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EXTENSIONS OF MEIR-KEELER CONTRACTION VIA w-DISTANCES WITH AN APPLICATION

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Abstract. In this article, we conceive the notion of a generalized (α, ψ, q)-Meir-Keeler contractive mapping and then we investigate a fixed point theorem involving such kind of contractions in the setting of a complete metric space via a w-distance. Our obtained result extends and generalizes some of the previously derived fixed point theorems in the literature via w-distances. In addition, to validate the novelty of our findings, we illustrate a couple of constructive numerical examples. Moreover, as an application, we employ the achieved result to earn the existence criteria of the solution of a kind of non-linear Fredholm integral equation.

1. Introduction and Preliminaries

In this paper, we introduce the notion of a generalized (α, ψ, q)-Meir-Keeler contractive mapping and investigate fixed points for such operators in the context of complete metric spaces via a w-distance. For this purpose we first recall the outstanding result of Meir-Keeler [14] (see also [10]).

Theorem 1.1 ([14])**.** *Let f be a self-map defined on a complete metric space (M, d). Also assume that for any $\varepsilon > 0$ we can find a $\delta > 0$ such that*

$$\varepsilon \leq d(\rho, \varrho) < \varepsilon + \delta \quad \textit{implies} \quad d(f\rho, f\varrho) < \varepsilon,$$

for all $\rho, \varrho \in M$. Then f has a unique fixed point.

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This result is also known as a uniform contraction and it has been studied and extended by a number of researchers in many directions (see [16, 20]). Now we recall the notion of w-distance introduced by Kada et al. [12].

Definition 1.1 ([12])**.** Let (M, d) be a metric space. A mapping $q : M \times M \to [0, \infty)$ is said to be a w-distance on M if

(i) $q(\rho, \sigma) \leq q(\rho, \varrho) + q(\varrho, \sigma)$ for any $\rho, \varrho, \sigma \in M$;
(ii) q is a lower semi-continuous map in the second variable, that is, when $\rho \in M$ and $\sigma_n \to \sigma$ in M, then we have $q(\rho, \sigma) \leq \liminf_n q(\rho, \sigma_n)$;
(iii) for every $\epsilon > 0$, there is a $\delta > 0$ which $q(\sigma, \rho) \leq \delta$ and $q(\sigma, \varrho) \leq \delta$ imply that $d(\rho, \varrho) \leq \epsilon$.

Let $T : M \to M$ and $\alpha : M \times M \to [0, \infty)$. We say that T is α-orbital admissible (see [17]) if

$$\alpha(p, Tp) \geq 1 \quad \text{implies} \quad \alpha(Tp, T^2p) \geq 1,$$

for all $p \in M$. By using this auxiliary function, it is possible to combine several existing results in the literature, see, e.g. [9, 15, 18, 19] and the related references therein. In particular, Lakzian et al. [13] introduced the concept of (α, ψ, q)-contractive mappings in metric spaces via w-distances and proved fixed point results via this notion.

On the other hand, inspired by the notion of Meir-Keeler contractions, Chen [11] introduced the concept of a weaker Meir-Keeler function as follows.

Definition 1.2 ([11])**.** A mapping $\psi : [0, \infty) \to [0, \infty)$ is said to be a weaker Meir-Keeler function if, for every $\epsilon > 0$, there is a $\delta > 0$ such that for every $\tau \in [0, \infty)$ with $\epsilon \leq \tau < \epsilon + \delta$, we have an $n_0 \in \mathbb{N}$ satisfying $\psi^{n_0}(\tau) < \epsilon$.

Regarding [11], we also consider the family Ψ of weaker Meir-Keeler functions $\psi : [0, \infty) \to [0, \infty)$ fulfilling the subsequent properties:

(ψ_1) $\psi(\tau) > 0$ whenever $\tau > 0$ and $\psi(0) = 0$;
(ψ_2) $\sum_{n=1}^{\infty} \psi^n(\tau) < \infty$, $\tau \in (0, \infty)$;
(ψ_3) for each $y_n \in [0, \infty)$, the following hold:
  (i) when $\lim_{n\to\infty} y_n = \ell > 0$, then $\lim_{n\to\infty} \psi(y_n) < \ell$;
  (ii) whenever $\lim_{n\to\infty} y_n = 0$, we have $\lim_{n\to\infty} \psi(y_n) = 0$.

Along with the aforementioned terminologies, the following lemma is also playing a crucial role in our subsequent studies.

Lemma 1.1 ([12])**.** *Suppose that (M, d) is a metric space with a w-distance q.*

(i) *For any sequence $\{\rho_n\}$ in M with $\lim_n q(\rho_n, \rho) = \lim_n q(\rho_n, \varrho) = 0$, we have $\rho = \varrho$. Additionally, $q(\sigma, \rho) = q(\sigma, \varrho) = 0$ implies $\rho = \varrho$.*
(ii) *For two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $[0, \infty)$ converging to 0, whenever $q(\rho_n, \varrho_n) \leq \alpha_n$, $q(\rho_n, \varrho) \leq \beta_n$ hold for each $n \in \mathbb{N}$, then the sequence $\{\varrho_n\}$ converges to ϱ.*

(iii) *Suppose that* $\{\rho_n\}$ *is a sequence in* M *such that for every* $\varepsilon > 0$ *there is an* $N_\varepsilon \in \mathbb{N}$ *with* $m > n > N_\varepsilon$ *implies that* $q(\rho_n, \rho_m) < \varepsilon$ (*or* $\lim_{m,n} q(\rho_n, \rho_m) = 0$). *Then* $\{\rho_n\}$ *is a Cauchy sequence.*

In this paper, we define the concept of generalized (α, ψ, q)-Meir-Keeler contractive mappings and by using this new concept, we give some fixed point results. Furthermore, some significant non-trivial numerical examples are investigated to authenticate our findings. Moreover, as an application, the existence of the solution for a non-linear Fredholm integral equation is investigated.

2. (α, ψ, q)-Meir-Keeler Contractions

This section brings the idea of generalized (α, ψ, q)-Meir-Keeler contractive mappings with the help of a weaker Meir-Keeler function. Also, we conceive a fixed point result concerning such kinds of mappings. Now we consider the following expressions:

$$M_q(\rho, \varrho) = \max\left\{q(\rho, \varrho), q(\rho, f\rho), q(\varrho, f\varrho), \frac{q(\rho, f\varrho) + q(f\rho, \varrho)}{2}\right\}$$

and

$$m(\rho, \varrho) = \max\left\{d(\rho, \varrho), d(\rho, f\rho), d(\varrho, f\varrho), \frac{d(\rho, f\varrho) + d(f\rho, \varrho)}{2}\right\}.$$

Here, we propose the idea of generalized (α, ψ, q)-Meir-Keeler contractive mappings.

Definition 2.1. Suppose that (M, d) is a metric space with a w-distance q and consider the functions $\psi \in \Psi$, $\alpha : M \times M \to [0, \infty)$ and an α-orbital admissible map f. Then f is called a generalized (α, ψ, q)-Meir-Keeler contractive mapping if for every $\eta > 0$, there exists a $\delta > 0$ such that for $\rho, \varrho \in M$, when $\eta \leq \psi(M_q(\rho, \varrho)) < \eta + \delta$, we have $\alpha(\rho, \varrho)q(f\rho, f\varrho) < \eta$.

In addition, for $q = d$ and $M_q(\rho, \varrho) = m(\rho, \varrho)$, the mapping f is said to be a generalized (α, ψ)-Meir-Keeler-contractive. Furthermore, f is a (α, ψ, q)-Meir-Keeler contractive map, when $M_q(\rho, \varrho) = q(\rho, \varrho)$ for each $\rho, \varrho \in M$.

The succeeding theorem deals with an interesting fixed point result involving the previously discussed type of maps.

Theorem 2.1. *Suppose that* (M, d) *is a complete metric space with a* w*-distance* q*. Also assume that* f *is a generalized* (α, ψ, q)*-Meir-Keeler contractive map such that there is* $\rho_0 \in M$ *with* $q(f^n\rho_0, f^n\rho_0) = 0$ *for all non-negative integers* n *and* $\alpha(\rho_0, f\rho_0) \geq 1$*. Suppose that one of the following conditions holds.*

(i) *For each* $w \in M$ *satisfying* $w \neq fw$*, we have* $\inf\{q(\rho, w) + q(\rho, f\rho) : \rho \in M\} > 0$.

(ii) f *is continuous.*

(iii) *If for some sequence* $\{\rho_n\}$, $\lim_{n\to\infty} q(\rho_n, \rho) = \lim_{n\to\infty} q(f\rho_n, \rho)$*, then* $f\rho = \rho$.

Then f *owns a fixed point* $u \in M$*, with* $q(u, u) = 0$.

Proof. We construct a sequence $\{\rho_n\}$ in M such that $\rho_{n+1} = f\rho_n = f^{n+1}\rho_0$ for each $n \in \mathbb{N}$. When $\rho_{n_0} = \rho_{n_0+1}$ for some positive integer n_0, then $u = \rho_{n_0}$ is a fixed point of f. Hence, without loss of generality consider that,

$$\rho_n \neq \rho_{n+1}, \quad \text{for all } n \in \mathbb{N}.$$

As f is α-orbital admissible, we have

$$\alpha(\rho_0, \rho_1) = \alpha(\rho_0, f\rho_0) \geq 1 \quad \text{implies} \quad \alpha(f\rho_0, f\rho_1) = \alpha(\rho_1, \rho_2) \geq 1.$$

Using mathematical induction, it follows that $\alpha(\rho_n, \rho_{n+1}) \geq 1$ for each $n \in \mathbb{N}$. Now, we divide the entire proof into four steps and discuss one by one.

Step 1. We first prove that for each $n \in \mathbb{N}$

$$q(\rho_n, \rho_{n+1}) < M_q(\rho_{n-1}, \rho_n).$$

Note that for every natural number n, we have $q(\rho_n, \rho_{n+1}) > 0$. Since, otherwise by the combination of $q(\rho_n, \rho_{n+1}) = 0$ and the assumption $q(\rho_n, \rho_n) = 0$ and applying Lemma 1.1 we get $\rho_n = \rho_{n+1}$, which is a contradiction. Therefore, we find that

$$\begin{aligned} M_q(\rho_{n-1}, \rho_n) = \max &\left\{ q(\rho_{n-1}, \rho_n), q(\rho_{n-1}, \rho_n), q(\rho_n, \rho_{n+1}), \frac{q(\rho_{n-1}, \rho_{n+1}) + q(\rho_n, \rho_n)}{2} \right\} \\ &> 0. \end{aligned}$$

Hence, we obtain $\psi(M_q(\rho_{n-1}, \rho_n)) > 0$. Now, from the hypothesis and Definition 2.1 for $\eta = \psi(M_q(\rho_{n-1}, \rho_n))$, there exists a $\delta > 0$ such that for $\rho, \varrho \in M$, when $\eta \leq \psi(M_q(\rho, \varrho)) < \eta + \delta$, we have $\alpha(\rho, \varrho)q(f\rho, f\varrho) < \eta$.

In particular, since for each $\tau > 0$, $\psi(\tau) < \tau$, we have

$$q(\rho_n, \rho_{n+1}) \leq \alpha(\rho_{n-1}, \rho_n)q(\rho_n, \rho_{n+1}) < \eta = \psi(M_q(\rho_{n-1}, \rho_n)) < M_q(\rho_{n-1}, \rho_n). \tag{2.1}$$

Since

$$\frac{q(\rho_{n-1}, \rho_{n+1})}{2} \leq \frac{q(\rho_{n-1}, \rho_n) + q(\rho_n, \rho_{n+1})}{2} \leq \max\{q(\rho_{n-1}, \rho_n), q(\rho_n, \rho_{n+1})\},$$

we have

$$\begin{aligned} M_q(\rho_{n-1}, \rho_n) &= \max \left\{ q(\rho_{n-1}, \rho_n), q(\rho_{n-1}, \rho_n), q(\rho_n, \rho_{n+1}), \frac{q(\rho_{n-1}, \rho_{n+1}) + q(\rho_n, \rho_n)}{2} \right\} \\ &= \max \left\{ q(\rho_{n-1}, \rho_n), q(\rho_n, \rho_{n+1}), \frac{q(\rho_{n-1}, \rho_{n+1})}{2} \right\} \\ &= \max\{q(\rho_{n-1}, \rho_n), q(\rho_n, \rho_{n+1})\}. \end{aligned}$$

So, $q(\rho_n, \rho_{n+1}) < M_q(\rho_{n-1}, \rho_n) = \max\{q(\rho_{n-1}, \rho_n), q(\rho_n, \rho_{n+1})\}$ and this implies that

$$M_q(\rho_{n-1}, \rho_n) = q(\rho_{n-1}, \rho_n) \quad \text{and} \quad q(\rho_n, \rho_{n+1}) < q(\rho_{n-1}, \rho_n).$$

Now, since $\{q(\rho_{n-1}, \rho_n)\}$ is decreasing and bounded below, it is convergent to $t \geq 0$ such that $q(\rho_n, \rho_{n+1}) \geq t$ for each n. Assume that $t \neq 0$ and $\xi = \lim_n \psi(q(\rho_n, \rho_{n+1}))$. Then by (ψ_3), $0 < \xi < t$ and by Definition 2.1, we can find $\delta > 0$ satisfying

$$\text{when} \quad \xi \leq \psi(M_q(\rho, \varrho)) < \xi + \delta, \quad \text{we have} \quad \alpha(\rho, \varrho)q(f\rho, f\varrho) < \xi, \tag{2.2}$$

for $\rho, \varrho \in M$. Consider $k_0 \in \mathbb{N}$ such that $\frac{1}{k_0} < \delta$ and $\frac{1}{k_0} < \xi$. Then for each $k \geq k_0$ there is $\delta_k \leq \frac{1}{k}$ such that

$$\xi - \frac{1}{k} \leq \psi(M_q(\rho, \varrho)) < \xi - \frac{1}{k} + \delta_k \quad \text{implies} \quad \alpha(\rho, \varrho)p(f\rho, f\varrho) < \xi - \frac{1}{k} < \xi. \tag{2.3}$$

Also there is $k_2 \in \mathbb{N}$ such that for each $n \geq k_2$ one obtains

$$\xi - \frac{1}{k_0} < \psi(q(\rho_{n-1}, \rho_n)) = \psi(M_q(\rho_{n-1}, \rho_n)) < \xi + \frac{1}{k_0} < \xi + \delta.$$

Now, when $\xi \leq \psi(M_q(\rho_{n-1}, \rho_n)) \leq \xi + \frac{1}{k_0}$, by (2.2), we have

$$q(\rho_n, \rho_{n+1}) \leq \alpha(\rho_{n-1}, \rho_n)q(\rho_n, \rho_{n+1}) < \xi < t,$$

and when $\xi - \frac{1}{k_0} \leq \psi(M_q(\rho_{n-1}, \rho_n) < \xi$ by (2.3) and since

$$[\xi - \frac{1}{k_0}, \xi] \subseteq \cup_{k \geq k_0}[\xi - \frac{1}{k}, \xi - \frac{1}{k} + \delta_k),$$

we have $q(\rho_n, \rho_{n+1}) \leq \alpha(\rho_{n-1}, \rho_n)q(\rho_n, \rho_{n+1}) < \xi < t$, which is a contradiction. Therefore, $t = 0$ and so

$$\lim_n q(\rho_n, \rho_{n+1}) = 0. \tag{2.4}$$

Step 2. We prove that $\{\rho_n\}$ is a Cauchy sequence. Alternatively, from the inequality (2.1), we arrive at

$$q(\rho_n, \rho_{n+1}) \leq \psi(q(\rho_{n-1}, \rho_n)), \quad \text{for all } n \in \mathbb{N}. \tag{2.5}$$

Indeed, if there exists some n^* such that

$$q(\rho_{n^*}, \rho_{n^*+1}) \leq \psi(q(\rho_{n^*}, \rho_{n^*+1})) < q(\rho_{n^*}, \rho_{n^*+1}),$$

we get a contradiction. Hence, (2.5) holds. Inductively, we derive, from (2.5), that

$$q(\rho_n, \rho_{n+1}) \leq \psi^n(q(\rho_0, \rho_1)), \quad \text{for all } n \in \mathbb{N}.$$

Fix ε and let $n_\varepsilon \in \mathbb{N}$ such that $\sum_{k \geq n_\varepsilon} \psi^k(q(\rho_1, \rho_0)) < \varepsilon$. Furthermore, for $m > n > n_\varepsilon$ we can find that

$$\begin{aligned} q(\rho_n, \rho_m) &\leq q(\rho_n, \rho_{n+1}) + \cdots + q(\rho_{m-1}, \rho_m) \\ &\leq \sum_{k=n}^{m-1} \psi^k(q(\rho_1, \rho_0)) \\ &\leq \sum_{k \geq n_\varepsilon} \psi^k(q(\rho_1, \rho_0)). \end{aligned}$$

Hence, we conclude that the sequence $\{\rho_n\}$ is Cauchy. Now, since (M, d) is complete, we can get $u \in M$ with $\rho_n \to u$ in M.

Step 3. u is a fixed point of f.

Case (i). For each $\varrho \in M$ satisfying $\varrho \neq f\varrho$, we have $\inf\{q(\rho, \varrho) + q(\rho, f\rho) : \rho \in M\} > 0$. It implies that for every $\varepsilon > 0$, there is a natural number N such that for

$n > N_\varepsilon$, we have $q(\rho_{N_\varepsilon}, \rho_n) < \varepsilon$. Since, $\rho_n \to u$ and $q(\rho, \cdot)$ is a lower semi-continuous map, we have

$$q(\rho_{N_\varepsilon}, u) \leq \liminf_{n\to\infty} q(\rho_{N_\varepsilon}, \rho_n) \leq \varepsilon.$$

Putting $\varepsilon = \frac{1}{k}$ and $N_\varepsilon = n_k$, we have

$$\lim_{k\to\infty} q(\rho_{n_k}, u) = 0. \tag{2.6}$$

Assume that $u \neq fu$. Then

$$0 < \inf\{q(\sigma, u) + q(\sigma, f\sigma) : \sigma \in M\} \leq \inf\{q(\rho_{n_k}, u) + q(\rho_{n_k}, \rho_{n_k+1}) : k \in \mathbb{N}\}.$$

From (2.4) and (2.6), we derive $\inf\{q(\sigma, u)+q(\sigma, f\sigma) : \sigma \in M\} = 0$, which contradicts the given hypothesis. Therefore, $fu = u$.

Case (ii). Let f be continuous.

Using the triangular inequality, we have

$$q(\rho_n, f^2\rho_n) \leq q(\rho_n, f\rho_n) + q(f\rho_n, f^2\rho_n).$$

Accordingly, letting $n \to \infty$, we obtain $q(\rho_n, f^2\rho_n) \to 0$. Further, Lemma 1.1 confirms that $\{f^2\rho_n\} \to u$ as $n \to \infty$. As f is continuous, we have

$$fu = f(\lim_{n\to\infty} f\rho_n) = \lim_{n\to\infty} f^2\rho_n = u.$$

Hence, u is a fixed point of f.

Case (iii). Here, $\lim_{n\to\infty} q(f\rho_n, u) = \lim_{n\to\infty} q(\rho_{n+1}, u) = \lim_{n\to\infty} q(\rho_n, u)$. Hence, $fu = u$.

Step 4. u is a fixed point with $q(u, u) = 0$.

Conversely, suppose that $q(u, u) > 0$. Then from (2.1), we get

$$0 < q(u, u) = q(fu, fu) \leq \psi(M_q(u, u)) < M_q(u, u) = q(u, u),$$

and this is impossible. Hence, our claim is verified. □

The fixed point obtained in the previous theorem may be not unique. The following examples validate our claim.

Example 2.1. Suppose that G is a locally compact group, $M = L^1(G)$ and

$$q(f, g) = \|g\|_1, \quad f, g \in L^1(G).$$

Then q is a w-distance . Define $\psi(t) = \begin{cases} \frac{t}{2}, & t \in [0, 1], \\ \frac{1}{2}, & t \in (1, \infty), \end{cases}$ and

$$\alpha(f, g) = \begin{cases} 2, & g = 0 \quad (a.e.), \\ \dfrac{\psi(M_q(f, g))}{2\|g\|_1}, & \text{otherwise}, \end{cases}$$

and for an arbitrary $x \in G$

$$\begin{array}{rcl} T_x : L^1(G) & \to & L^1(G), \\ f & \mapsto & \frac{1}{8}L_x f, \end{array}$$

where $L_xf(y) = f(x^{-1}y)$. Then for each $f \in L^1(G)$ and $x \in G$, since $\|L_xf\|_1 = \|f\|_1$, we conclude that $M_q(f,g) = \max\{\frac{1}{8}\|f\|_1, \|g\|_1\}$ and so

$$\alpha(f,g) = \frac{\psi(M_q(f,g))}{2\|g\|_1} = \frac{\psi(\max\{\frac{1}{8}\|f\|_1, \|g\|_1\})}{2\|g\|_1} \geq 1.$$

In each of the cases $0 \leq \max\{\frac{1}{8}\|f\|_1, \|g\|_1\} \leq 1$, $1 \leq \max\{\frac{1}{8}\|f\|_1, \|g\|_1\} \leq 8$ and $8 \leq \max\{\frac{1}{8}\|f\|_1, \|g\|_1\}$ we conclude that

$$\alpha(T_xf, T_xg) = \frac{\psi(\max\{\frac{1}{64}\|f\|_1, \frac{1}{8}\|g\|_1\})}{\frac{2}{8}\|g\|_1} > 1.$$

So, T_x is α-orbital admissible. Now for each $\eta > 0$ and $\delta = \eta$, if $\eta \leq \psi(M_q(f,g)) < 2\eta$, then for $g \neq 0$ we have

$$\alpha(f,g)q\left(\frac{1}{8}L_xf, \frac{1}{8}L_xg\right) \leq \frac{\psi(M_q(f,g))}{2\|g\|_1}\left(\frac{1}{8}\right)\|g\|_1 \leq \frac{1}{8}\eta < \eta,$$

and for $g = 0$, since $q(T_xf, T_xg) = 0$, we are done. So, T_x is a generalized (α, ψ, q)-Meir-keeler contractive map. Moreover, $\alpha(0, T_x0) = \alpha(0,0) = 2 > 1$,

$$q(T_x^n0, T_x^n0) = q(0,0) = \|0\|_1 = 0,$$

and T_x is continuous. Therefore, all the hypotheses of Theorem 2.1 hold and so, T_x has a fixed point (which is $f = 0$, satisfying $q(0,0) = 0$). Note that for each $f \in L^1(G)$, we have

$$\lim \|T_x^nf\|_1 = \lim \frac{1}{8^n}\|f\|_1 = 0.$$

Therefore, T_x^nf converges to 0 and so 0 is the only fixed point of T_x.

Example 2.2. Suppose that $M = \{\frac{1}{2^n} : n \in \mathbb{N}\} \cup \{0\}$ is equipped with the usual metric on $\mathbb{R}$. Consider

$$q(\rho, \varrho) = \begin{cases} \frac{1}{n} + \frac{1}{m}, & \varrho = \frac{1}{2^m},\ \rho = \frac{1}{2^n}, \\ 0, & \rho = 0 \text{ or } \varrho = 0, \end{cases}$$

$$\alpha(\rho, \varrho) = \begin{cases} \frac{m}{n}, & \varrho = \frac{1}{2^m},\ \rho = \frac{1}{2^n} \text{ and } 2n \geq m \geq n, \\ 1, & \rho = 0 \text{ or } \varrho = 0, \\ \frac{1}{n}, & \text{otherwise}, \end{cases}$$

and $f\rho = \rho^8$. Then $\alpha(0, f0) = 1$, $q(f^n0, f^n0) = 0$ for each $n \in \mathbb{N}$ and f is continuous and also α-orbital admissible. Since if $\alpha(\rho, f\rho) \geq 1$, then $\rho = 0$, since if $\rho = \frac{1}{2^n}$ for some n, then $n \leq 8n \leq 2n$ is impossible. Therefore, $\alpha(f\rho, f^2\rho) \geq 1$. Also if

$$\psi(t) = \begin{cases} \frac{t}{2}, & t \in [0,1], \\ \frac{1}{2}, & t \in (1, \infty), \end{cases}$$

then, since $0 \le M_q(\rho,\varrho) \le 1$, we have $\psi(M_q(\rho,\varrho)) = \frac{1}{2}M_q(\rho,\varrho)$. On the other hand, for each $\eta > 0$ and for $\delta = \eta$, if $\eta \le \psi(M_q(\rho,\varrho)) < 2\eta$, then ρ or ϱ is non-zero. So if $\rho = \frac{1}{2^n}$ and $\varrho = \frac{1}{2^m}$, then since $\alpha(\rho,\varrho) \le 2$, we conclude that

$$\alpha(\rho,\varrho)q(f\rho,f\varrho) \le 2\left(\frac{1}{8n}+\frac{1}{8m}\right) = \frac{1}{4}q(\rho,\varrho) \le \frac{1}{4}M_p(\rho,\varrho) = \frac{1}{2}\psi(M_p(\rho,\varrho)) < \eta.$$

Also, if one of ρ or ϱ is zero, then $\alpha(\rho,\varrho)q(f\rho,f\varrho) = 0 \le \eta$. So, f is a generalized (α,ψ,q)-Meir-Keeler contractive map. Therefore, all the conditions of Theorem 2.1 hold. Hence, $\rho = 0$ is the unique fixed point of f.

Example 2.3. Let $M = [0,1]$ be equipped with the usual metric. Also let us consider the w-distance as $q(\rho,\varrho) = |\rho - \varrho|$ for each $\rho,\varrho \in M$. Further, we define

$$f\rho = \begin{cases} \dfrac{\rho}{20}, & \rho \in [0,1), \\ 1, & \rho = 1, \end{cases} \qquad \alpha(\rho,\varrho) = \begin{cases} 1, & \rho,\varrho \in [0,1), \\ 0, & \rho = 1, \end{cases} \qquad \psi(\rho) = \begin{cases} 0, & \rho = 0, \\ \dfrac{1}{3}, & \rho \in \left(0,\dfrac{1}{2}\right), \\ \dfrac{\rho}{2}, & \rho \in \left[\dfrac{1}{2},1\right], \\ \dfrac{1}{2}, & \rho \in (1,\infty). \end{cases}$$

Hence, for every $w \in M$ with $fw \ne w$, we obtain $w \ne 0,1$ and so

$$\lim_{\rho\to w}(|w-\rho| + |\rho - f\rho|) \ge \frac{19}{20}w > 0.$$

Again,

$$\lim_{\rho\to\varrho}(|w-\rho| + |\rho - f\rho|) \ge |w-\varrho| > 0, \quad \varrho \ne w.$$

Therefore, we have $\inf\{q(\rho,w) + q(\rho,f\rho) : \rho \in M\} > 0$ for each $w \in M$ satisfying $w \ne fw$. Besides, for every $\rho \in M$, we obtain $|f^n\rho - f^n\rho| = 0$. Now for each $\eta > 0$, put $\delta = \eta$. Then, $\rho = \varrho$ implies $M_q(\rho,\varrho) = 0$ and when $\rho \ne \varrho$, $M_q(\rho,\varrho) \ne 0$ and further, $\psi(M_q(\rho,\varrho)) \ge \frac{1}{4}$. Therefore, for $\eta > \frac{1}{8}$, there is no $\rho,\varrho \in M$ satisfying

$$\frac{1}{8} \le \psi(M_q(\rho,\varrho)) < \frac{1}{4}.$$

On the other hand, for $\eta \le \frac{1}{8}$, if $\eta \le \psi(M_q(\rho,\varrho)) < \eta + \eta = 2\eta$, we have

$$\alpha(\rho,\varrho)|f\rho - f\varrho| \le |f\rho - f\varrho| = \left|\frac{\rho}{20} - \frac{\varrho}{20}\right| \le \frac{2}{20} < \frac{1}{8} < \eta.$$

That is for each ρ,ϱ, if $\eta \le \psi(M_q(\rho,\varrho)) < \eta+\eta = 2\eta$, then $\alpha(\rho,\varrho)|f\rho - f\varrho| \le \eta$. Note that $0, 1$ are the fixed points of f.

Remark 2.1. In the case where $q(\rho,\varrho) = \varrho$ for each $\rho,\varrho \in M$, the assumption $q(f^n\rho, f^n\rho) = 0$, for some $\rho \in M$ and for each $n \in \mathbb{N}$, imply that $f^n\rho = 0$ for each n. Therefore, in this case without any another condition, since $\rho_n = 0 = \rho_{n+1}$, the first part of the Theorem 2.1 implies that f possesses a fixed point. For example, let $M = \{\frac{1}{2^n} : n \in \mathbb{N}\} \cup \{0\}$,

$$\alpha(\rho, \varrho) = \begin{cases} 0, & \varrho \in \left\{\frac{1}{2^{2k}} : k \in \mathbb{N}\right\}, \\ 1, & \text{otherwise,} \end{cases} \quad \text{and} \quad f\rho = \begin{cases} \frac{\rho}{2}, & \rho \in \left\{\frac{1}{2^{2k}} : k \in \mathbb{N}\right\}, \\ 1, & \text{otherwise.} \end{cases}$$

Then f is continuous, $q(f^n 0, f^n 0) = 0$ for each $n \in \mathbb{N}$ and $\rho, \varrho \in M$ and $\eta, \delta > 0$, if $\eta \leq \psi(M_q(\rho, \varrho)) < \eta + \delta$, then we have

$$0 = \alpha(\rho, \varrho) q(f\rho, f\varrho) \leq \eta.$$

Note that 0 is a fixed point of f, since here we require only $q(f^n 0, f^n 0) = 0$.

Now we put down the following additional hypothesis. To attest the uniqueness of the fixed point of f, this condition along with those of Theorem 2.1 is required.
Property U. Let $\alpha(u, v) < 1$, implies that at least one of u or v is not a fixed point of f.

For example if $\alpha(u, v) \geq 1$ for each $u, v \in M$, then the property U is valid.

Theorem 2.2. *Suppose that (M, d) is a metric space with a w-distance q. Also assume that f is a generalized (α,ψ, p)-Meir-Keeler contractive mapping and satisfies all the hypotheses of Theorem* 2.1 *along with the additional property U. Then we can claim the uniqueness of the fixed point of f obtained in Theorem* 2.1.

Proof. We suppose that $u, v \in M$ are two distinct fixed points of f. Then $\alpha(u, v) \geq 1$, $fu = u$, $fv = v$, $q(u, u) = 0$ and $q(v, v) = 0$. Using the aforementioned criteria and (2.1), we obtain

$$q(u, v) = q(fu, fv) \leq \alpha(u, v) q(fu, fv) \leq \psi(M_q(u, v)) = \psi(q(u, v)) < q(u, v),$$

and this is impossible. Hence, f possesses a unique fixed point. ☐

3. Consequences

This section deals with a few immediate corollaries of our obtained Theorem 2.1. First, we give the following important result for an (α, ψ, q)-Meir-Keeler contractive mapping.

Corollary 3.1. *Suppose that (M, d) is a complete metric space with a w-distance q. Also let f be an (α, ψ, q)-Meir-Keeler contractive mapping with the fact that there is some $\rho_0 \in M$, with $q(f^n \rho_0, f^n \rho_0) = 0$ for all non-negative integers n and $\alpha(\rho_0, f\rho_0) \geq 1$. Suppose that one of the following holds.*

- (i) *For each $w \in M$ satisfying $w \neq fw$, we have* $\inf\{q(\rho, w) + q(\rho, f\rho) : \rho \in M\} > 0$.
- (ii) *f is continuous.*
- (iii) *If for some sequence $\{\rho_n\}$, $\lim_{n\to\infty} q(\rho_n, \rho) = \lim_{n\to\infty} q(f\rho_n, \rho)$, then $f\rho = \rho$.*

Then f possesses a fixed point $u \in M$, with $q(u, u) = 0$.

Putting $\alpha \equiv 1$ in Theorem 2.1, we obtain the trailing important corollary.

Corollary 3.2. *Suppose that (M, d) is a complete metric space with a w-distance q. Also let f be a (ψ, q)-Meir-Keeler contractive mapping with the fact that there is some $\rho_0 \in M$, with $q(f^n\rho_0, f^n\rho_0) = 0$ for all non-negative integers n. Suppose that one of the following conditions holds.*

(i) *For each $w \in M$ satisfying $w \neq fw$, we have* $\inf\{q(\rho, w) + q(\rho, f\rho) : \rho \in M\} > 0$.

(ii) *f is continuous.*

(iii) *If for some sequence $\{\rho_n\}$, $\lim_{n\to\infty} q(\rho_n, \rho) = \lim_{n\to\infty} q(f\rho_n, \rho)$, then $f\rho = \rho$.*

Then f possesses a fixed point $u \in M$.

Considering $q = d$ in Theorem 2.1, we deduce the subsequent corollary.

Corollary 3.3. *Suppose that (M, d) is a complete metric space and f be an (α,ψ)-Meir-Keeler contractive mapping with the fact that there is some $\rho_0 \in M$ with $\alpha(\rho_0, f\rho_0) \geq 1$ or $\alpha(f\rho_0, \rho_0) \geq 1$. Suppose that one of the following conditions holds.*

(i) *For each $w \in M$ satisfying $w \neq fw$, we have* $\inf\{d(\rho, w) + d(\rho, f\rho) : \rho \in M\} > 0$.

(ii) *f is continuous.*

(iii) *For some sequence $\{\rho_n\}$ in M with $\alpha(\rho_n, \rho_{n+1}) \geq 1$ for all natural numbers n and $\rho_n \to \rho \in M$ as $n \to \infty$, then $\alpha(\rho_n, \rho) \geq 1$ for every $n \in \mathbb{N}$.*

Then f possesses a fixed point $u \in M$.

Taking $\alpha \equiv 1$ in Corollary 3.3, we get the succeeding consequence.

Corollary 3.4. *Suppose that (M, d) is a complete metric space and f be a ψ-Meir-Keeler contractive mapping. Suppose that either f is continuous or* $\inf\{d(\rho, w) + d(\rho, f\rho) : \rho \in M\} > 0$ *for each $w \in M$ with $w \neq fw$. Then f possesses a fixed point $u \in M$.*

Definition 3.1. Suppose that (M, d) is a metric space with a w-distance q and consider the functions $\psi \in \Psi$, $\alpha : M \times M \to [0, \infty)$ and a self-map f. Then f is said to be a generalized (α, ψ, q)-Meir-Keeler contractive mapping of

(a) Banach type if for every $\eta > 0$, there exists a $\delta > 0$ such that for $\rho, \varrho \in M$

$$\text{when} \quad \eta \leq \psi(q(\rho, \varrho)) < \eta + \delta, \quad \text{we have} \quad \alpha(\rho, \varrho)p(f\rho, f\varrho) < \eta;$$

(b) Kannan type I if for every $\eta > 0$, there exists a $\delta > 0$ such that for $\rho, \varrho \in M$

$$\text{when} \quad \eta \leq \psi\left(\frac{q(\rho, f\rho) + q(\varrho, f\varrho)}{2}\right) < \eta + \delta, \quad \text{we have} \quad \alpha(\rho, \varrho)p(f\rho, f\varrho) < \eta;$$

(c) Kannan type II if for every $\eta > 0$, there exists a $\delta > 0$ such that for $\rho, \varrho \in M$

$$\text{when} \quad \eta \leq \psi(\max\{q(\rho, f\rho), q(\varrho, f\varrho)\}) < \eta + \delta, \quad \text{we have} \quad \alpha(\rho, \varrho)p(f\rho, f\varrho) < \eta;$$

(d) Chatterjea type I if for every $\eta > 0$, there exists a $\delta > 0$ such that for $\rho, \varrho \in M$

$$\text{when} \quad \eta \leq \psi\left(\frac{q(\rho, f\varrho) + q(\varrho, f\rho)}{2}\right) < \eta + \delta, \quad \text{we have} \quad \alpha(\rho, \varrho)p(f\rho, f\varrho) < \eta;$$

(e) Chatterjea type II if for every $\eta > 0$, there exists a $\delta > 0$ such that for $\rho, \varrho \in M$

$$\text{when} \quad \eta \leq \psi(\max\{q(\rho, f\varrho), q(\varrho, f\rho)\}) < \eta + \delta, \quad \text{we have} \quad \alpha(\rho, \varrho)p(f\rho, f\varrho) < \eta;$$

(f) Reich type I if for every $\eta > 0$, there exists a $\delta > 0$ such that for $\rho, \varrho \in M$

$$\text{when} \quad \eta \leq \psi\left(\frac{q(\rho, \varrho) + q(\rho, f\rho) + q(\varrho, f\varrho)}{3}\right) < \eta+\delta, \quad \text{we have} \quad \alpha(\rho, \varrho)p(f\rho, f\varrho) < \eta;$$

(g) Reich type II if for every $\eta > 0$, there exists a $\delta > 0$ such that for $\rho, \varrho \in M$

$$\text{when} \quad \eta \leq \psi(\max\{q(\rho, \varrho), q(\rho, f\rho), q(\varrho, f\varrho)\}) < \eta+\delta, \quad \text{we have} \quad \alpha(\rho, \varrho)p(f\rho, f\varrho) < \eta;$$

(h) Reich type III if for every $\eta > 0$, there exists a $\delta > 0$ such that for $\rho, \varrho \in M$

$$\text{when} \quad \eta \leq \psi(\max\{q(\rho, \varrho), q(\rho, f\varrho), q(\varrho, f\rho)\}) < \eta+\delta, \quad \text{we have} \quad \alpha(\rho, \varrho)p(f\rho, f\varrho) < \eta.$$

In addition, for taking $q = d$ in the inequalities above, we can get several other kind of contractions in the context of metric spaces.

If in Theorem 2.1, we change the contraction condition 'generalized (α, ψ, q)-Meir-Keeler contractive mapping' with one of the new contractions defined in Definition 3.1, then we may obtain a similar result as Theorem 2.1. Furthermore, as in Corollary 3.3 and Corollary 3.4, we may get some more results by letting $q = d$. Also, notice that by choosing the auxiliary function α in a proper way in Theorem 2.1, we can deduce more consequences related to cyclic contractions and results in metric spaces endowed with a partially ordered set, see for example [1–8].

4. An Application

In this section, we discuss an application of our obtained fixed point result to a certain kind of non-linear Fredholm integral equations. First of all, we prove a proposition which is going to play a crucial role here.

Proposition 4.1. *Suppose that (M, d) is a metric space with a w-distance q. Also, assume that f is a self-mapping on M satisfying*

$$\alpha(\rho, \varrho)q(f\rho, f\varrho) \leq k\psi(M_q(\rho, \varrho)), \tag{4.1}$$

for all $\rho, \varrho \in M$ and for some $k \in (0, 1)$. Then f is a generalized (α, ψ, q)-Meir-Keeler contractive mapping.

Proof. Consider $\delta = (\frac{1}{k} - 1)\eta$ in Definition 2.1. Accordingly, we derive

$$\eta \leq \psi(M_q(\rho, \varrho)) < \eta + \delta < \eta + \left(\frac{1}{k} - 1\right)\eta = \frac{\eta}{k},$$

and so, for every $\rho, \varrho \in M$, we obtain $k\eta \leq k\psi(M_q(\rho, \varrho)) < \eta$. Using (4.1), we get

$$\alpha(\rho, \varrho)q(f\rho, f\varrho) \leq k\psi(M_q(\rho, \varrho)) < \eta.$$

Hence, $\alpha(\rho, \varrho)q(f\rho, f\varrho) < \eta$ and therefore, f is an (α, ψ, q)-Meir-Keeler contractive mapping. □

Now, we try to obtain a criterion to ensure the existence of a solution for a type of non-linear Fredholm integral equation.

Theorem 4.1. *Let us consider the non-linear Fredholm integral equation*

$$(fx)(t) = g(t) + \int_a^b H(t, s, x(s))ds, \tag{4.2}$$

for some $a, b \in \mathbb{R}$, with $a < b$, $g : [a, b] \to \mathbb{R}$ and $H : [a, b]^2 \times \mathbb{R} \to \mathbb{R}$ be two continuous maps. Also, assume that the subsequent properties hold:

(i) *$f : C[a, b] \to C[a, b]$ is a continuous mapping;*
(ii) *there exists a weaker Meir-Keeler function ψ and $k \in [0, 1)$ satisfying*

$$|H(t, s, x(s))| + |H(t, s, y(s))|$$
$$\leq \frac{k\left[\psi\left(\max\left\{ |x(t)| + |y(t)|, |x(t)| + |(fx)(t)|, |y(t)| + |(fy)(t)|, \right.\right.\right.}{b-a}$$
$$\frac{\left.\left.\left.\frac{(|x(t)|+|(fy)(t)|)+(|(fx)(t)|+|y(t)|)}{2}\right\}\right)\right] - 2|g(t)|}{b-a},$$

for all $t, s \in [a, b]$. Then the non-linear Fredholm integral equation (4.2) *owns a unique solution in $C[a, b]$.*

Proof. Suppose $M = C[a, b]$. Obviously, M is complete with respect to the metric $d : M \times M \to \mathbb{R}^+$ defined as

$$d(x, y) = \sup_{t\in[a,b]} |x(t) - y(t)|,$$

where $x, y \in M$. Now, we consider the map $q : M \times M \to \mathbb{R}^+$ given by

$$q(x, y) = \sup_{t\in[a,b]} |x(t)| + \sup_{t\in[a,b]} |y(t)|,$$

where $x, y \in M$. One can easily check that, q is a w-distance on M. Here we have

$$\begin{aligned}
&|(fx)(t)| + |(fy)(t)| \\
=&\left|g(t) + \int_a^b H(t, s, x(s))ds\right| + \left|g(t) + \int_a^b H(t, s, y(s))ds\right| \\
\leq& |g(t)| + \left|\int_a^b H(t, s, x(s))ds\right| + |g(t)| + \left|\int_a^b H(t, s, y(s))ds\right| \\
\leq& 2\,|g(t)| + \left|\int_a^b H(t, s, x(s))ds\right| + \left|\int_a^b H(t, s, y(s))ds\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq 2\,|g(t)| + \int_a^b |H(t,s,x(s))|\,ds + \int_a^b |H(t,s,y(s))|\,ds\\
&\leq 2\,|g(t)| + \int_a^b \left(|H(t,s,x(s))| + |H(t,s,y(s))|\right) ds\\
&\leq 2\,|g(t)| + \int_a^b \left(\frac{k\left[\psi\left(\max\left\{ |x(t)|+|y(t)|, |x(t)|+|(fx)(t)|, |y(t)|+|(fy)(t)|,\right.\right.\right.}{b-a}\right.\\
&\qquad \left.\frac{\left.\left.\left.\frac{(|x(t)|+|(fy)(t)|)+(|(fx)(t)|+|y(t)|)}{2}\right\}\right)\right] - 2|g(t)|}{b-a}\right) ds\\
&= 2\,|g(t)| + \frac{k\left[\psi\left(\max\left\{ |x(t)|+|y(t)|, |x(t)|+|(fx)(t)|, |y(t)|+|(fy)(t)|,\right.\right.\right.}{b-a}\\
&\qquad \frac{\left.\left.\left.\frac{(|x(t)|+|(fy)(t)|)+(|(fx)(t)|+|y(t)|)}{2}\right\}\right)\right] - 2|g(t)|}{b-a}\int_a^b ds\\
&= k\left[\psi\left(\max\left\{ |x(t)|+|y(t)|, |x(t)|+|(fx)(t)|, |y(t)|+|(fy)(t)|,\right.\right.\right.\\
&\qquad \left.\left.\left.\frac{(|x(t)|+|(fy)(t)|)+(|(fx)(t)|+|y(t)|)}{2}\right\}\right)\right]\\
&\leq k\left[\psi\left(\max\left\{q(x,y), q(x,fx), q(y,fy), \frac{q(x,fy)+q(y,fx)}{2}\right\}\right)\right]\\
&= k\left[\psi\left(M_q(x,y)\right)\right],
\end{aligned}
$$

for all $x, y \in M$ and $t \in [0, \infty]$. Thus,

$$\sup_{t\in[a,b]} |(fx)(t)| + \sup_{t\in[a,b]} |(Ty)(t)| \leq k\left[\psi\left(M_q(x,y)\right)\right],$$

and therefore for each $x, y \in M$

$$q(fx, fy) \leq k\left[\psi\left(M_q(x,y)\right)\right].$$

This implies that f satisfies Proposition 4.1 and hence it is an (α, ψ, q)-Meir-Keeler contractive mapping. Therefore, by Theorem 2.1, the non-linear Fredholm integral equation (4.2) has a solution. □

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