a lattice. A function $d : L \to L$ is called a *derivation* on L if it satisfies the following two conditions:

(D_1) $d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y))$ for any $x, y \in L$;

(D_2) $d(x \vee y) = d(x) \vee d(y)$ for any $x, y \in L$.

Later on, Xin et al. [27] have reduced the above conditions by considering only the condition (D_1). Moreover, they have shown that the condition (D_2) obviously holds for the isotone derivations on a distributive lattice.

Definition 3.2 ([27])**.** Let $(L, \leqslant, \wedge, \vee)$ be a lattice. A function $d : L \to L$ is called a *derivation* on L if it satisfies the (D_1) condition, i.e.,

$$d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y)), \quad \text{for any } x, y \in L.$$

Inspired by the above Definition 3.2, we introduce the core definition of this paper. It is based on two arbitrary binary operations on a lattice.

Definition 3.3. Let $(L, \leqslant, \wedge, \vee)$ be a lattice and F, G be two binary operations on L. A function $d : L \to L$ is called an (F, G)-*derivation* on L if it satisfies the following condition:

$$d(F(x, y)) = G(F(d(x), y), F(x, d(y))), \quad \text{for any } x, y \in L.$$

In the rest of the paper, we shortly write dx instead of $d(x)$ and $dF(x, y)$ instead of $d(F(x, y))$.

In the following, we present some illustrative examples of (F, G)-derivations on a lattice.

Example 3.1. Let $(L, \leqslant, \wedge, \vee)$ be a lattice and F, G be two binary operations on L such that G is idempotent. The identity function of L is an (F, G)-derivation on L. Indeed, suppose that d is the identity function of L. The fact that G is idempotent implies that $dF(x, y) = F(x, y) = G(F(x, y), F(x, y)) = G(F(dx, y), F(x, dy))$ for any $x, y \in L$. Hence, d is an (F, G)-derivation on L.

Example 3.2. Let $(\mathbb{N}^*, \leqslant, \min, \max)$ be the lattice of positive integers and $\alpha \in \mathbb{N}^*$. Then the following hold.

- (i) The null function of $\mathbb{N}$ is a $(\cdot, +)$-derivation on $\mathbb{N}$, but it is not a $(+, \cdot)$-derivation on $\mathbb{N}$.
- (ii) The translation function d_1 of $\mathbb{N}^*$ defined by $d_1(x) = x + \alpha$ for any $x \in \mathbb{N}^*$, it is both $(+, \gcd)$-derivation and $(+, \operatorname{lcm})$-derivation on $\mathbb{N}^*$.
- (iii) The homothety function d_2 of $\mathbb{N}^*$ defined by $d_2(x) = \alpha \cdot x$ for any $x \in \mathbb{N}^*$, it is both $(\cdot, \gcd)$-derivation and $(\cdot, \operatorname{lcm})$-derivation on $\mathbb{N}^*$.

Remark 3.1. We note that any derivation on a lattice L is a $(\wedge, \vee)$-derivation on L.

Definition 3.4. Let $(L, \leqslant, \wedge, \vee)$ be a lattice. An (F, G)-derivation d on L is called *isotone* if it satisfies the following condition:

$$x \leqslant y \quad \text{implies} \quad dx \leqslant dy, \quad \text{for any } x, y \in L.$$

Example 3.3. The translation (resp. homothety) function given in Example 3.2 is both isotone $(+, \gcd)$-derivation and isotone $(+, \operatorname{lcm})$-derivation (resp. isotone $(\cdot, \gcd)$-derivation and isotone $(\cdot, \operatorname{lcm})$-derivation) on $\mathbb{N}^*$.

3.2. **Properties of (F, G)-derivations on a lattice.** In this subsection, we investigate some properties of (F, G)-derivations on a lattice.

Proposition 3.1. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and d be an (F, G)-derivation on L. The following implications hold.*

(*i*) *If G is conjunctive, then $dF(x,y) \leqslant F(dx,y) \wedge F(x,dy)$ for any $x,y \in L$.*
(*ii*) *If G is disjunctive, then $F(dx,y) \vee F(x,dy) \leqslant dF(x,y)$ for any $x,y \in L$.*

Proof. (*i*) The fact that d is an (F,G)-derivation on L and G is conjunctive imply that $dF(x,y) = G(F(dx,y), F(x,dy)) \leqslant F(dx,y) \wedge F(x,dy)$, for any $x,y \in L$. Thus, $dF(x,y) \leqslant F(dx,y) \wedge F(x,dy)$ for any $x,y \in L$.
(*ii*) The proof is similar to that of (*i*). □

The above proposition leads to the following corollary.

Corollary 3.1. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and d be an (F,G)-derivation on L. The following implications hold.*

(*i*) *If F and G are conjunctive, then $dF(x,y) \leqslant dx \wedge dy \wedge x \wedge y$ for any $x,y \in L$.*
(*ii*) *If F and G are disjunctive, then $dx \vee dy \vee x \vee y \leqslant dF(x,y)$ for any $x,y \in L$.*

Proposition 3.2. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and d be an (F,G)-derivation on L. The following implications hold.*

(*i*) *If F is conjunctive and G is increasing, then $dF(x,y) \leqslant G(dx,dy) \wedge G(y,x) \wedge G(dx,x) \wedge G(y,dy)$ for any $x,y \in L$.*
(*ii*) *If F is disjunctive and G is increasing, then $G(dx,dy) \vee G(y,x) \vee G(dx,x) \vee G(y,dy) \leqslant dF(x,y)$ for any $x,y \in L$.*

Proof. (*i*) Let $x,y \in L$, the conjunctivity of F guarantees that $F(dx,y) \leqslant dx \wedge y$ and $F(x,dy) \leqslant x \wedge dy$. Since G is increasing, it holds that

$$G(F(dx,y), F(x,dy)) \leqslant G(dx,dy) \wedge G(y,x) \wedge G(dx,x) \wedge G(y,dy).$$

The fact that d is an (F,G)-derivation on L implies that

$$dF(x,y) \leqslant G(dx,dy) \wedge G(y,x) \wedge G(dx,x) \wedge G(y,dy).$$

(*ii*) The proof is similar to that of (*i*). □

Theorem 3.1. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and d be an (F,G)-derivation on L. If F and G are increasing and conjunctive, then the following statements hold for any $x,y \in L$:*

(*i*) *$G(F(dx, dF(y,y)), F(dF(x,x), dy)) \leqslant dF(x,y) \wedge F(dx,dy)$;*
(*ii*) *$G(F(dF(x,x), y), F(x, dF(y,y))) \leqslant dF(x,y) \wedge F(x,y)$;*
(*iii*)

$$\begin{aligned} &G(F(dF(x,x), dF(y,y)), F(dF(x,x), dF(y,y))) \\ \leqslant &dF(x,y) \wedge F(x,y) \wedge F(x,dy) \wedge F(dx,y) \wedge F(dx,dy). \end{aligned}$$

Proof. To prove (*i*), suppose that d is an (F,G)-derivation on L and let $x,y \in L$. Corollary 3.1 (*i*) guarantees that $dF(x,x) \leqslant dx \wedge x$ and $dF(y,y) \leqslant dy \wedge y$. Then

$$\begin{cases} ((dx, dF(y,y)), (dF(x,x), dy)) \leqslant_{L^2 \times L^2} ((dx,y), (x,dy)), \\ ((dx, dF(y,y)), (dF(x,x), dy)) \leqslant_{L^2 \times L^2} ((dx,dy), (dx,dy)). \end{cases}$$

The fact that F is increasing implies that

$$\begin{cases}(F(dx,dF(y,y)),F(dF(x,x),dy))\leqslant_{L\times L}(F(dx,y),F(x,dy)),\\(F(dx,dF(y,y)),F(dF(x,x),dy))\leqslant_{L\times L}(F(dx,dy),F(dx,dy)).\end{cases}$$

Since G is increasing and conjunctive, and d is an (F,G)-derivation on L, then it follows that

$$\begin{cases}G(F(dx,dF(y,y)),F(dF(x,x),dy))\leqslant G(F(dx,y),F(x,dy))=dF(x,y),\\G(F(dx,dF(y,y)),F(dF(x,x),dy))\leqslant G(F(dx,dy),F(dx,dy))\leqslant F(dx,dy).\end{cases}$$

Thus, $G(F(dx,dF(y,y)),F(dF(x,x),dy))\leqslant dF(x,y)\wedge F(dx,dy)$.

To demonstrate (ii), let $x,y\in L$. Corollary 3.1 (i) guarantees that $dF(x,x)\leqslant dx\wedge x$ and $dF(y,y)\leqslant dy\wedge y$. Then

$$\begin{cases}((dF(x,x),y),(x,dF(y,y)))\leqslant_{L^2\times L^2}((dx,y),(x,dy)),\\((dF(x,x),y),(x,dF(y,y)))\leqslant_{L^2\times L^2}((x,y),(x,y)).\end{cases}$$

Since F is increasing, it holds that

$$\begin{cases}(F(dF(x,x),y),F(x,dF(y,y)))\leqslant_{L\times L}(F(dx,y),F(x,dy)),\\(F(dF(x,x),y),F(x,dF(y,y)))\leqslant_{L\times L}(F(x,y),F(x,y)).\end{cases}$$

The fact that G is increasing and conjunctive, and d is an (F,G)-derivation on L, it implies that

$$\begin{cases}G(F(dF(x,x),y),F(x,dF(y,y)))\leqslant G(F(dx,y),F(x,dy))=dF(x,y),\\G(F(dF(x,x),y),F(x,dF(y,y)))\leqslant G(F(x,y),F(x,y))\leqslant F(x,y).\end{cases}$$

Therefore, $G(F(dF(x,x),y),F(x,dF(y,y)))\leqslant dF(x,y)\wedge F(x,y)$.

For proving (iii), let $x,y\in L$. Corollary 3.1 (i) guarantees that $dF(x,x)\leqslant dx\wedge x$ and $dF(y,y)\leqslant dy\wedge y$. Then

$$\begin{cases}((dF(x,x),dF(y,y)),(dF(x,x),dF(y,y)))\leqslant_{L^2\times L^2}((dx,y),(x,dy)),\\((dF(x,x),dF(y,y)),(dF(x,x),dF(y,y)))\leqslant_{L^2\times L^2}((x,y),(x,y)),\\((dF(x,x),dF(y,y)),(dF(x,x),dF(y,y)))\leqslant_{L^2\times L^2}((x,dy),(x,dy)),\\((dF(x,x),dF(y,y)),(dF(x,x),dF(y,y)))\leqslant_{L^2\times L^2}((dx,y),(dx,y)),\\((dF(x,x),dF(y,y)),(dF(x,x),dF(y,y)))\leqslant_{L^2\times L^2}((dx,dy),(dx,dy)).\end{cases}$$

The fact that F is increasing implies that

$$\begin{cases}(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y)))\leqslant_{L\times L}(F(dx,y),F(x,dy)),\\(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y)))\leqslant_{L\times L}(F(x,y),F(x,y)),\\(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y)))\leqslant_{L\times L}(F(x,dy),F(x,dy)),\\(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y)))\leqslant_{L\times L}(F(dx,y),F(dx,y)),\\(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y)))\leqslant_{L\times L}(F(dx,dy),F(dx,dy)).\end{cases}$$

Since G is increasing, it holds that

$$\begin{cases} G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant G(F(dx,y),F(x,dy)), \\ G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant G(F(x,y),F(x,y)), \\ G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant G(F(x,dy),F(x,dy)), \\ G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant G(F(dx,y),F(dx,y)), \\ G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant G(F(dx,dy),F(dx,dy)). \end{cases}$$

The conjunctivity of G and the fact that d is an (F,G)-derivation on L assure that

$$\begin{cases} G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant dF(x,y), \\ G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant F(x,y), \\ G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant F(x,dy), \\ G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant F(dx,y), \\ G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant F(dx,dy). \end{cases}$$

Thus, $G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant dF(x,y) \wedge F(x,y) \wedge F(x,dy) \wedge F(dx,y) \wedge F(dx,dy)$. □

Analogously, we obtain the following result for increasing and disjunctive binary operations.

Theorem 3.2. *Let $(L,\leqslant,\wedge,\vee)$ be a lattice and d be an (F,G)-derivation on L. If F and G are increasing and disjunctive, then the following statements hold for any $x,y \in L$:*

- (i) $dF(x,y) \vee F(dx,dy) \leqslant G(F(dx,dF(y,y)),F(dF(x,x),dy))$;
- (ii) $dF(x,y) \vee F(x,y) \leqslant G(F(dF(x,x),y),F(x,dF(y,y)))$;
- (iii) $dF(x,y) \vee F(x,y) \vee F(x,dy) \vee F(dx,y) \vee F(dx,dy) \leqslant G(F(dF(x,x),dF(y,y)), F(dF(x,x),dF(y,y)))$.

Proposition 3.3. *Let $(L,\leqslant,\wedge,\vee)$ be a lattice and d be an (F,G)-derivation on L. The following implications hold:*

- (i) *If F is conjunctive, G is increasing and idempotent, then $dF(x,x) \leqslant x \wedge dx$ for any $x \in L$;*
- (ii) *If F is disjunctive, G is increasing and idempotent, then $x \vee dx \leqslant dF(x,x)$ for any $x \in L$.*

Proof. (i) Let $x \in L$, since F is conjunctive, it holds that $(F(dx,x),F(x,dx)) \leqslant_{L\times L} (x \wedge dx, x \wedge dx)$. The fact that G is increasing implies that

$$G(F(dx,x),F(x,dx)) \leqslant G(x \wedge dx, x \wedge dx).$$

Since d is an (F,G)-derivation on L and G is idempotent, it follows that $dF(x,x) \leqslant x \wedge dx$.

(ii) The proof is similar to that of (i). □

The following theorems give alternative conditions to that of Theorems 3.1 and 3.2. Their proofs can be done in a similar way.

Theorem 3.3. *Let* $(L, \leqslant, \wedge, \vee)$ *be a lattice and* d *be an* (F, G)*-derivation on* L*. If* F *is increasing and conjunctive,* G *is increasing and idempotent, then the following statements hold for any* $x, y \in L$*:*

(i) $G(F(dx, dF(y, y)), F(dF(x, x), dy)) \leqslant dF(x, y) \wedge F(dx, dy)$;
(ii) $G(F(dF(x, x), y), F(x, dF(y, y))) \leqslant dF(x, y) \wedge F(x, y)$;
(iii) $F(dF(x, x), dF(y, y)) \leqslant dF(x, y) \wedge F(x, y) \wedge F(x, dy) \wedge F(dx, y) \wedge F(dx, dy)$.

Theorem 3.4. *Let* $(L, \leqslant, \wedge, \vee)$ *be a lattice and* d *be an* (F, G)*-derivation on* L*. If* F *is increasing and disjunctive,* G *is increasing and idempotent, then the following statements hold for any* $x, y \in L$*:*

(i) $dF(x, y) \vee F(dx, dy) \leqslant G(F(dx, dF(y, y)), F(dF(x, x), dy))$;
(ii) $dF(x, y) \vee F(x, y) \leqslant G(F(dF(x, x), y), F(x, dF(y, y)))$;
(iii) $dF(x, y) \vee F(x, y) \wedge F(x, dy) \vee F(dx, y) \vee F(dx, dy) \leqslant F(dF(x, x), dF(y, y))$.

Proposition 3.4. *Let* $(L, \leqslant, \wedge, \vee)$ *be a lattice and* d *be an* (F, G)*-derivation on* L*. If* F *is commutative and* G *is idempotent, then*

$$dF(x, x) = F(dx, x) = F(x, dx), \quad \text{for any } x \in L.$$

Proof. Since d is an (F, G)-derivation on L, F is commutative and G is idempotent, it holds that $dF(x, x) = G(F(dx, x), F(x, dx)) = G(F(dx, x), F(dx, x)) = F(dx, x)$ for any $x \in L$. □

Proposition 3.4 leads to the following corollary.

Corollary 3.2. *Let* $(L, \leqslant, \wedge, \vee)$ *be a lattice and* d *be an* (F, G)*-derivation on* L*. The following implications hold:*

(i) *If* F *is commutative and conjunctive and* G *is idempotent, then*

$$dF(x, x) \leqslant x \wedge dx, \quad \text{for any } x \in L;$$

(ii) *If* F *is commutative and disjunctive and* G *is idempotent, then*

$$x \vee dx \leqslant dF(x, x), \quad \text{for any } x \in L.$$

Proposition 3.5. *Let* $(L, \leqslant, \wedge, \vee, 0)$ *be a lattice with the least element* $0 \in L$ *and* d *be an* (F, G)*-derivation on* L*. The following implications hold.*

(i) *If* F *and* G *are conjunctive, then* $d0 = 0$ *(i.e.,* 0 *is a fixed point of* d*).*
(ii) *If* F *is conjunctive,* G *is increasing and idempotent, then* $d0 = 0$*.*
(iii) *If* F *is commutative and conjunctive, and* G *is idempotent, then* $d0 = 0$*.*

Proof. (i) The conjunctivity of F implies that $F(0, 0) = 0$. Then $d0 = dF(0, 0)$. From Corollary 3.1 (i), it holds that $dF(0, 0) \leqslant 0$. Thus, $d0 = 0$.

(ii) The proof is similar to that of (i) by using Proposition 3.3 (i).

(iii) The proof is similar to that of (i) by using Corollary 3.2 (i). □

In the same line, we obtain the following result.

Proposition 3.6. *Let $(L, \leqslant, \wedge, \vee, 1)$ be a lattice with the greatest element $1 \in L$ and d be an (F, G)-derivation on L. The following implications hold.*

- (i) *If F and G are disjunctive, then $d1 = 1$ (i.e., 1 is a fixed point of d).*
- (ii) *If F is disjunctive, G is increasing and idempotent, then $d1 = 1$.*
- (iii) *If F is commutative and disjunctive and G is idempotent, then $d1 = 1$.*

Proposition 3.7. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and d be an (F, G)-derivation on L. If F has a right- (resp. a left-) neutral element $e \in L$, then $dx = G(dx, F(x, de))$ (resp. $dx = G(F(de, x), dx))$ for any $x \in L$.*

Proof. The fact that e is a right- (resp. a left-) neutral element of F and d is an (F, G)-derivation on L imply that $dx = dF(x, e) = G(F(dx, e), F(x, de)) = G(dx, F(x, de))$ (resp. $dx = dF(e, x) = G(F(de, x), F(e, dx)) = G(F(de, x), dx))$ for any $x \in L$. ☐

Proposition 3.8. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and d be an (F, G)-derivation on L. If F has a right- (resp. a left-) absorbing element $k \in L$, then $dk = G(k, F(x, dk))$ (resp. $dk = G(F(dk, x), k))$ for any $x \in L$.*

Proof. Since k is a right- (resp. a left-) absorbing element of F and d is an (F, G)-derivation on L, then $dk = dF(x, k) = G(F(dx, k), F(x, dk)) = G(k, F(x, dk))$ (resp. $dk = dF(k, x) = G(F(dk, x), F(k, dx)) = G(F(de, x), k))$ for any $x \in L$. ☐

The above Proposition 3.8 leads to the following corollary.

Corollary 3.3. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and d be an (F, G)-derivation on L. If F and G have a right- or a left-absorbing element $k \in L$, then $dk = k$ (i.e., k is a fixed point of d).*

4. Principal (F, G)-Derivations on a Lattice

In this section, we introduce the notion of principal (F, G)-derivation on a lattice and investigate their various properties.

4.1. **Definitions and auxiliary results.**

Let $(L, \leqslant, \wedge, \vee)$ be a lattice and F be a binary operation on L. For any element $\alpha \in L$, there exists an F-function $f_\alpha : L \to L$ defined as:

$$f_\alpha(x) = F(\alpha, x), \quad \text{for any } x \in L.$$

Let $\mathcal{A}_F(L)$ be the set of the f_α functions on L, i.e., $\mathcal{A}_F(L) = \{f_\alpha \mid \alpha \in L\}$. One can easily verify that $\mathcal{A}_F(L)$ equipped with the usual order of functions (i.e., $f_\alpha \preceq f_\beta$ if and only if $f_\alpha(x) \leqslant f_\beta(x)$ for any $x \in L$) is a poset.

The following propositions show some cases that the poset $(\mathcal{A}_F(L), \preceq)$ has a lattice structure.

Proposition 4.1 ([28]). *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and F be a binary operation on L. If F is increasing with respect to the first variable and having a right-neutral element $e \in L$, then the poset $(\mathcal{A}_F(L), \preceq)$ is a lattice. Where, the meet $\frown$ and the join $\smile$ operations of $\mathcal{A}_F(L)$ are defined as $f_\alpha \frown f_\beta = f_{\alpha \wedge \beta}$ and $f_\alpha \smile f_\beta = f_{\alpha \vee \beta}$ for any $f_\alpha, f_\beta \in \mathcal{A}_F(L)$.*

Proposition 4.2 ([28]). *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and F be a binary operation on L. If F is the meet (resp. the join) operation of L, then $(\mathcal{A}_F(L), \preceq, \frown, \smile)$ is a lattice.*

Proposition 4.3 ([28]). *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and F be a binary operation on L. Then the following hold.*

(*i*) *If F is increasing with respect to the first variable and having a right-neutral element $e \in L$, then for any $\alpha, \beta \in L$, we obtain that*

$$\alpha \leqslant \beta \quad \text{if and only if} \quad f_\alpha \preceq f_\beta.$$

(*ii*) *If F is idempotent, conjunctive or disjunctive and increasing with respect to the first variable, then for any $\alpha, \beta \in L$, we obtain that*

$$\alpha \leqslant \beta \quad \text{if and only if} \quad f_\alpha \preceq f_\beta.$$

The following proposition provides some conditions that the elements of $\mathcal{A}_F(L)$ are (F, G)-derivations on L.

Proposition 4.4. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and F, G be two binary operations on L such that F is commutative and associative, and G is idempotent. Then any elements of $\mathcal{A}_F(L)$ is an (F, G)-derivation on L.*

Proof. Let $f_\alpha \in \mathcal{A}_F(L)$ and $x, y \in L$. We will show that

$$f_\alpha(F(x, y)) = G(F(f_\alpha(x), y), F(x, f_\alpha(y)))\,.$$

Since F is commutative and associative, and G is idempotent, it follows that

$$\begin{aligned} f_\alpha(F(x, y)) &= F(\alpha, F(x, y)) \\ &= G(F(\alpha, F(x, y)), F(\alpha, F(x, y))) \\ &= G(F(\alpha, F(x, y)), F(F(\alpha, x), y)) \\ &= G(F(\alpha, F(x, y)), F(F(x, \alpha), y)) \\ &= G(F(F(\alpha, x), y), F(x, F(\alpha, y))) \\ &= G(F(f_\alpha(x), y), F(x, f_\alpha(y)))\,. \end{aligned}$$

Thus, f_α is an (F, G)-derivation on L. □

In view of Proposition 4.4, we introduce the notion of principal (F, G)-derivation on a given lattice.

Definition 4.1. Let $(L, \leqslant, \wedge, \vee)$ be a lattice and F, G be two binary operations on L such that F is commutative and associative, and G is idempotent. Any f_α function is called *principal (F, G)-derivation* on L. In this case, $\mathcal{A}_F(L)$ denotes the set of the principal (F, G)-derivations on L.

Example 4.1. Let $(L, \leqslant, \wedge, \vee, 0, 1)$ be a bounded lattice, $\alpha \in L$, T be a t-norm on L and S a t-conorm on L. The function t_α (resp. s_α) defined for any $x \in L$ by $t_\alpha(x) = T(\alpha, x)$ (resp. $s_\alpha(x) = S(\alpha, x)$) is a principal (T, G)-derivation (resp. principal (S, G)-derivation) on L, for any idempotent binary operation G on L.

4.2. **Properties of principal (F, G)-derivations on a lattice.** In this subsection, we investigate some properties of principal (F, G)-derivations on a given lattice.

Proposition 4.5. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $f_\alpha \in \mathcal{A}_F(L)$ be a principal (F, G)-derivation on L. Then it holds that*

$$f_\alpha \circ f_\alpha(F(x, y)) = F(f_\alpha(x), f_\alpha(y)), \quad \text{for any } x, y \in L\,.$$

Proof. Let $f_\alpha \in \mathcal{A}_F(L)$ be a principal (F, G)-derivation on L and $x, y \in L$. The facts that F is commutative and associative imply that

$$\begin{aligned} f_\alpha \circ f_\alpha(F(x, y)) &= f_\alpha(f_\alpha(F(x, y))) = f_\alpha F(\alpha, F(x, y)) = F(\alpha, F(\alpha, F(x, y))) \\ &= F(\alpha, F(F(\alpha, x), y)) = F(F(\alpha, F(\alpha, x)), y) = F(F(F(\alpha, x), \alpha), y) \\ &= F(F(\alpha, x), F(\alpha, y)) = F(f_\alpha(x), f_\alpha(y)). \end{aligned}$$ □

Proposition 4.6. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $f_\alpha \in \mathcal{A}_F(L)$ be a principal (F, G)-derivation on L. If F is increasing, then f_α is isotone, i.e., if $x \leqslant y$, then $f_\alpha(x) \leqslant f_\alpha(y)$, for any $x, y \in L$.*

Proof. Let $x, y \in L$ such that $x \leqslant y$. Since F is increasing, it holds that $F(\alpha, x) \leqslant F(\alpha, y)$, i.e., $f_\alpha(x) \leqslant f_\alpha(y)$. Thus, the principal (F, G)-derivation f_α is isotone. □

One can easily verify that if the principal (F, G)-derivations on L are isotone, then F is increasing.

Proposition 4.6 leads to the following corollary.

Corollary 4.1. *Let $(L, \leqslant, \wedge, \vee, 0, 1)$ be a bounded lattice, T be a t-norm and S be a t-conorm on L. Then the principal (T, G)-derivations (resp. principal (S, G)-derivations) on L are isotone for any idempotent binary operation G on L.*

Proposition 4.7. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $f_\alpha \in \mathcal{A}_F(L)$ be a principal (F, G)-derivation on L. If F is increasing, then f_α satisfies that if $x \leqslant y$, then $f_\alpha(F(x, x)) \leqslant f_\alpha(F(y, y))$ for any $x, y \in L$.*

Proof. Let $x, y \in L$ such that $x \leqslant y$. The fact that F is associative implies that $f_\alpha(F(x, x)) = F(\alpha, F(x, x)) = F(F(\alpha, x), x)$ and $f_\alpha(F(y, y)) = F(F(\alpha, y), y)$. Since $x \leqslant y$ and F is increasing, it follows that $F(F(\alpha, x), x) \leqslant F(F(\alpha, y), y)$. Thus, $f_\alpha(F(x, x)) \leqslant f_\alpha(F(y, y))$. □

5. Representations of a Lattice in Terms of its Principal (F, G)-Derivations

In this section, we provide two representations of a given lattice in terms of its principal (F, G)-derivations. We start by the first one.

Theorem 5.1. *Let* $(L,\leqslant,\wedge,\vee)$ *be a lattice and* F,G *be two binary operations on* L *such that* F *is commutative and associative, and* G *is idempotent. If* F *is increasing and having a neutral element* $e\in L$*, then the lattice* $(L,\leqslant,\wedge,\vee)$ *is isomorphic to the lattice* $(\mathcal{A}_F(L),\preceq,\frown,\smile)$ *of principal* (F,G)*-derivations on* L.

Proof. Proposition 4.1 guarantees that the poset $(\mathcal{A}_F(L),\preceq,\frown,\smile)$ of principal (F,G)-derivations on L is a lattice. Next, let ψ be a mapping from L into $\mathcal{A}_F(L)$ defined by $\psi(\alpha)=f_\alpha$, for any $\alpha\in L$. One can easily verify that ψ is surjective. Furthermore, from Proposition 4.3 (i), it holds that

$$\alpha\leqslant\beta \quad \text{if and only if} \quad \psi(\alpha)\preceq\psi(\beta), \quad \text{for any } \alpha,\beta\in L.$$

Now, ψ is an order isomorphism between L and $\mathcal{A}_F(L)$. Thus, Proposition 2.1 guarantees that ψ is a lattice isomorphism. Therefore, the lattices $(L,\leqslant,\wedge,\vee)$ and $(\mathcal{A}_F(L),\preceq,\frown,\smile)$ are isomorphic. □

In the following, we present an illustrative example of Theorem 5.1.

Example 5.1. Let $L=\mathbb{R}^*_+$ be the lattice of the positive real numbers ordered by the usual order, and let F,G be two binary operations on $\mathbb{R}^*_+$ defined for any $x,y\in\mathbb{R}^*_+$ as $F(x,y)=x\cdot y$ and

$$G(x,y)=\begin{cases} x, & \text{if } x=y,\\ x+y, & \text{otherwise.}\end{cases}$$

One can easily verify that F is commutative and associative, and G is idempotent. Furthermore, F is increasing and having the neutral element $1\in\mathbb{R}^*_+$. Theorem 5.1 guarantees that the lattice $(\mathbb{R}^*_+,\leqslant,\min,\max)$ is isomorphic to the lattice $(\mathcal{A}_F(\mathbb{R}^*_+),\preceq,\frown,\smile)$ of principal (F,G)-derivations on $\mathbb{R}^*_+$.

Theorem 5.1 leads to the following corollary.

Corollary 5.1. *Let* $(L,\leqslant,\wedge,\vee,0,1)$ *be a bounded lattice,* G *be an idempotent binary operation,* T *be a t-norm and* S *be a t-conorm on* L*. Then the following hold.*

(i) *The bounded lattice* $(L,\leqslant,\wedge,\vee,0,1)$ *is isomorphic to the bounded lattice* $(\mathcal{A}_T(L),\preceq,\frown,\smile,t_0,t_1)$ *of principal* (T,G)*-derivations on* L*, where* $t_0(x)=T(0,x)=0$ *and* $t_1(x)=T(1,x)=x$ *for any* $x\in L$.

(ii) *The bounded lattice* $(L,\leqslant,\wedge,\vee,0,1)$ *is isomorphic to the bounded lattice* $(\mathcal{A}_S(L),\preceq,\frown,\smile,s_0,s_1)$ *of principal* (S,G)*-derivations on* L*, where* $s_0(x)=S(0,x)=x$ *and* $s_1(x)=S(1,x)=1$ *for any* $x\in L$.

The following theorem provides the second representation of a lattice in terms of its principal (F,G)-derivations.

Theorem 5.2. *Let* $(L,\leqslant,\wedge,\vee)$ *be a lattice and* F,G *be two binary operations on* L *such that* G *is idempotent. If* F *is the meet (resp. the join) operation of* L*, then the lattice* $(L,\leqslant,\wedge,\vee)$ *is isomorphic to the lattice* $(\mathcal{A}_F(L),\preceq,\frown,\smile)$ *of principal* (F,G)*-derivations on* L.

Proof. Since F is the meet (resp. the join) operation of L, it follows from Proposition 4.2 that the poset $(\mathcal{A}_F(L), \preceq, \frown, \smile)$ of principal (F, G)-derivations on L is a lattice. Now, let ψ be a mapping from L into $\mathcal{A}_F(L)$ defined by $\psi(\alpha) = f_\alpha$, for any $\alpha \in L$. Using the same steps as Theorem 5.2, we obtain that ψ is a lattice isomorphism. Thus, the lattices $(L, \leqslant, \wedge, \vee)$ and $(\mathcal{A}_F(L), \preceq, \frown, \smile)$ are isomorphic. □

Remark 5.1. We note the following.

(i) If $(L, \leqslant, \wedge, \vee, 0, 1)$ is a bounded lattice, F is the meet (resp. the join) operation of L and G is an idempotent binary operation on L, then $(L, \leqslant, \wedge, \vee, 0, 1)$ can be represented by using both Theorems 5.1 and 5.2.

(ii) If $(L, \leqslant, \wedge, \vee)$ is a latticea and it does not have the least element 0 (resp. greatest element 1), F is the meet (resp. the join) operation of L and G is an idempotent binary operation on L, then $(L, \leqslant, \wedge, \vee)$ can be represented only by Theorem 5.2.

(iii) If $(L, \leqslant, \wedge, \vee)$ is a distributive lattice and F, G are respectively the meet and the join operations of L, then Theorem 5.2 coincides with representation theorem given by Xin et al. (Theorem 3.29 in [27]). Thus, Theorem 5.2 is a generalization of Theorem 3.29 to an arbitrary lattice.

6. Conclusion

In this work, based on two arbitrary binary operations F and G on a given lattice, we have introduced the notion of (F, G)-derivation on a lattice as a generalization to the notion of $(\wedge, \vee)$-derivation. Also, we have investigated their various properties. We have defined and studied the principal (F, G)-derivations as a particular class of (F, G)-derivations on a lattice. As applications, we have provided two representations of a given lattice in terms of its principal (F, G)-derivations.

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References

[1] A. Y. Abdelwanis and A. Boua, *On generalized derivations of partially ordered sets*, Commun. Math. **27** (2019), 69–78.
[2] M. Ashraf, S. Ali and C. Haetinger, *On derivations in rings and their applications*, Aligarh Bull. Math. **25** (2006), 79–107.
[3] B. Bede, *Mathematics of Fuzzy Sets and Fuzzy Logic*, Springer, Berlin, 2013.
[4] G. D. Cooman and E. E. Kerre, *Order norms on bounded partially ordered sets*, The Journal of Fuzzy Mathematics **2** (1994), 281–310.
[5] B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order*, 2nd edition, Cambridge University Press, London, 2002.
[6] D. S. Dummit and R. M. Foote, *Abstract Algebra*, 3rd edition, Wiley, Hoboken, 2004.
[7] L. Ferrari, *On derivations of lattices*, Pure Math. Appl. (PU.M.A.) **12** (2001), 365–382.
[8] G. Grätzer and F. Wehrung, *Lattice Theory: Special Topics and Applications. Volume* 2, Springer International Publishing, Chaim, 2016.

[9] R. Halaš and J. Pócs, *On the clone of aggregation functions on bounded lattices*, Inform. Sci. **329** (2016), 381–389.
[10] P. He, X. L. Xin and J. Zhan, *On derivations and their fixed point sets in residuated lattices*, Fuzzy Sets and Systems **303** (2016), 97–113.
[11] T. Jwaid, B. De Baets, J. Kalická and R. Mesiar, *Conic aggregation functions*, Fuzzy Sets and Systems **167** (2011), 3–20.
[12] F. Karaçal and M. N. Kesicioğlu, *A t-partial order obtained from t-norms*, Kybernetika **47** (2011), 300–314.
[13] F. Karaçal and R. Mesiar, *Aggregation functions on bounded lattices*, Int. J. Gen. Syst. **46** (2017), 37–51.
[14] K. H. Kim and B. Davvaz, *On f-derivations of BE-algebras*, J. Chungcheong Math. Soc. **28** (2015), 127–138.
[15] B. Kolman, R. C. Busby and S. C. Ross, *Discrete Mathematical Structures*, 4th Edition, Prentice Hall PTR, Upper Saddle River, New Jersey, 2000.
[16] M. Komorníková and R. Mesiar, *Aggregation functions on bounded partially ordered sets and theirs classification*, Fuzzy Sets and Systems **175** (2011), 48–56.
[17] S. M. Lee and K. H. Kim, *A note on f-derivations of BCC-algebras*, Pure Mathematical Sciences **1** (2012), 87–93.
[18] R. Mesiar and M. Komorníková, *Aggregation functions on bounded posets*, in: C. Cornelis, G. Deschrijver, M. Nachtegael, S. Schockaert and Y. Shi Y. (Eds.) 35 *Years of Fuzzy Set Theory*, Springer, Berlin, Heidelberg, 2010.
[19] Ş. A. Özbal and A. Firat, *On f-derivations of incline algebras*, International Electronic Journal of Pure and Applied Mathematics **3** (2011), 83–90.
[20] M. S. Rao, *Congruences and ideals in a distributive lattice with respect to a derivation*, Bulletin of the Section of Logic **42** (2013), 1–10.
[21] S. Roman, *Lattices and Ordered Sets*, Springer Science+Business Media, New York, 2008.
[22] B. S. Schröder, *Ordered Sets*, Birkhauser, Boston, 2003.
[23] G. Szász, *Derivations of lattices*, Acta Sci. Math. **37** (1975), 149–154.
[24] J. Wang, Y. Jun, X. L. Xin, T. Y. Li and Y. Zou, *On derivations of bounded hyperlattices*, J. Math. Res. Appl. **36** (2016), 151–161.
[25] Q. Xiao and W. Liu, *On derivations of quantales*, Open Math. **14** (2016), 338–346.
[26] X. L. Xin, *The fixed set of a derivation in lattices*, Fixed Point Theory Appl. **218** (2012), 1–12.
[27] X. L. Xin, T. Y. Li and J. H. Lu, *On derivations of lattices*, Inform. Sci. **178** (2008), 307–316.
[28] M. Yettou, A. Amroune and L. Zedam, *A binary operation-based representation of a lattice*, Kybernetika **55** (2019), 252–272.
[29] Y. H. Yon and K. H. Kim, *On f-derivations from semilattices to lattices*, Commun. Korean Math. Soc. **29** (2014), 27–36.
[30] L. Zedam, M. Yettou and A. Amroune, *f-Fixed points of isotone f-derivations on a lattice*, Discuss. Math. Gen. Algebra Appl. **39** (2019), 69–89.
[31] H. Zhang and Q. Li, *On derivations of partially ordered sets*, Math. Slovaca **67** (2017), 17–22.

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