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STABILITY OF NONLINEAR NEUTRAL MIXED TYPE LIVEN-NOHEL INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we use the contraction mapping theorem to obtain asymptotic stability results about the zero solution for a nonlinear neutral mixed type Levin-Nohel integro-differential equation. An asymptotic stability theorem with a necessary and sufficient condition is proved. An example is also given to illustrate our main results.

1. Introduction

The Lyapunov direct method has been very effective in establishing stability results and the existence of periodic solutions for wide variety of ordinary, functional and partial differential equations. Nevertheless, in the application of Lyapunov's direct method to problems of stability in delay differential equations, serious difficulties occur if the delay is unbounded or if the equation has unbounded terms. In recent years, several investigators have tried stability by using a new technique. Particularly, Burton, Furumochi, Zhang and others began a study in which they noticed that some of this difficulties vanish or might be overcome by means of fixed point theory, see [1–28] and the references therein. The fixed point theory does not only solve the problems on stability but have other significant advantage over Lyapunov's direct method. The conditions of the former are often average but those of the latter are usually pointwise, see [8] and the references therein.

Key words and phrases. Asymptotic stability, contraction mapping theorem, neutral integro-differential equations, mixed type.

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In paper, we consider the following nonlinear neutral mixed type Levin-Nohel integro-differential equation

$$\begin{aligned}\frac{d}{dt}x(t) = &-\sum_{j=1}^{m}\int_{t-\tau_j(t)}^{t} a_j(t,s)\,x(s)\,ds - \sum_{j=1}^{m}\int_{t}^{t+\sigma_j(t)} b_j(t,s)\,x(s)\,ds \\ &+\frac{d}{dt}g\left(t, x(t-\tau_1(t)), \ldots, x(t-\tau_m(t))\right),\end{aligned} \tag{1.1}$$

with an assumed initial condition

$$x(t) = \phi(t), \quad t \in [m(t_0), t_0],$$

where $\phi \in C([m(t_0), t_0], \mathbb{R})$ and

$$m_j(t_0) = \inf\{t - \tau_j(t)\}, \quad m(t_0) = \min\{m_j(t_0) : 1 \le j \le m\}.$$

Throughout this chapter, we assume that $a_j \in C([t_0, +\infty) \times [m(t_0), +\infty), \mathbb{R})$, $b_j \in C([t_0, +\infty) \times [t_0, +\infty), \mathbb{R})$ and $\tau_j, \sigma_j \in C([t_0, +\infty), \mathbb{R}^+)$, with $t - \tau_j(t) \to +\infty$ as $t \to +\infty$ and $t + \sigma_j(t) \to +\infty$ as $t \to +\infty$, $(1 \le j \le m)$. The functions g is globally Lipschitz continuous in x. That is, there are positive constants E_j, $1 \le j \le m$, such that

$$|g(t, x_1, x_2, \ldots, x_m) - g(t, y_1, y_2, \ldots, y_m)| \le \sum_{j=1}^{m} E_j |x_j - y_j|, \quad g(t, 0, \ldots, 0) = 0. \tag{1.2}$$

In this paper, our purpose is to use the contraction mapping theorem [26] to show the asymptotic stability of the zero solution for (1.1). An asymptotic stability theorem with a necessary and sufficient condition is proved. In the special case $b_j(t,s) = 0$, $1 \le j \le m$ and $g(t, x_1, x_2, \ldots, x_m) = \sum_{j=1}^{m} g_j(t, x_j)$, Bessioud, Ardjouni and Djoudi [5] proved the zero solution of (1.1) is asymptotically stable with a necessary and sufficient condition by using the contraction mapping theorem. Then, the results presented in this paper extend the main results in [5]. An example is also given to illustrate our main results.

2. Main Results

For each t_0, we denote $C(t_0)$ the space of continuous functions on $[m(t_0), t_0]$ with the supremum norm $\|\cdot\|_{t_0}$. For each $(t_0, \phi) \in [0, +\infty) \times C(t_0)$, denote by $x(t) = x(t, t_0, \phi)$ the unique solution of (1.1).

Definition 2.1. The zero solution of (1.1) is called

(i) stable if for each $\epsilon > 0$ there exists a $\delta > 0$ such that $|x(t, t_0, \phi)| < \epsilon$ for all $t \ge t_0$ if $\|\phi\|_{t_0} < \delta$,

(ii) asymptotically stable if it is stable and $\lim_{t \to +\infty} |x(t, t_0, \phi)| = 0$.

In order to be able to construct a new fixed mapping, we transform the Levin-Nohel equation into an equivalent equation. For this, we use the variation of parameter formula and the integration by parts.

Lemma 2.1. *x is a solution of* (1.1) *if and only if*

$$\begin{aligned} x(t) = & \left(\phi(t_0) - G_\phi(t_0)\right) e^{-\int_{t_0}^t A(z)dz} + G_x(t) \\ & - \int_{t_0}^t \left[L_x(s) + A(s) G_x(s)\right] e^{-\int_s^t A(z)dz} ds - \int_{t_0}^t N_x(s) e^{-\int_s^t A(z)dz} ds, \end{aligned} \tag{2.1}$$

where

$$G_x(t) = g\left(t, x(t-\tau_1(t)), \ldots, x(t-\tau_m(t))\right), \tag{2.2}$$

$$\begin{aligned} L_x(t) = & \sum_{j=1}^m \int_{t-\tau_j(t)}^t a_j(t,s) \left(\int_s^t \left(\sum_{k=1}^m \int_{u-\tau_k(u)}^u a_k(u,\nu) x(\nu) d\nu \right.\right. \\ & \left.\left. + \sum_{k=1}^m \int_u^{u+\sigma_k(u)} b_k(u,\nu) x(\nu) d\nu \right) du + G_x(s) - G_x(t) \right) ds, \end{aligned} \tag{2.3}$$

$$\begin{aligned} N_x(t) = & \sum_{j=1}^m \int_t^{t+\sigma_j(t)} b_j(t,s) \left(\int_s^t \left(\sum_{k=1}^m \int_{u-\tau_k(u)}^u a_k(u,\nu) x(\nu) d\nu \right.\right. \\ & \left.\left. + \sum_{k=1}^m \int_u^{u+\sigma_k(u)} b_k(u,\nu) x(\nu) d\nu \right) du + G_x(s) - G_x(t) \right) ds, \end{aligned} \tag{2.4}$$

and

$$A(t) = \sum_{j=1}^m \int_{t-\tau_j(t)}^t a_j(t,s) ds + \sum_{j=1}^m \int_t^{t+\sigma_j(t)} b_j(t,s) ds. \tag{2.5}$$

Proof. Obviously, we have

$$x(s) = x(t) - \int_s^t \frac{\partial}{\partial u} x(u) du.$$

Inserting this relation into (1.1), we get

$$\begin{aligned} & \frac{d}{dt} x(t) + \sum_{j=1}^m \int_{t-\tau_j(t)}^t a_j(t,s) \left(x(t) - \int_s^t \frac{\partial}{\partial u} x(u) du \right) ds \\ & + \sum_{j=1}^m \int_t^{t+\sigma_j(t)} b_j(t,s) \left(x(t) - \int_s^t \frac{\partial}{\partial u} x(u) du \right) ds - \frac{d}{dt} G_x(t) = 0, \end{aligned}$$

where G_x is given by (2.2). Or equivalently

$$\begin{aligned} & \frac{d}{dt} x(t) + x(t) \left(\sum_{j=1}^m \int_{t-\tau_j(t)}^t a_j(t,s) ds + \sum_{j=1}^m \int_t^{t+\sigma_j(t)} b_j(t,s) ds \right) \\ & - \sum_{j=1}^m \int_{t-\tau_j(t)}^t a_j(t,s) \left(\int_s^t \frac{\partial}{\partial u} x(u) du \right) ds \\ & - \sum_{j=1}^m \int_t^{t+\sigma_j(t)} b_j(t,s) \left(\int_s^t \frac{\partial}{\partial u} x(u) du \right) ds - \frac{d}{dt} G_x(t) = 0. \end{aligned}$$

Substituting $\frac{\partial x}{\partial u}$ from (1.1), we obtain

$$\begin{aligned}
&\frac{d}{dt}x(t)+x(t)\left(\sum_{j=1}^{m}\int_{t-\tau_j(t)}^{t}a_j(t,s)\,ds+\sum_{j=1}^{m}\int_{t}^{t+\sigma_j(t)}b_j(t,s)\,ds\right)\\
&-\sum_{j=1}^{m}\int_{t-\tau_j(t)}^{t}a_j(t,s)\left[\int_s^t\left(-\sum_{k=1}^{m}\int_{u-\tau_k(u)}^{u}a_k(u,\nu)\,x(\nu)\,d\nu\right.\right.\\
&\left.\left.-\sum_{k=1}^{m}\int_{u}^{u+\sigma_k(u)}b_k(u,\nu)\,x(\nu)\,d\nu+\frac{\partial}{\partial u}G_x(u)\right)du\right]ds\\
&-\sum_{j=1}^{m}\int_{t}^{t+\sigma_j(t)}b_j(t,s)\left[\int_s^t\left(-\sum_{k=1}^{m}\int_{u-\tau_k(u)}^{u}a_k(u,\nu)\,x(\nu)\,d\nu\right.\right.\\
&\left.\left.-\sum_{k=1}^{m}\int_{u}^{u+\sigma_k(u)}b_k(u,\nu)\,x(\nu)\,d\nu+\frac{\partial}{\partial u}G_x(u)\right)du\right]ds-\frac{d}{dt}G_x(t)\;=\;0.
\end{aligned}\tag{2.6}$$

By performing the integration, we have

$$\int_s^t\frac{\partial}{\partial u}G_x(u)\,du=G_x(t)-G_x(s)\,.\tag{2.7}$$

Substituting (2.7) into (2.6), we have

$$\frac{d}{dt}x(t)+A(t)\,x(t)+L_x(t)+N_x(t)-\frac{d}{dt}G_x(t)=0,\quad t\geq t_0,$$

where A and L_x and N_x are given by (2.5) and (2.3) and (2.4) respectively. By the variation of constants formula, we get

$$\begin{aligned}
x(t)=&\phi(t_0)\,e^{-\int_{t_0}^{t}A(z)dz}-\int_{t_0}^{t}\left[L_x(s)+N_x(s)\right]e^{-\int_s^tA(z)dz}ds\\
&+\int_{t_0}^{t}\left(\frac{\partial}{\partial s}G_x(s)\right)e^{-\int_s^tA(z)dz}ds.
\end{aligned}\tag{2.8}$$

By using the integration by parts, we obtain

$$\begin{aligned}
&\int_{t_0}^{t}\left(\frac{\partial}{\partial s}G_x(s)\right)e^{-\int_s^tA(z)dz}ds\\
=&G_x(t)-G_x(t_0)\,e^{-\int_{t_0}^{t}A(z)dz}-\int_{t_0}^{t}A(s)\,G_x(s)\,e^{-\int_s^tA(z)dz}ds.
\end{aligned}\tag{2.9}$$

Finally, we obtain (2.1) by substituting (2.9) in (2.8). Since each step is reversible, the converse follows easily. This completes the proof. □

Theorem 2.1. *Let* (1.2) *holds and suppose that the following two conditions hold*

$$\lim_{t\to+\infty}\inf\int_0^tA(z)\,dz>-\infty,\tag{2.10}$$

$$\sup_{t\geq0}\left(\sum_{j=1}^{m}E_j+\int_0^t\omega(s)\,e^{-\int_s^tA(z)dz}ds\right)=\alpha<1,\tag{2.11}$$

where

$$\begin{aligned}\omega(s) =& \sum_{j=1}^{m} \int_{s-\tau_j(s)}^{s} |a_j(s,w)| \left(\int_w^s \left(\sum_{k=1}^{m} \int_{u-\tau_k(u)}^{u} |a_k(u,\nu)| \, d\nu \right.\right. \\ &\left.\left. + \sum_{k=1}^{m} \int_u^{u+\sigma_k(u)} |b_k(u,\nu)| \, d\nu \right) du + 2\sum_{k=1}^{m} E_k \right) dw \\ &+ \sum_{j=1}^{m} \int_s^{s+\sigma_j(s)} |b_j(s,w)| \left(\int_w^s \left(\sum_{k=1}^{m} \int_{u-\tau_k(u)}^{u} |a_k(u,\nu)| \, d\nu \right.\right. \\ &\left.\left. + \sum_{k=1}^{m} \int_u^{u+\sigma_k(u)} |b_k(u,\nu)| \, d\nu \right) du + 2\sum_{k=1}^{m} E_k \right) dw + |A(s)| \sum_{j=1}^{m} E_j.\end{aligned}$$

Then the zero solution of (1.1) *is asymptotically stable if and only if*

$$\int_0^t A(z)\, dz \to +\infty \quad as \quad t \to +\infty. \tag{2.12}$$

Proof. **Sufficient condition.** Suppose that (2.12) holds. Denoted by C the space of continuous bounded functions $x : [m(t_0), +\infty) \to \mathbb{R}$ such that $x(t) = \phi(t)$, $t \in [m(t_0), t_0]$. It is known that C is a complete metric space endowed with a metric $\|x\| = \sup_{t \geq m(t_0)} |x(t)|$. Define the operator P on C by $(Px)(t) = \phi(t)$, $t \in [m(t_0), t_0]$ and

$$\begin{aligned}(Px)(t) =& (\phi(t_0) - G_\phi(t_0))\, e^{-\int_{t_0}^t A(z)dz} + G_x(t) \\ &- \int_{t_0}^t [L_x(s) + A(s) G_x(s)]\, e^{-\int_s^t A(z)dz} ds - \int_{t_0}^t N_x(s)\, e^{-\int_s^t A(z)dz} ds.\end{aligned}$$

Obviously, Px is continuous for each $x \in C$. Moreover, it is a contraction operator. Indeed, let $x, y \in C$

$$\begin{aligned}&|(Px)(t) - (Py)(t)| \\ \leq& |G_x(t) - G_y(t)| + \int_{t_0}^t [|L_x(s) - L_y(s)| + |N_x(s) - N_y(s)| \\ &+ |A(s)|\, |G_x(s) - G_y(s)|]\, e^{-\int_s^t A(z)dz} ds.\end{aligned}$$

Since $x(t) = y(t) = \phi(t)$ for all $t \in [m(t_0), t_0]$, this implies that

$$\begin{aligned}&|L_x(s) - L_y(s)| \\ \leq& \sum_{j=1}^{m} \int_{s-\tau_j(s)}^{s} |a_j(s,w)| \left(\int_w^s \left(\sum_{k=1}^{m} \int_{u-\tau_k(u)}^{u} |a_k(u,\nu)| \, d\nu \right.\right. \\ &\left.\left. + \sum_{k=1}^{m} \int_u^{u+\sigma_k(u)} |b_k(u,\nu)| \, d\nu \right) du + 2\sum_{k=1}^{m} E_k \right) dw \, \|x - y\|\end{aligned}$$

and

$$\begin{aligned}
&\left|N_x(s)-N_y(s)\right| \\
\leq&\sum_{j=1}^{m}\int_{s}^{s+\sigma_j(s)}\left|b_j(s,w)\right|\left(\int_{w}^{s}\left(\sum_{k=1}^{m}\int_{u-\tau_k(u)}^{u}\left|a_k(u,\nu)\right|d\nu\right.\right. \\
&\left.\left.+\sum_{k=1}^{m}\int_{u}^{u+\sigma_k(u)}\left|b_k(u,\nu)\right|d\nu\right)du+2\sum_{k=1}^{m}E_k\right)dw\left\|x-y\right\|.
\end{aligned}$$

Consequently, it holds for all $t \geq t_0$ that

$$\begin{aligned}
&\left|(Px)(t)-(Py)(t)\right| \\
\leq&\sum_{j=1}^{m}E_j\left\|x-y\right\|+\int_{t_0}^{t}\left(\sum_{j=1}^{m}\int_{s-\tau_j(s)}^{s}\left|a_j(s,w)\right|\left(\int_{w}^{s}\left(\sum_{k=1}^{m}\int_{u-\tau_k(u)}^{u}\left|a_k(u,\nu)\right|d\nu\right.\right.\right. \\
&\left.\left.+\sum_{k=1}^{m}\int_{u}^{u+\sigma_k(u)}\left|b_k(u,\nu)\right|d\nu\right)du+2\sum_{k=1}^{m}E_k\right)dw \\
&+\sum_{j=1}^{m}\int_{s}^{s+\sigma_j(s)}\left|b_j(s,w)\right|\left(\int_{w}^{s}\left(\sum_{k=1}^{m}\int_{u-\tau_k(u)}^{u}\left|a_k(u,\nu)\right|d\nu\right.\right. \\
&\left.\left.\left.+\sum_{k=1}^{m}\int_{u}^{u+\sigma_k(u)}\left|b_k(u,\nu)\right|d\nu\right)du+2\sum_{k=1}^{m}E_k\right)dw+\left|A(s)\right|\sum_{j=1}^{m}E_j\right) \\
&\times e^{-\int_s^t A(z)dz}ds\left\|x-y\right\|.
\end{aligned}$$

Hence, it follows from (2.11) that

$$\left|(Px)(t)-(Py)(t)\right| \leq \alpha\left\|x-y\right\|, \quad t \geq t_0.$$

Thus P is a contraction operator on C. We now consider a closed subspace S of C that is defined by

$$S=\left\{x \in C : \left|x(t)\right| \to 0 \text{ as } t \to +\infty\right\}.$$

We will show that $P(S) \subset S$. To do this, we need to point out that for each $x \in S$, $\left|(Px)(t)\right| \to 0$ as $t \to +\infty$. Let $x \in S$, by the definition of P we have

$$\begin{aligned}
(Px)(t)=&\left(\phi(t_0)-G_\phi(t_0)\right)e^{-\int_{t_0}^t A(z)dz}+G_x(t) \\
&-\int_{t_0}^{t}\left[L_x(s)+A(s)G_x(s)\right]e^{-\int_s^t A(z)dz}ds-\int_{t_0}^{t}N_x(s)e^{-\int_s^t A(z)dz}ds \\
=&I_1+I_2+I_3+I_4.
\end{aligned}$$

The first term I_1 tends to 0 by (2.12) and I_2 tends to 0 by (1.2) and $t-\tau_j(t) \to +\infty$ as $t \to +\infty$, and $t+\sigma_j(t) \to +\infty$ as $t \to +\infty$. For any $T \in (t_0, t)$, we have the following estimate for the third term

$$
\begin{aligned}
|I_3| \leq & \left| \int_{t_0}^{T} \left[L_x(s) + A(s) G_x(s) \right] e^{-\int_s^t A(z)dz} ds \right| \\
& + \left| \int_{T}^{t} \left[L_x(s) + A(s) G_x(s) \right] e^{-\int_s^t A(z)dz} ds \right| \\
\leq & \left[\int_{t_0}^{T} \left(\sum_{j=1}^{m} \int_{s-\tau_j(s)}^{s} |a_j(s,w)| \left(\int_{w}^{s} \left(\sum_{k=1}^{m} \int_{u-\tau_k(u)}^{u} |a_k(u,\nu)| \, d\nu \right. \right. \right. \right. \\
& \left. + \sum_{k=1}^{m} \int_{u}^{u+\sigma_k(u)} b_k(u,\nu) \, d\nu \right) du \\
& \left. \left. \left. + 2 \sum_{k=1}^{m} E_k \right) dw + |A(s)| \sum_{j=1}^{m} E_j \right) e^{-\int_s^t A(z)dz} ds \right] \left(\|x\| + \|\phi\|_{t_0} \right) \\
& + \int_{T}^{t} \left(\sum_{j=1}^{m} \int_{s-\tau_j(s)}^{s} |a_j(s,w)| \left(\int_{w}^{s} \left(\sum_{k=1}^{m} \int_{u-\tau_k(u)}^{u} |a_k(u,\nu)| \, |x(\nu)| \, d\nu \right. \right. \right. \\
& \left. + \sum_{k=1}^{m} \int_{u}^{u+\sigma_k(u)} |b_k(u,\nu)| \, |x(v)| \, d\nu \right) du \\
& \left. + \sum_{k=1}^{m} E_k \, |x(s-\tau_k(s))| + \sum_{k=1}^{m} E_k \, |x(w-\tau_k(w))| \right) dw \\
& \left. + |A(s)| \sum_{j=1}^{m} E_j \, |x(s-\tau_j(s))| \right) e^{-\int_s^t A(z)dz} ds \\
= & I_{31} + I_{32}.
\end{aligned}
$$

Since $t - \tau_j(t) \to +\infty$ as $t \to +\infty$, and $t + \sigma_j(t) \to +\infty$ as $t \to +\infty$, this implies that $u - \tau_k(u) \to +\infty$, and $u + \sigma_k(u) \to +\infty$ as $T \to +\infty$. Thus, from the fact $|x(\nu)| \to 0$, $\nu \to +\infty$, we can infer that for any $\epsilon > 0$ there exists $T_1 = T > t_0$ such that

$$
\begin{aligned}
I_{32} < & \frac{\epsilon}{2\alpha} \int_{T_1}^{t} \left(\sum_{j=1}^{m} \int_{s-\tau_j(s)}^{s} |a_j(s,w)| \left(\int_{w}^{s} \left(\sum_{k=1}^{m} \int_{u-\tau_k(u)}^{u} |a_k(u,\nu)| \, d\nu \right. \right. \right. \\
& \left. \left. \left. + \sum_{k=1}^{m} \int_{u}^{u+\sigma_k(u)} |b_k(u,\nu)| \, d\nu \right) du + 2 \sum_{k=1}^{m} E_k \right) dw + |A(s)| \sum_{j=1}^{m} E_j \right) e^{-\int_s^t A(z)dz} ds,
\end{aligned}
$$

and hence, $I_{32} < \frac{\epsilon}{2}$ for all $t \geq T_1$. On the other hand, $\|x\| < +\infty$ because $x \in S$. This combined with (2.12) yields $I_{31} \to 0$ as $t \to +\infty$. As a consequence, there exists $T_2 \geq T_1$ such that $I_{31} < \frac{\epsilon}{2}$ for all $t \geq T_2$. Thus, $I_3 < \epsilon$ for all $t \geq T_2$, that is, $I_3 \to 0$ as $t \to +\infty$. Similarly, $I_4 \to 0$ as $t \to +\infty$. So, $P(S) \subset S$.

By the Contraction Mapping Principle, P has a unique fixed point x in S which is a solution of (1.1) with $x(t) = \phi(t)$ on $[m(t_0), t_0]$ and $x(t) = x(t, t_0, \phi) \to 0$ as

$t \to +\infty$. To obtain the asymptotic stability, we need to show that the zero solution of (1.1) is stable. By condition (2.10), we can define

$$K = \sup_{t\geq 0} e^{-\int_0^t A(z)dz} < +\infty. \tag{2.13}$$

Using the formula (2.1) and condition (2.11), we can obtain

$$|x(t)| \leq K\left(1+\sum_{j=1}^{m} E_j\right)\|\phi\|_{t_0} e^{\int_0^{t_0} A(z)dz} + \alpha\left(\|x\| + \|\phi\|_{t_0}\right), \quad t \geq t_0,$$

which leads us to

$$\|x\| \leq \frac{K\left(1+\sum\limits_{j=1}^{m} E_j\right)e^{\int_0^{t_0} A(z)dz} + \alpha}{1-\alpha}\|\phi\|_{t_0}. \tag{2.14}$$

Thus, for every $\epsilon > 0$, we can find $\delta > 0$ such that $\|\phi\|_{t_0} < \delta$ implies that $\|x\| < \epsilon$. This shows that the zero solution of (1.1) is stable and hence, it is asymptotically stable.

Necessary condition. Suppose that the zero solution of (1.1) is asymptotically stable and that the condition (2.12) fails. It follows from (2.10) that there exists a sequence $\{t_n\}$, $t_n \to +\infty$ as $n \to +\infty$ such that $\lim_{n\to+\infty}\int_0^{t_n} A(z)\,dz$ exists and is finite. Hence, we can choose a positive constant L satisfying

$$-L < \lim_{n\to+\infty}\int_0^{t_n} A(z)\,dz < L, \quad \text{for all } n \geq 1. \tag{2.15}$$

Then condition (2.11) gives us

$$c_n = \int_0^{t_n} \omega(s)\, e^{\int_0^s A(z)dz} ds \leq \alpha e^{\int_0^{t_n} A(z)dz} \leq e^L.$$

The sequence $\{c_n\}$ is increasing and bounded, so it has a finite limit. For any $\delta_0 > 0$, there exists $n_0 > 0$ such that

$$\int_{t_{n_0}}^{t_n} \omega(s)\, e^{\int_0^s A(z)dz} ds < \frac{\delta_0}{2K}, \quad \text{for all } n \geq n_0, \tag{2.16}$$

where K is as in (2.13). We choose δ_0 such that $\delta_0 < \frac{1-\alpha}{K\left(1+\sum_{j=1}^{m} E_j\right)e^L+1}$, and consider the solution $x(t) = x(t, t_{n_0}, \phi)$ of (1.1), with the initial data $\phi(t_{n_0}) = \delta_0$ and $|\phi(s)| \leq \delta_0$, $s < t_{n_0}$. It follows from (2.14) that

$$|x(t)| \leq 1-\delta_0, \quad \text{for all } t \geq t_{n_0}. \tag{2.17}$$

Applying the fundamental inequality $|a-b| \geq |a|-|b|$ and then using (2.17), (2.16) and (2.15), we get

$$\begin{aligned}
&|x(t_n)-G_x(t_n)| \\
\geq &\delta_0 e^{-\int_{t_{n_0}}^{t_n} A(z)dz} - \int_{t_{n_0}}^{t_n} \omega(s) e^{-\int_s^{t_n} A(z)dz} ds \\
\geq &e^{-\int_{t_{n_0}}^{t_n} A(z)dz} \left(\delta_0 - e^{-\int_0^{t_{n_0}} A(z)dz} \int_{t_{n_0}}^{t_n} \omega(s) e^{\int_0^s A(z)dz} ds\right) \\
\geq &e^{-\int_{t_{n_0}}^{t_n} A(z)dz} \left(\delta_0 - K \int_{t_{n_0}}^{t_n} \omega(s) e^{\int_0^s A(z)dz} ds\right) \\
\geq &\frac{1}{2}\delta_0 e^{-\int_{t_{n_0}}^{t_n} A(z)dz} \geq \frac{1}{2}\delta_0 e^{-2L} > 0,
\end{aligned}$$

which is a contradiction because, then $(x(t_n)-G_x(t_n)) \to 0$ as $t_n \to +\infty$. The proof is complete. □

Letting $G_x(t_n)=0$, we get the following result.

Corollary 2.1. *Suppose that the following two conditions hold:*

$$\liminf_{t\to+\infty} \int_0^t A_0(z)\,dz > -\infty \tag{2.18}$$

and

$$\begin{aligned}
&\sup_{t\geq 0} \int_0^t \Bigg[\sum_{j=1}^m \int_{s-\tau_j(s)}^s |a_j(s,w)| \left(\int_w^s \left(\sum_{k=1}^m \int_{u-\tau_k(u)}^u |a_k(u,\nu)|\,d\nu\right.\right. \\
&\left.\left.+\sum_{k=1}^m \int_u^{u+\sigma_k(u)} |b_k(u,\nu)|\,d\nu\right) du\right) dw \\
&+\sum_{j=1}^m \int_s^{s+\sigma_j(s)} |b_j(s,w)| \int_w^s \left(\left(\sum_{k=1}^m \int_{u-\tau_k(u)}^u |a_k(u,\nu)|\,d\nu\right.\right. \\
&\left.\left.+\sum_{k=1}^m \int_u^{u+\sigma_k(u)} |b_k(u,\nu)|\,d\nu\right) du\right) dw\Bigg] e^{-\int_s^t A_0(z)dz} ds \\
&=\alpha < 1,
\end{aligned}\tag{2.19}$$

where

$$A_0(z) = \sum_{j=1}^m \int_{z-\tau_j(z)}^z a_j(z,s)\,ds + \sum_{j=1}^m \int_z^{z+\sigma_j(z)} b_j(z,s)\,ds.$$

Then the zero solution of

$$\frac{d}{dt}x(t) + \sum_{j=1}^m \int_{t-\tau_j(t)}^t a_j(t,s)\,x(s)\,ds + \sum_{j=1}^m \int_t^{t+\sigma_j(t)} b_j(t,s)\,x(s)\,ds = 0$$

is asymptotically stable if and only if

$$\int_0^t A_0(z)\,dz \to +\infty \quad as \quad t \to +\infty. \tag{2.20}$$

Example 2.1. Consider the following nonlinear neutral mixed type Levin-Nohel integro-differential equation

$$\frac{d}{dt}x(t) = \int_{t-\tau(t)}^{t} a(t,s)\,ds + \int_t^{t+\sigma(t)} b(t,s)\,x(s)\,ds + \frac{d}{dt}g(t, x(t-\tau(t))), \tag{2.21}$$

where $a(t,s) = \frac{2}{t^2+1}$, $\tau(t) = \frac{t}{2}$, $b(t,s) = \frac{1}{t^2+1}$, $\sigma(t) = t$, $g(t,x) = 0.08\,(x\cos t + 3) + 0.09x\sin t^2$. Then the zero solution of (2.21) is asymptotically stable.

Proof. We have

$$A(t) = \int_{t-\tau(t)}^{t} a(t,s)\,ds + \int_t^{t+\sigma(t)} b(t,s)\,ds = 2\frac{t}{t^2+1}, \quad \int_0^t A(z)\,dz = \ln\left(t^2+1\right),$$

$$\begin{aligned}
\omega(s) =& \int_{\frac{s}{2}}^{s} \frac{2}{s^2+1}\left(\int_w^s \left(\int_{\frac{u}{2}}^{u} \frac{2}{u^2+1}d\nu + \int_u^{2u}\frac{1}{u^2+1}d\nu\right)du + 0.34\right)dw \\
&+ \int_s^{2s} \frac{1}{s^2+1}\left(\int_w^s \left(\int_{\frac{u}{2}}^{u} \frac{2}{u^2+1}d\nu + \int_u^{2u}\frac{1}{u^2+1}d\nu\right)du + 0.34\right)dw + \frac{0.34s}{s^2+1} \\
=& \frac{1}{s^2+1}\Bigg[s\ln\left(s^2+1\right) + 4\arctan\frac{s}{2} - 2\arctan s \\
&+ s\ln\left(\frac{s^2}{4}+1\right) - 2\arctan 2s - 2s\ln\left(4s^2+1\right) + 5.02s\Bigg]
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t \omega(s)\,e^{-\int_s^t A(z)dz}dz \\
=& \frac{1}{t^2+1}\int_0^t \Bigg[s\ln\left(s^2+1\right) + 4\arctan\frac{s}{2} - 2\arctan s \\
&+ s\ln\left(\frac{s^2}{4}+1\right) - 2\arctan 2s - 2s\ln\left(4s^2+1\right) + 5.02s\Bigg]ds \\
=& \frac{1}{t^2+1}\Bigg[\frac{1}{2}\left(t^2+3\right)\ln\left(t^2+1\right) + 2\left(\frac{t^2}{4}-1\right)\ln\left(\frac{t^2}{4}+1\right) - \frac{1}{4}\left(4t^2-1\right)\ln\left(4t^2+1\right) \\
&- 2t\arctan t + 4t\arctan\frac{t}{2} - 2t\arctan 2t + 2.51t^2\Bigg] \\
\leq& 0.43056.
\end{aligned}$$

Then

$$\sup_{t\geq 0}\left(E + M + \int_0^t \omega(s)\,e^{-\int_s^t A(z)dz}dz\right) \leq 0.60.$$

It is easy to see that all conditions of Theorem 2.1 hold for $\alpha = 0.60 < 1$. Thus, Theorem 2.1 implies that the zero solution of (2.21) is asymptotically stable. $\square$

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