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OBTAINING VOIGT FUNCTIONS VIA QUADRATURE FORMULA FOR THE FRACTIONAL IN TIME DIFFUSION AND WAVE PROBLEM

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Abstract. In many given physical problems and in the course of dispersion curve through a spectral line under the influence of the Doppler-effect and in collision damping, the Voigt functions have been widely utilized. By taking advantage of the fractional calculus in spectral theory and the Sturm-Liouville problems, in this paper, we obtain the Voigt functions via the quadrature formulae of one dimensional fractional in time evolution diffusion and wave problems consisting of different initial and inhomogeneous boundary conditions.

1. Introduction

The Voigt functions $V_{\gamma,\nu}(x, y, z)$ in generalized form have been studied by many authors (e.g., [10, 19, 25] and [26]) for getting various connections with a class of special functions and the numbers. In astrophysics the fundamental equations of stellar statistics are of this type. Other remarkable examples are the Voigt functions which occurs and utilized frequently in the course of the dispersion curve through a spectral line under the influence of the Doppler-effect and collision damping. The Voigt functions $V_{\gamma,\nu}(x, y, z)$ which play an essential role in spectroscopy, neutron physics and in several diverse field of physics and harmonic analysis are generally investigated from the viewpoint of integral operators.

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In 1991, Klusch [10] defined the generalized Voigt function of the second kind by the Hankel integral transform

$$V_{\gamma,\nu}(x,y,z)=\sqrt{\frac{x}{2}}\int_0^\infty t^\gamma e^{-yt-zt^2}J_\nu(xt)dt,\quad x,y,z\in\mathbb{R}^+,\ \Re(\gamma+\nu)>-1, \tag{1.1}$$

where $J_\nu(\cdot)$ is the classical Bessel function (see [1,22] and [24]) defined by

$$J_v(z)=\sum_{n=0}^{\infty}\frac{(-1)^n(z/2)^{v+2n}}{\Gamma(n+1)\Gamma(v+n+1)},\quad |z|<\infty.$$

Again, we note that $J_v(z)$ is the defining oscillatory kernel of Hankel's integral transform

$$(H_vf)(x)=\int_0^\infty f(t)J_v(xt)dt.$$

Furthermore, the relation of the Bessel functions with the trigonometrical functions is given by

$$J_{\frac{1}{2}}(z)=\sqrt{\frac{2}{\pi}}z^{-\frac{1}{2}}\sin z\quad\text{and}\quad J_{-\frac{1}{2}}(z)=\sqrt{\frac{2}{\pi}}z^{\frac{-1}{2}}\cos z.$$

To explore new ideas for representing the relation of the Voigt functions (1.1) with the quadrature formula of the solution of fractional in time diffusion and wave problem, in our current investigation, we present following fractional in time Sturm-Liouville type diffusion and wave equation in the form:

$${}_t^CD_{0+}^\alpha Y(x,t)=\frac{\partial}{\partial x}\left[p(x)\frac{\partial}{\partial x}\right]Y(x,t)-q(x)Y(x,t)+f(x,t),\quad 0<\alpha\leq 2, \tag{1.2}$$

for all $(x,t)\in(0,l)\times(0,\infty)$, for the function defined by $f:[0,l]\times[0,\infty)\to\mathbb{R}$, $[0,l]\subset\mathbb{R}$.

Throughout this paper l is taken greater than zero, and also subjected to the initial and inhomogeneous boundary values

$$\begin{aligned}
Y(x,0)=&g(x)+\left(\frac{x}{l}-1\right)\varphi_1(0)-\frac{x}{l}\varphi_2(0), \\
\frac{\partial}{\partial t}Y(x,t)|_{t=0}=&\left(\frac{x}{l}-1\right)\varphi_1^{'}(0)-\frac{x}{l}\varphi_2^{'}(0),\quad (x,t)\in[0,l]\times\{0\}, \\
Y(0,t)+\varphi_1(t)=&0,\quad \frac{\partial}{\partial x}Y(x,t)|_{x=0}=1+\frac{1}{l}(\varphi_1(t)-\varphi_2(t)),\quad (x,t)\in\{0\}\times[0,\infty), \\
Y(l,t)+\varphi_2(t)=&0,\quad \text{for all } (x,t)\in\{l\}\times[0,\infty).
\end{aligned} \tag{1.3}$$

Here in (1.2), the Caputo fractional derivative ${}_t^CD_{0+}^\alpha$, $m-1<\alpha\leq m$, of function $Y(t)$ is given by

$$({}_t^CD_{0+}^\alpha Y)(t)=(I^{m-\alpha}Y^{(m)})(t),\quad \text{for all } m\in\mathbb{N}, \tag{1.4}$$

where $Y^{(m)}(t) = \frac{d^m Y}{dt^m}(t)$, $I^{m-\alpha}$ being the Riemann-Liouville fractional integral (see, Diethelm [2, p. 49])

$$(I^{m-\alpha}Y)(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} Y(\tau) d\tau, & t > 0,\, m-1 < \alpha \le m, \\ Y(t), \alpha = m, & \text{for all } m \in \mathbb{N}. \end{cases}$$

In this work, we also use the Laplace transformation of Caputo derivative (1.4), for $L[Y(t)] = \overline{Y}(s)$, $s > 0$, (see, Kilbas, Srivastava and Trujillo [8, p. 312]), given by

$$\begin{aligned} L[({}_t^C D_{0+}^\alpha Y)(t)] =& s^\alpha \overline{Y}(s) - s^{\alpha-1} Y(0) - s^{\alpha-2} Y^{(1)}(0) - \cdots - s^{\alpha-m} Y^{(m-1)}(0), \\ & m-1 < \alpha \le m. \end{aligned} \tag{1.5}$$

It may be observed that for $\alpha = 1$, the equation (1.2) converts into a linear second order parabolic partial differential equation and a diffusion problem with initial and boundary conditions given in (1.3). For $\alpha = 2$, equation (1.2) reduces to a linear second order elliptic partial differential equation of wave problem with given initial and inhomogeneous boundary conditions (see Evans [3]). On the other hand, when $0 < \alpha \le 1$, the above problem becomes identical to the initial-boundary value problem for the one dimensional time fractional diffusion equation because of the availability of the vast literature due to the researchers and authors (e.g., [4, 9, 14, 15]) with some additional boundary conditions. The analytic solutions of the space-time fractional differential equations with initial and boundary value problems are computed by the authors ([11, 14]). The computation of anomalous diffusion problems in the form of integral equations can be found in ([5, 12] and [13]). For the theory and analysis of the fractional differential equations, we refer the work of the researchers including authors (e.g., [2, 6–8, 18, 21] and [23]).

We will focus on the relations of the Voigt functions with the quadrature formula of the solution of fractional in time diffusion and wave problem. We first convert this fractional in time problem into the Sturm-Liouville problem and then find out its solution on using Green function in the form of Mercer formula [20]. The theory and applications of Sturm-Liouville problems are studied and computed by various authors (e.g., [5, 17, 28]).

2. Solution of the Problem (1.2)–(1.3)

We solve our problem (1.2)–(1.3) by setting $Y(x,t) = y(x,t) + \frac{x}{l}(\varphi_1(t) - \varphi_2(t)) - \varphi_1(t)$ and to get

$${}_t^C D_{0+}^\alpha y(x,t) = \frac{\partial}{\partial x}\left[p(x)\frac{\partial}{\partial x}\right] y(x,t) - q(x)y(x,t) + f_1(x,t), \quad 0 < \alpha \le 2, \tag{2.1}$$

for all $(x,t) \in (0,l) \times (0,\infty)$, where

$$f_1(x,t) = \left(1 - \frac{x}{l}\right) {}_t^C D_{0+}^\alpha \varphi_1(t) + \frac{x}{l} {}_t^C D_{0+}^\alpha \varphi_2(t) + \left[\left(1 - \frac{x}{l}\right) q(x) + \frac{p'(x)}{l}\right] \varphi_1(t)$$

$$+\left[\frac{x}{l}q(x)-\frac{p^{'}(x)}{l}\right]\varphi_2(t)+f(x,t),$$

along with initial and homogeneous boundary conditions, given by

$$\begin{aligned}(2.2)\qquad & y(x,0)=g(x),\quad \frac{\partial}{\partial t}y(x,t)|_{t=0}=0,\quad \text{for all } (x,t)\in[0,l]\times\{0\},\\ & y(0,t)=0,\quad \frac{\partial}{\partial x}y(x,t)|_{x=0}=1,\quad \text{for all } (x,t)\in\{0\}\times[0,\infty),\\ & y(l,t)=0,\quad \text{for all } (x,t)\in\{l\}\times[0,\infty).\end{aligned}$$

Then consider $L\{y(x,t)\}=\bar{y}(x,s)$ for $s>0$. Now using the result (1.5), and then taking Laplace transformation of (2.1) and (2.2), we find that in the form of Sturm-Liouville problem [1]

$$(2.3)\qquad \frac{\partial}{\partial x}\left[p(x)\frac{\partial}{\partial x}\right]\bar{y}(x,s)-\{q(x)+s^{\alpha}\}\bar{y}(x,s)=\bar{f_1}(x,s),$$

where,

$$\begin{aligned}\bar{f_1}(x,s)=&-s^{\alpha-1}\left\{g(x)-\left(1-\frac{x}{l}\right)\varphi_1(0)-\frac{x}{l}\varphi_2(0)\right\}\\ &+s^{\alpha-2}\left\{\left(1-\frac{x}{l}\right)\varphi_1^{'}(0)+\frac{x}{l}\varphi_2^{'}(0)\right\}-s^{\alpha}\left\{\left(1-\frac{x}{l}\right)\bar{\varphi_1}(s)+\frac{x}{l}\bar{\varphi_2}(s)\right\}\\ &-\left\{\left(1-\frac{x}{l}\right)q(x)+\frac{p^{'}(x)}{l}\right\}\bar{\varphi_1}(s)-\left\{\frac{x}{l}q(x)-\frac{p^{'}(x)}{l}\right\}\bar{\varphi_2}(s)-\bar{f}(x,s),\end{aligned}$$

$0<\alpha\leq 2$ for all $x\in(0,l)$ and $s>0$, along with homogeneous boundary conditions

$$\begin{aligned}(2.4)\qquad & \bar{y}(0,s)=0,\quad \text{for all } (x,s)\in\{0\}\times(0,\infty),\ s>0,\\ & \bar{y}(l,s)=0,\quad \text{for all } (x,s)\in\{l\}\times(0,\infty),\ s>0.\end{aligned}$$

Again, letting $\mathfrak{L}\bar{y}(x,s)=\{\frac{\partial}{\partial x}[p(x)\frac{\partial}{\partial x}]-q(x)\}\bar{y}(x,s)$, we may write the problem (2.3)–(2.4) in the form

$$(2.5)\qquad \mathfrak{L}\bar{y}(x,s)-s^{\alpha}\bar{y}(x,s)=\bar{f_1}(x,s),\quad 0<\alpha\leq 2,$$

for all $x\in(0,l)$, $(0,l)\subset\mathbb{R}$, and $s>0$, along with the boundary conditions given in (2.4).

Now, to solve the differential equation (2.5), with boundary conditions (2.4), first we construct a Green function and consider the normalized eigenfunctions (see, Churchill [1, p. 291]) $\Psi_n(x)$, for all $n=1,2,3,\ldots$, where $\Psi_n(x)=\frac{\bar{y}_n(x,s_n)}{\|\bar{y}_n(x,s_n)\|}$ for $s\geq s_n$, $s_n>0$ for all $n=1,2,3,\ldots$, and the orthonormalized property, given by

$$\int_0^l \Psi_n(t)\Psi_m(t)dt=\begin{cases}0, & m\neq n,\\ 1, & m=n.\end{cases}$$

Thus, by differential equation (2.5) with boundary conditions (2.4), we have the following homogeneous differential equation

$$\text{(2.6)}\quad \mathfrak{L}\Psi_n(x) - s_n^\alpha \Psi_n(x) = 0, \Psi_n(0) = 0, \Psi_n(l) = 0, \quad \text{for all } x \in [0, l],\ n = 1, 2, 3, \dots$$

Again then, in (2.5) and (2.6), we introduce two series

$$\bar{f}_1(x, s) = \sum_{n=1}^{\infty} A_n \Psi_n(x), \bar{y}(x, s) = \sum_{n=1}^{\infty} C_n \Psi_n(x), \tag{2.7}$$

for all $s \geq s_n,\ s_n > 0,\ A_n \neq 0,\ n = 1, 2, 3, \dots$

Then on using the relations from (2.5) to (2.7), we find following equations

$$\sum_{n=1}^{\infty} \mathfrak{L} C_n \Psi_n(x) - s^\alpha \sum_{n=1}^{\infty} C_n \Psi_n(x) = \sum_{n=1}^{\infty} A_n \Psi_n(x) \tag{2.8}$$

and

$$\sum_{n=1}^{\infty} \mathfrak{L} C_n \Psi_n(x) - \sum_{n=1}^{\infty} s_n^\alpha C_n \Psi_n(x) = 0. \tag{2.9}$$

Therefore, on use of (2.8) and (2.9), we find that

$$-\sum_{n=1}^{\infty} (s^\alpha - s_n^\alpha) C_n \int_0^l \Psi_n(x) \Psi_m(x) dx = \sum_{n=1}^{\infty} A_n \int_0^l \Psi_n(x) \Psi_m(x) dx. \tag{2.10}$$

Now, for obtaining the solution of the problem (1.2)-(1.3), for $s \geq s_n,\ s_n > 0$, we use the orthogonal property given in (2.5) and consider that $A_n[[\alpha]]_{s,s_n} = \left(\frac{B_{n_o}}{s - s_n}\right)$, $B_{n_0} \neq 0$, when $s \to s_n$ for all $n \geq n_0$ and

$$C_n = -\frac{A_n}{H(\alpha; s, s_n)}, \quad \alpha > 0, s\ \geq s_n,\ s_n > 0 \text{ for all } n = 1, 2, 3, \dots,$$

where

$$H(\alpha; s, s_n) = \begin{cases} (s - s_n)[[\alpha]]_{s,s_n}, & s > s_n > 0, \\ (s - s_n)^{-2}, & s \to s_n,\ s_n > 0, \end{cases}$$

for all $n = 1, 2, 3, \dots,\ \alpha > 0$. Here $[[\alpha]]_{s,s_n} = (s^{m-1} + s^{m-2} s_n + ... + s s_n^{m-2} + s_n^{m-1})$, $[\alpha] = m$, m is the smallest integer greater than or equal to α, then

$$C_n = \begin{cases} \dfrac{-A_n}{s^\alpha - s_n^\alpha}, & s > s_n, s_n > 0, \\ 0, & s \to s_n,\ \text{for all } n = 1, 2, 3, \dots \end{cases}$$

Again then, for $s \geq s_n,\ s_n > 0$ for all $n = 1, 2, 3, \dots$, by (2.7) and (2.10), and the orthogonal property (2.5), we may write

$$\bar{y}(x, s) = -\sum_{n=1}^{\infty} \frac{A_n}{(s^\alpha - s_n^\alpha)} \Psi_n(x), \tag{2.11}$$

and further for all $s \geq s_n$, $s_n > 0$ for all $n = 1, 2, 3, \ldots$, and by relation (2.7), we get an equality as

$$\sum_{m=1}^{\infty} \int_0^l \frac{\bar{f}_1(\xi, s)}{(s^\alpha - s_n^\alpha)} \Psi_m(x)\Psi_m(\xi) d\xi = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{A_n}{(s^\alpha - s_n^\alpha)} \Psi_m(x) \int_0^l \Psi_n(\xi)\Psi_m(\xi) d\xi.$$

Therefore, for all $m = n$, by using the orthogonal property (2.5), and the relations given in (2.11), we obtain an identity

$$\bar{y}(x, s) = \int_0^l G(x, \xi, s) \bar{f}_1(\xi, s) d\xi, \tag{2.12}$$

where the following Green function in form of Mercer formula [20] is obtained as

$$G(x, \xi, s) = -\sum_{n=1}^{\infty} \frac{\Psi_n(x)\Psi_n(\xi)}{(s^\alpha - s_n^\alpha)}, \quad s \geq s_n,\ s_n > 0 \text{ for all } n = 1, 2, 3, \ldots \tag{2.13}$$

Here in (2.12), the value of $\bar{f}_1(x, s)$ is given in (2.3) and the functions $\Psi_n(x)$ for all $n = 1, 2, 3, \ldots$, are found by the problem (2.6). Thus, by (2.3), (2.12) and (2.13), the solution of the problem (2.5) with the conditions (2.4) may be computed in the form

(2.14)

$$\begin{aligned}
&\bar{y}(x, s) \\
=&\sum_{n=1}^{\infty} \frac{\bar{y}_n(x, s_n) s^{\alpha-1}}{\|\bar{y}_n(x, s_n)\|^2 (s^\alpha - (s_n)^\alpha)} \int_0^l \bar{y}_n(\xi, s_n)(g(\xi) - \left(1 - \frac{\xi}{l}\right) \varphi_1(0) - \frac{\xi}{l}\varphi_2(0)) d\xi \\
&- \sum_{n=1}^{\infty} \frac{\bar{y}_n(x, s_n) s^{\alpha-1}}{\|\bar{y}_n(x, s_n)\|^2 s (s^\alpha - (s_n)^\alpha)} \int_0^l \bar{y}_n(\xi, s_n) \left(\left(1 - \frac{\xi}{l}\right) \varphi_1^{'}(0) + \frac{\xi}{l}\varphi_2^{'}(0)\right) d\xi \\
&+ \sum_{n=1}^{\infty} \frac{\bar{y}_n(x, s_n)}{\|\bar{y}_n(x, s_n)\|^2} \int_0^l \bar{y}_n(\xi, s_n) \left(\left(1 - \frac{\xi}{l}\right) \bar{\varphi}_1(s) + \frac{\xi}{l}\bar{\varphi}_2(s)\right) d\xi \\
&+ \sum_{n=1}^{\infty} \frac{(s_n)^\alpha \bar{y}_n(x, s_n)}{\|\bar{y}_n(x, s_n)\|^2 (s^\alpha - (s_n)^\alpha)} \int_0^l \bar{y}_n(\xi, s_n) \left(\left(1 - \frac{\xi}{l}\right) \bar{\varphi}_1(s) + \frac{\xi}{l}\bar{\varphi}_2(s)\right) d\xi \\
&+ \sum_{n=1}^{\infty} \frac{\bar{y}_n(x, s_n)(s_n)^\alpha}{(s_n)^\alpha \|\bar{y}_n(x, s_n)\|^2 (s^\alpha - (s_n)^\alpha)} \int_0^l \bar{y}_n(\xi, s_n) \left(\left(\left(1 - \frac{\xi}{l}\right) q(\xi) + \frac{p^{'}(\xi)}{l}\right) \bar{\varphi}_1(s)\right. \\
&+ \left.\left(\frac{\xi}{l} q(\xi) - \frac{p^{'}(\xi)}{l}\right) \bar{\varphi}_2(s) + \bar{f}(x, s)\right) d\xi.
\end{aligned}$$

Now, to take the inverse Laplace transformation on both of the sides of result (2.14), we have the following formulae. For $0 < \alpha \leq 2$, $|\theta| < |s^\alpha|$ and $s \geq s_n$, $s_n > 0$ for all $n = 1, 2, 3, \ldots$, the inverse Laplace transformation formula of Mittag-Leffler function $E_\alpha(z)$, where $E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}$ (see Mathai and Haubold [16, p. 80], and Kilbas, Srivastava and Trujillo [8, p. 313]), is given by

$$L^{-1}\left\{\frac{s^{\alpha-1}}{s^\alpha - \theta}\right\} = E_\alpha(\theta t^\alpha), \quad 0 < |\theta| < |s^\alpha|. \tag{2.15}$$

Again, the Laplace transformation formula of the derivative of the Mittag-Leffler function (see in (2.15)), with the aid of formula (1.5), is found in the form

$$L\left\{\frac{d}{dt}E_\alpha(\theta t^\alpha)\right\} = \frac{s^\alpha}{s^\alpha-\theta} - 1 = \frac{\theta}{s^\alpha-\theta}, \tag{2.16}$$

so that the relation (2.16) gives us

$$L^{-1}\left\{\frac{1}{s^\alpha-\theta}\right\} = \frac{1}{\theta}\frac{d}{dt}E_\alpha(\theta t^\alpha), \quad 0<|\theta|<|s^\alpha|. \tag{2.17}$$

Finally, on making an application of the results (2.15)–(2.17), we obtain

$$L^{-1}\left\{\frac{s^{\alpha-2}}{s^\alpha-\theta}\right\} = L^{-1}\left\{\frac{1}{s}\frac{s^{\alpha-1}}{s^\alpha-\theta}\right\} = \int_0^t E_\alpha(\theta\tau^\alpha)d\tau. \tag{2.18}$$

Thus, on using above results of (2.15)–(2.18) into the result (2.14), we obtain the solution of the problem (2.3)–(2.4) for all $x\in(0,l)$, and $t>0$, $s\geq s_n$, $s_n>0$ for all $n=1,2,3,\ldots$, in the form

$$\begin{aligned}
(2.19)\quad & y(x,t)\\
&=\sum_{n=1}^{\infty}\frac{y_n(x,s_n)}{\|y_n(x,s_n)\|^2}E_\alpha((s_n)^\alpha t^\alpha)\int_0^l y_n(\xi,s_n)(g(\xi)-\left(1-\frac{\xi}{l}\right)\varphi_1(0)-\frac{\xi}{l}\varphi_2(0))d\xi\\
&-\sum_{n=1}^{\infty}\frac{y_n(x,s_n)}{\|y_n(x,s_n)\|^2}\int_0^t E_\alpha((s_n)^\alpha\tau^\alpha)d\tau\int_0^l y_n(\xi,s_n)\left\{\left(1-\frac{\xi}{l}\right)\varphi_1'(0)+\frac{\xi}{l}\varphi_2'(0)\right\}d\xi\\
&+\sum_{n=1}^{\infty}\frac{y_n(x,s_n)}{\|y_n(x,s_n)\|^2}\int_0^l y_n(\xi,s_n)\left\{\left(1-\frac{\xi}{l}\right)\varphi_1(t)+\frac{\xi}{l}\varphi_2(t)\right\}d\xi\\
&+\sum_{n=1}^{\infty}\frac{y_n(x,s_n)}{\|y_n(x,s_n)\|^2}\int_0^l y_n(\xi,s_n)\left\{\left(1-\frac{\xi}{l}\right)\int_0^t\varphi_1(t-\tau)\frac{d}{d\tau}E_\alpha((s_n)^\alpha\tau^\alpha)d\tau\right.\\
&\left.+\frac{\xi}{l}\int_0^t\varphi_2(t-\tau)\frac{d}{d\tau}E_\alpha((s_n)^\alpha\tau^\alpha)d\tau\right\}d\xi\\
&+\sum_{n=1}^{\infty}\frac{y_n(x,s_n)}{(s_n)^\alpha\|y_n(x,s_n)\|^2}\int_0^l y_n(\xi,s_n)\left\{\left(\left(1-\frac{\xi}{l}\right)q(\xi)+\frac{p'(\xi)}{l}\right)\right.\\
&\times\int_0^t\varphi_1(t-\tau)\frac{d}{d\tau}E_\alpha((s_n)^\alpha\tau^\alpha)d\tau+\left(\frac{\xi}{l}q(\xi)-\frac{p'(\xi)}{l}\right)\\
&\left.\times\int_0^t\varphi_2(t-\tau)\frac{d}{d\tau}E_\alpha((s_n)^\alpha\tau^\alpha)d\tau+\int_0^t f(x,t-\tau)\frac{d}{d\tau}E_\alpha((s_n)^\alpha\tau^\alpha)d\tau\right\}d\xi.
\end{aligned}$$

Finally, putting

$$y(x,t)=Y(x,t)-\frac{x}{l}(\varphi_1(t)-\varphi_2(t))+\varphi_1(t)$$

in solution (2.19), we obtain the solution of the problem (1.2)–(1.3) for all $x\in(0,l)$ and $t>0$, $s_n>0$, for all $n=1,2,3,\ldots$, in the form

(2.20) $\quad Y(x,t)$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{y_n(x,s_n)}{\|y_n(x,s_n)\|^2} E_\alpha((s_n)^\alpha t^\alpha) \int_0^l y_n(\xi,s_n) \left(g(\xi) - \left(1-\frac{\xi}{l}\right)\varphi_1(0) - \frac{\xi}{l}\varphi_2(0)\right) d\xi \\
&- \sum_{n=1}^{\infty} \frac{y_n(x,s_n)}{\|y_n(x,s_n)\|^2} \int_0^t E_\alpha((s_n)^\alpha \tau^\alpha) d\tau \int_0^l y_n(\xi,s_n) \left(\left(1-\frac{\xi}{l}\right)\varphi_1'(0) + \frac{\xi}{l}\varphi_2'(0)\right) d\xi \\
&+ \sum_{n=1}^{\infty} \frac{y_n(x,s_n)}{\|y_n(x,s_n)\|^2} \int_0^l y_n(\xi,s_n) \left\{ \left(1-\frac{\xi}{l}\right)\varphi_1(t) + \frac{\xi}{l}\varphi_2(t)\right\} d\xi \\
&+ \sum_{n=1}^{\infty} \frac{y_n(x,s_n)}{\|y_n(x,s_n)\|^2} \int_0^l y_n(\xi,s_n) \left\{ \left(1-\frac{\xi}{l}\right) \int_0^t \varphi_1(t-\tau)\frac{d}{d\tau}E_\alpha((s_n)^\alpha \tau^\alpha) d\tau \right. \\
&\left. + \frac{\xi}{l}\int_0^t \varphi_2(t-\tau)\frac{d}{d\tau}E_\alpha((s_n)^\alpha \tau^\alpha) d\tau \right\} d\xi \\
&+ \sum_{n=1}^{\infty} \frac{y_n(x,s_n)}{(s_n)^\alpha \|y_n(x,s_n)\|^2} \int_0^l y_n(\xi,s_n) \{ \left(\left(1-\frac{\xi}{l}\right) q(\xi) + \frac{p'(\xi)}{l}\right) \\
&\times \int_0^t \varphi_1(t-\tau)\frac{d}{d\tau}E_\alpha((s_n)^\alpha \tau^\alpha) d\tau \\
&+ \left(\frac{\xi}{l} q(\xi) - \frac{p'(\xi)}{l}\right) \int_0^t \varphi_2(t-\tau)\frac{d}{d\tau}E_\alpha((s_n)^\alpha \tau^\alpha) d\tau \\
&+ \int_0^t f(x,t-\tau)\frac{d}{d\tau}E_\alpha((s_n)^\alpha \tau^\alpha) d\tau \} d\xi + \frac{x}{l}(\varphi_1(t)-\varphi_2(t)) - \varphi_1(t).
\end{aligned}
$$

Hence, from the study of the results (1.2) to (2.20), we find the following.

Theorem 2.1. *If $l > 0$ and f is a function defined by $f : [0,l] \times [0,\infty) \to \mathbb{R}$, $[0,l] \subset \mathbb{R}$, $q(x)$, $p(x)$, $p'(x)$, $p''(x)$ are continuous real valued functions of x on $0 \le x \le l$. Then there exists the normalized eigenfunctions*

$$\Psi_n(x) = \frac{y_n(x,s_n)}{\|y_n(x,s_n)\|}, \quad s_n > 0 \text{ for all } n \in \mathbb{N},$$

by the solution of boundary value problem (2.5) and with boundary conditions (2.4) simultaneously give the Green's function

$$G(x,\xi,s) = -\sum_{n=1}^{\infty} \frac{\Psi_n(x)\Psi_n(\xi)}{(s^\alpha - s_n^\alpha)},$$

provided that $s \ge s_n$, $s_n > 0$ for all $n \in \mathbb{N}$ and $0 < \alpha \le 2$, for all $x, \xi \in (0,l)$, $(0,l) \subset \mathbb{R}$ and $s > 0$, which gives the solution of the problem (2.3) with boundary values (2.4), in the form

$$\bar{y}(x,s) = -\sum_{n=1}^{\infty} \int_0^l \bar{f}_1(\xi,s) \frac{\Psi_n(x)\Psi_n(\xi)}{(s^\alpha - s_n^\alpha)} d\xi, \quad s \ge s_n,\ s_n > 0, \text{ for all } n = 1,2,3\ldots,$$

the function $\bar{f}_1(x,s)$ is given in the (2.3).

Finally, its inverse Laplace transformation gives the solution (2.20) of the fractional in time Sturm-Liouville type diffusion and wave problem (1.2)–(1.3) for all x, $0 < x < l$, $t > 0$, $s \ge s_n$, $s_n > 0$ for all $n \in \mathbb{N}$.

3. The Voigt Functions via Solution of the Problem (1.2)–(1.3) in Various Conditions

In any given physical problem, a numerical, computational or analytical evaluation of the Voigt functions (or of their variants) is required. We begin our study of Voigt functions and their relations with quadrature formula of the solution of the fractional in time Sturm-Liouville type diffusion and wave problem (1.2)–(1.3) in different particular cases and conditions.

3.1. The Voigt functions via non-homogeneous Bessel type diffusion and wave problem, when $0 < \alpha \leq 2$.

Theorem 3.1. *If we put* $f = 0$, $q(x) = 0$, $p(x) = x$ *for all* x, $0 \leq x \leq l \subset \mathbb{R}$, $\varphi_1(t) = t$, $\varphi_2(t) = t^2$, $0 < \alpha \leq 2$, $t \geq 0$, *in the problem* (1.2)–(1.3), *then our problem becomes Bessel type fractional in time diffusion-wave problem of the form*

$$
{}_t^C D_{0^+}^{\alpha} Y(x,t) = \frac{\partial}{\partial x}\left[x\frac{\partial}{\partial x}\right] Y(x,t), \quad 0 < \alpha \leq 2, \tag{3.1}
$$

for all $(x,t) \in (0,l) \times (0,\infty)$, $(0,l) \subset \mathbb{R}$, *subjected to the initial and inhomogeneous boundary values*

$$
\begin{aligned}
Y(x,0) =& g(x), \quad \frac{\partial}{\partial t} Y(x,t)|_{t=0} = \left(\frac{x}{l} - 1\right), \quad \textit{for all } (x,t) \in [0,l] \times \{0\},\ [0,l] \subset \mathbb{R}, \\
Y(0,t) + t =& 0, \quad \frac{\partial}{\partial x} Y(x,t)|_{x=0} = 1 + \frac{1}{l}(t - t^2), \quad \textit{for all } (x,t) \in \{0\} \times [0,\infty), \\
Y(l,t) + t^2 =& 0, \quad \textit{for all } (x,t) \in \{l\} \times [0,\infty).
\end{aligned} \tag{3.2}
$$

Then, solution of problem (3.1)–(3.2) *has the form*

$$
\begin{aligned}
& Y(x,t) \\
=& \frac{1}{l}\sum_{n=1}^{\infty} \frac{J_0(-\mu_n\sqrt{\frac{x}{l}})}{[J_1(-\mu_n)]^2} E_\alpha\left(-\frac{(\mu_n)^2}{4l}t^\alpha\right)\int_0^l J_0\left(-\mu_n\sqrt{\frac{\xi}{l}}\right) g(\xi)d\xi \\
&+ \frac{2}{l}\sum_{n=1}^{\infty} \frac{J_0(-\mu_n\sqrt{\frac{x}{l}})}{[J_1(-\mu_n)]^2}\int_0^t (t-\tau)\frac{d}{d\tau}E_\alpha\left(-\frac{(\mu_n)^2}{4l}\tau^\alpha\right)d\tau \int_0^l J_0\left(-\mu_n\sqrt{\frac{\xi}{l}}\right)\xi d\xi \\
&+ 8l\sum_{n=1}^{\infty} \frac{J_0(-\mu_n\sqrt{\frac{x}{l}})}{(\mu_n)^3 J_1(-\mu_n)} \\
&\times \left[\int_0^t (t-\tau)\frac{d}{d\tau}E_\alpha\left(-\frac{(\mu_n)^2}{4l}\tau^\alpha\right)d\tau - \int_0^t (t-\tau)^2\frac{d}{d\tau}E_\alpha\left(-\frac{(\mu_n)^2}{4l}\tau^\alpha\right)d\tau\right] \\
&+ \left(\frac{x}{l} - 1\right)t - \frac{x}{l}t^2, \quad \textit{for all } \mu_n \in \mathbb{R}^-,\ n = 1,2,3,\ldots
\end{aligned} \tag{3.3}
$$

Proof. Here, put $Y(x,t) = y(x,t) + \frac{x}{l}(t-t^2) - t$, in differential equation (3.1) and boundary values (3.2), and then make an appeal to the techniques applied for finding out the solution (2.20) of the problem (1.2)–(1.3) and with the aid of Theorem 2.1, we obtain the solution (3.3) of the problem (3.1)–(3.2). □

Corollary 3.1. *If $J_0(-\mu_n)=0$ for all $n=1,2,3\dots$, and for all $x,y,z\in\mathbb{R}^+$, $\Re(\gamma+\nu)>-1$, then for all $\mu_n\in\mathbb{R}^-$ under the conditions given in the differential equation* (3.1) *and boundary values* (3.2)*, the quadrature formula of the solution* (3.3) *exists and is given by the relation*

(3.4)
$$\begin{aligned}&\int_0^\infty u^\gamma e^{-yu-zu^2}J_\nu\left(-\mu_n u\sqrt{\frac{x}{l}}\right)Y(xu^2,t)du\\
&=\sum_{n=1}^{\infty}\{H_1(t;l,\mu_n)+H_2(t;l,\mu_n)+H_3(t;l,\mu_n)\}\\
&\times\left[\left(-\frac{\mu_n}{2}\sqrt{\frac{x}{l}}\right)^{\nu}\sum_{m=0}^{\infty}\frac{\Gamma(1+\nu+2m)}{(\Gamma(1+\nu+m)(1)_m)^2}\left(-(\mu_n)^2\frac{x}{4l}\right)^m I_m^{(1)}(\gamma,\nu,y,z)\right]+x\left(\frac{t}{l}-\frac{t^2}{l}\right)\\
&\times\left(-\frac{\mu_n}{2}\sqrt{\frac{x}{l}}\right)^{-1/2}V_{\gamma+2,\nu}\left(-\mu_n\sqrt{\frac{x}{l}},y,z\right)-t\left(-\frac{\mu_n}{2}\sqrt{\frac{x}{l}}\right)^{-1/2}V_{\gamma,\nu}\left(-\mu_n\sqrt{\frac{x}{l}},y,z\right).\end{aligned}$$

Here in (3.4)*, we have*

$$\begin{aligned}H_1(t;l,\mu_n)=&\frac{1}{l[J_1(-\mu_n)]^2}E_\alpha\left(-\frac{(\mu_n)^2}{4l}t^\alpha\right)\int_0^l J_0\left(-\mu_n\sqrt{\frac{\xi}{l}}\right)g(\xi)d\xi,\\
H_2(t;l,\mu_n)=&\frac{2}{l[J_1(-\mu_n)]^2}\int_0^t(t-\tau)\frac{d}{d\tau}E_\alpha\left(-\frac{(\mu_n)^2}{4l}\tau^\alpha\right)d\tau\int_0^l J_0\left(-\mu_n\sqrt{\frac{\xi}{l}}\right)\xi d\xi,\\
H_3(t;l,\mu_n)=&\frac{8l}{(\mu_n)^3J_1(-\mu_n)}\left[\int_0^t(t-\tau)\frac{d}{d\tau}E_\alpha\left(-\frac{(\mu_n)^2}{4l}\tau^\alpha\right)d\tau\right.\\
&\left.\int_0^t(t-\tau)^2\frac{d}{d\tau}E_\alpha\left(-\frac{(\mu_n)^2}{4l}\tau^\alpha\right)d\tau\right],\quad \mu_n\in\mathbb{R}^-,n=1,2,3,\dots,\end{aligned}\tag{3.5}$$

and
$$I_m^{(1)}(\gamma,\nu,\theta,\phi)=\int_0^\infty u^{\gamma+\nu+2m}e^{-\theta u-\phi u^2}du,\quad \textit{for all } m\in\mathbb{N},\ \theta,\phi\in\mathbb{R}^+,$$
(*see* [10]).

Proof. In both sides of (3.3), replace x by xu^2 and then multiply by $u^\gamma e^{-yu-zu^2}J_\nu(-\mu_{n'}u\sqrt{\frac{x}{l}})$ and then integrate both the sides with respect to u from 0 to ∞, and use (1.1) and (3.5), to get the relation

$$\begin{aligned}&\int_0^\infty u^\gamma e^{-yu-zu^2}J_\nu\left(-\mu_{n'}u\sqrt{\frac{x}{l}}\right)Y(xu^2,t)du\\
&=\sum_{n=1}^{\infty}H_1(t;l,\mu_n)\int_0^\infty u^\gamma e^{-yu-zu^2}J_\nu\left(-\mu_{n'}u\sqrt{\frac{x}{l}}\right)J_0\left(-\mu_n u\sqrt{\frac{x}{l}}\right)du\\
&+\sum_{n=1}^{\infty}H_2(t;l,\mu_n)\int_0^\infty u^\gamma e^{-yu-zu^2}J_\nu\left(-\mu_{n'}u\sqrt{\frac{x}{l}}\right)J_0\left(-\mu_n u\sqrt{\frac{x}{l}}\right)du\\
&+\sum_{n=1}^{\infty}H_3(t;l,\mu_n)\int_0^\infty u^\gamma e^{-yu-zu^2}J_\nu\left(-\mu_{n'}u\sqrt{\frac{x}{l}}\right)J_0\left(-\mu_n u\sqrt{\frac{x}{l}}\right)du\end{aligned}\tag{3.6}$$

$$+ x\left(\frac{t}{l}-\frac{t^2}{l}\right)\left(-\frac{\mu_{n'}}{2}\sqrt{\frac{x}{l}}\right)^{-1/2} V_{\gamma+2,\nu}\left(-\mu_{n'}\sqrt{\frac{x}{l}}, y, z\right)$$

$$- t\left(-\frac{\mu_{n'}}{2}\sqrt{\frac{x}{l}}\right)^{-1/2} V_{\gamma,\nu}\left(-\mu_{n'}\sqrt{\frac{x}{l}}, y, z\right), \quad \mu_n, \mu_{n'} \in \mathbb{R}^-,\ n, n' = 1, 2, 3, \ldots.$$

Now in both sides of equation (3.6), replacing n' by n and then using the following result given in Rainville [22, p. 121]

$$J_\nu\left(-\mu_n u\sqrt{\frac{x}{l}}\right) J_0\left(-\mu_n u\sqrt{\frac{x}{l}}\right) = \frac{\left(-u(\frac{\mu_n}{2})\sqrt{\frac{x}{l}}\right)^\nu}{\Gamma(\nu+1)} {}_2F_3\left[\begin{matrix}\frac{1}{2}(1+\nu), \frac{1}{2}(2+\nu); \\ 1, 1+\nu, 1+\nu;\end{matrix} - u^2(\mu_n)^2\frac{x}{l}\right],$$

and the sequence of functions of mathematical physics due to [14], given by

$$I_m^{(1)}(\gamma, \nu, \theta, \phi) = \int_0^\infty u^{\gamma+\nu+2m} e^{-\theta u - \phi u^2} du, \quad \text{for all } m \in \mathbb{N},\ \theta, \phi \in \mathbb{R}^+,$$

to obtain the result (3.4). □

3.2. **The Voigt functions via homogeneous Bessel type diffusion problem, when $0 < \alpha \leq 1$.** In a similar manner of the Theorem 3.1, we present and prove the following.

Theorem 3.2. *If we put $f = 0$, $q(x) = 0$, $p(x) = x$ for all x, $0 \leq x \leq l \subset \mathbb{R}$, $\varphi_1(t) = 0$, $\varphi_2(t) = 0$, $0 < \alpha \leq 1$, $t > 0$, in the equations* (1.2)*–*(1.3)*, then we have Bessel type one dimensional time fractional diffusion problem*

$$\text{(3.7)} \qquad {}_t^C D_{0+}^\alpha Y(x,t) = \frac{\partial}{\partial x}\left[x\frac{\partial}{\partial x}\right] Y(x,t), \quad 0 < \alpha \leq 1,$$

for all $(x,t) \in (0,l) \times (0,\infty)$, $(0,l) \subset \mathbb{R}$, *subjected to the initial and inhomogeneous boundary values*

$$\text{(3.8)} \qquad \begin{aligned} &Y(x,0) = g(x), \quad \frac{\partial}{\partial t} Y(x,t)|_{t=0} = 0, \quad \textit{for all } (x,t) \in [0,l] \times \{0\},\ [0,l] \subset \mathbb{R}, \\ &Y(0,t) = 0, \quad \frac{\partial}{\partial x} Y(x,t)|_{x=0} = 1, \quad \textit{for all } (x,t) \in \{0\} \times [0,\infty), \\ &Y(l,t) = 0, \quad \textit{for all } (x,t) \in \{l\} \times [0,\infty). \end{aligned}$$

Then there exists

$$\text{(3.9)} \qquad Y(x,t) = \frac{1}{l}\sum_{n=1}^\infty \frac{J_0\left(-\mu_n\sqrt{\frac{x}{l}}\right)}{[J_1(-\mu_n)]^2} E_\alpha\left(-\frac{(\mu_n)^2}{4l} t^\alpha\right) \int_0^l J_0\left(-\mu_n\sqrt{\frac{\xi}{l}}\right) g(\xi) d\xi,$$

$0 < x < l$, $t > 0$.

Proof. With the aid of Theorem 2.1 and Subsection 3.1, the solution of the problems (3.7)–(3.8) is found by result (3.9). □

Corollary 3.2. *If $J_0(-\mu_n) = 0$ for all $n = 1, 2, 3 \ldots$, and for all $x, y, z \in \mathbb{R}^+$, $\Re(\gamma+\nu) > -1$, then for all $\mu_n \in \mathbb{R}^-$ under the conditions given in* (3.7) *and* (3.8)*, the quadrature formula of the solution* (3.9) *exists and is given by the relation*

(3.10)

$$\int_0^\infty u^\gamma e^{-yu - zu^2} J_\nu\left(-\mu_n u\sqrt{\frac{x}{l}}\right) Y(xu^2, t) du = \sum_{n=1}^\infty H_1(t; l, \mu_n)\left(-\frac{\mu_n}{2}\sqrt{\frac{x}{l}}\right)^\nu$$

$$\times \sum_{m=0}^{\infty} \frac{\Gamma(1+\nu+2m)}{(\Gamma(1+\nu+m)(1)_m)^2} \left(-(\mu_n)^2 \frac{x}{4l}\right)^m$$
$$\times I_m^{(1)}(\gamma, \nu, y, z),$$

where $H_1(t; l, \mu_n)$ *and* $I_m^{(1)}(\gamma, \nu, \theta, \phi)$ *are given in* (3.5).

4. Numerical Example

In this section, we consider more briefly a computational formula starting from $Y(x,t)$, $0 < x < l$, $t \in \mathbb{R}^+$ and using the Theorem 3.2.

If we set $g(x) = \frac{1}{2}$ for all x, $0 < x < l$, in (3.9), we find a numerical formula

$$Y(x,t) = \sum_{n=1}^{\infty} \frac{J_0\left(-\mu_n\sqrt{\frac{x}{l}}\right)}{(-\mu_n)J_1(-\mu_n)} E_\alpha\left(-\frac{(\mu_n)^2}{4l}t^\alpha\right). \tag{4.1}$$

A fairly immediate consequence of this result is its use for obtaining the approximate various real values of $Y(x,t)$. According to our formalism we now in (4.1), introduce the approximate value of $E_\alpha(-x)$, given by (see [27])

$$E_\alpha(-x) = \frac{1 + \frac{1}{\Gamma(1-\alpha)q_0^*}x}{1 + \frac{q_1^*}{q_0^*}x + \frac{1}{q_0^*}x^2},$$

where

$$q_0^* = \frac{\frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)} - \frac{\Gamma(1+\alpha)\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}}{\Gamma(1+\alpha)\Gamma(1-\alpha) - 1}$$

and

$$q_1^* = \frac{\Gamma(1+\alpha) - \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}}{\Gamma(1+\alpha)\Gamma(1-\alpha) - 1}.$$

Again from the formula (1.1), it follows that

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2}\left(\frac{x}{2}\right)^{2m}, \quad J_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)(m!)^2}\left(\frac{x}{2}\right)^{2m+1}.$$

Now putting the zeros of $J_0(x)$ as μ_i, $i = 1, 2, \ldots, n$, together with the values of α and l such that $0 < \alpha < 1$ and $l > 0$, we can provide several examples with selected values of n to compute and approximate various real values of $Y(x,t)$, for all $x, t \in \mathbb{R}^+$. We omit them due to lack of space and left them for further researchers in the field of computer science and technology.

Conclusion

Explicit expressions for the generalized Voigt functions [10,19,25] and [26] of the second kind defined by the Hankel integral transform (1.1) are given in terms of relatively more familiar special functions of one and more variables, indeed, each of these representations will naturally lead to various other needed properties of the Voigt functions. Here, in our work, we have obtained the relations of the Voigt functions with the quadrature formula of the solution of fractional in time diffusion and wave problem by first converting it into the Sturm-Liouville problems and then looked out for its solutions. This concept may provide the basis of investigations and further extensions for a high voltage technology to compute the fractional differential equations, anomalous diffusion problems and fractional in time and space diffusion and wave problems with the help of Voigt functions.

To explore new ideas for representing the relation of the Voigt functions (1.1) with the quadrature formula of the solution of fractional in time diffusion and wave problem, in our current investigation, we have presented fractional in time Sturm-Liouville type diffusion and wave equation. In the paper of Luchko [14] (see also [15]), some initial-boundary-value problems with the Dirichlet boundary conditions for the time-fractional diffusion equation were considered. Of course, the same method can be applied for the initial boundary value problems with the Neumann, Robin, or mixed boundary conditions.

Besides establishing some interesting integral and series representations of special functions, the results given in [13] and [14] may provide a new way of solution of a space-time fractional anomalous diffusion problem using the series of bilateral eigenfunctions and series solution for initial value problems of time fractional generalized anomalous diffusion equations as on the lines of [11, 12] and [13].

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