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HYPERGROUPS DEFINED ON HYPERGRAPHS AND THEIR REGULAR RELATIONS

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Abstract. The notion of hypergraphs, introduced around 1960, is a generalization of that of graphs and one of the initial concerns was to extend some classical results of graph theory. In this paper, we present some connections between hypergraph theory and hypergroup theory. In this regard, we construct two hypergroupoids by defining two new hyperoperations on $\mathbb{H}$, the set of all hypergraphs. We prove that our defined hypergroupoids are commutative hypergroups and we define hyperrings on $\mathbb{H}$ by using the two defined hyperoperations. Moreover, we study the fundamental group, complete parts, automorphism group and strongly regular relations of one of our hypergroups.

1. Introduction

Hypergraphs generalize standard graphs by defining edges between multiple vertices instead of only two vertices. Hence some properties must be a generalization of graph properties. Formally, a hypergraph is a pair $\Gamma = (X, E)$, where X is a finite set of vertices and $E = \{E_1, \dots, E_n\}$ is a set of hyperedges, which are non-empty subsets of X. The term hypergraph was coined by Berge [2,4], following a remark by Jean-Marie Pal who had used the word hyperedge in a seminar. In 1976, Berge enriched the field once more with his lecture notes [5], also see [3]. The hyperstructure theory was born in 1934, when Marty introduced the notion of a hypergroup [16]. Since then, many papers and several books have been written on this topic (see, for instance [6,8–10,18]). Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. After that, many researchers in the field of hyperstructure theory tried to make connections

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between hypergraphs and hyperstructures, for example see [7, 12–14]. Corsini in [7] associated to every hypergraph Γ a commutative quasihypergroup H_Γ and found a necessary and sufficient condition on Γ so that H_Γ is associative. In this paper we continue the study between hypergraphs and algebraic hyperstructures.

Our paper is organized as follows. After an introduction, Section 2 presents some basic definitions concerning hypergroups and hypergraphs that are used throughout this paper. Section 3 defines a new hyperoperation $(\star)$ on $\mathbb{H}$, the set of all hypergraphs and proves some interesting results about $(\mathbb{H}, \star)$. Section 4 presents the fundamental group of our defined hypergroup $(\mathbb{H}, \star)$ and studies its regular relations, complete parts and its automorphism group. Section 5 defines another new hyperoperation $(\circ)$ on $\mathbb{H}$, studies homomorphisms between $(\mathbb{H}, \star)$ and $(\mathbb{H}, \circ)$ and defines hyperrings on $\mathbb{H}$.

2. BASIC DEFINITIONS

In this section, we present some definitions related to hypergroups and hypergraphs that are used throughout the paper.

Let H be a non-empty set. Then, a mapping $\circ : H \times H \to \mathcal{P}^*(H)$ is called a *hyperoperation* on H, where $\mathcal{P}^*(H)$ is the family of all non-empty subsets of H. The couple $(H, \circ)$ is called a *hypergroupoid*. In the above definition, if A and B are two non-empty subsets of H and $x \in H$, then we define:

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b, \quad x \circ A = \{x\} \circ A \quad \text{and} \quad A \circ x = A \circ \{x\}.$$

An element $e \in H$ is called an *identity* of $(H, \circ)$ if $x \in x \circ e \cap e \circ x$ for all $x \in H$ and it is called a *scalar identity* of $(H, \circ)$ if $x \circ e = e \circ x = \{x\}$, for all $x \in H$. If e is a scalar identity of $(H, \circ)$, then e is the unique identity of $(H, \circ)$. An element $x \in H$ is called *idempotent* if $x \circ x = x$. An element $y \in H$ is said to be an *inverse* of $x \in H$ if $e \in x \circ y \cap y \circ x$, where e is an identity in $(H, \circ)$. The hypergroupoid $(H, \circ)$ is said to be *commutative* if $x \circ y = y \circ x$ for all $x, y \in H$. A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if it is associative, i.e., if for every $x, y, z \in H$, we have $x \circ (y \circ z) = (x \circ y) \circ z$ and is called a *quasihypergroup* if for every $x \in H$, $x \circ H = H = H \circ x$. This condition is called the reproduction axiom. The couple $(H, \circ)$ is called a *hypergroup* if it is a semihypergroup and a quasihypergroup. A subset S of a hypergroup $(H, \circ)$ is called *subhypergroup* of H if it is a hypergroup under $\circ$. A subhypergroup K of a hypergroup $(H, \circ)$ is *normal* if $a \circ K = K \circ a$ for all $a \in H$. A hypergroup $(H, \circ)$ is called a *regular hypergroup* if it has at least one identity and each of its elements admit at least one inverse. A subset I of H is called a *hyperideal* of H if $IH \subseteq H$. A hypergroup H is said to be *simple* if H has no proper hyperideals.

Cyclic semihypergroups have been studied by Desalvo and Freni [11], Vougiouklis [19], Leoreanu [15]. Cyclic semihypergroups are important not only in the sphere of finitely generated semihypergroups but also for interesting combinatorial implications.

A hypergroup $(H, \circ)$ is cyclic if there exists $h \in H$ such that

$$H = h \cup h^2 \cup \cdots \cup h^i \cup \cdots .$$

If there exists $s \in \mathbb{N}$ such that $H = h \cup h^2 \cup \cdots \cup h^s$ then H is cyclic hypergroup with finite period. Otherwise, H is called cyclic hypergroup with infinite period. Here, $h^i = \underbrace{h \circ h \circ \cdots \circ h}_{i \text{ times}}$. It is a *single-power cyclic hypergroup* if there exists $h \in H$ such that

$$H = h \cup h^2 \cup \cdots \cup h^i \cup \cdots \quad \text{and} \quad h \cup h^2 \cup \cdots \cup h^{i-1} \subset h^i, \quad \text{for all } i \in \mathbb{N}.$$

Let $(H, \star)$ and $(H', \star')$ be two hypergroups. A function $f : (H, \star) \to (H', \star')$ is said to be a *weak homomorphism* if $f(x_1 \star x_2) \cap f(x_1) \star' f(x_2) \neq \emptyset$ for all $x_1, x_2 \in H$. It is called *homomorphism* if $f(x_1 \star x_2) \subseteq f(x_1) \star' f(x_2)$ for all $x_1, x_2 \in H$. And it is called a *good homomorphism* if $f(x_1 \star x_2) = f(x_1) \star' f(x_2)$ for all $x_1, x_2 \in H$.

Two hypergroups are said to be *isomorphic* if there exists a bijective good homomorphism between them. An isomorphism from $(H, \star)$ to itself is called an *automorphism*. The set of all automorphisms of $(H, \star)$ is written as $\mathrm{Aut}(H, \star)$.

3. HYPERGROUP $(\mathbb{H}, \star)$ ASSOCIATED TO HYPERGRAPHS

In this section, we define a new hyperoperation $(\star)$ on the set of all hypergraphs $\mathbb{H}$ and we study some properties of $(\mathbb{H}, \star)$.

A partial hypergraph is a hypergraph with some edges removed.

Definition 3.1. Let $\mathbb{H}$ be the set of all hypergraphs and define $\star$ as follows. For all $H_1, H_2 \in \mathbb{H}$,

$$H_1 \star H_2 = \bigcup \{K \in \mathbb{H} : K \text{ is a partial hypergraph of } H_1 \cup H_2\}.$$

$H_1 \cup H_2$ is the union of all hyperedges from H_1 and H_2. If the same hyperedge corresponding to the same set of vertices occur in both H_1 and H_2 then we consider it once in $H_1 \cup H_2$.

Example 3.1. We present an example on the union of two hypergraphs illustrated in Figures 1, 2 and 3.

Proposition 3.1. *Let $H_1, H_2 \in \mathbb{H}$. Then $\{H_1, H_2\} \subseteq H_1 \star H_2$.*

Proof. The proof results from having H_1, H_2 partial hypergraphs of $H_1 \cup H_2$. □

Proposition 3.2. *Let $H \in \mathbb{H}$. Then $H^m = H^2$ for all $m \geq 2$.*

Proof. For $m \geq 2$, we have that

$$\begin{aligned} H^m &= \{K \in \mathbb{H} : K \text{ is a partial hypergraph of } \underbrace{H \cup H \cdots \cup H}_{m \text{ times}}\} \\ &= \{K \in \mathbb{H} : K \text{ is a partial hypergraph of } H\} \\ &= H^2. \end{aligned}$$

Therefore, $H^m = H^2$ for all $m \geq 2$. □

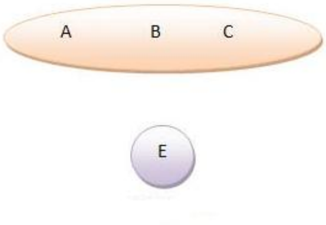

FIGURE 1. Hypergraph H_1

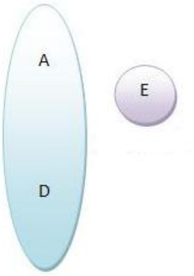

FIGURE 2. Hypergraph H_2

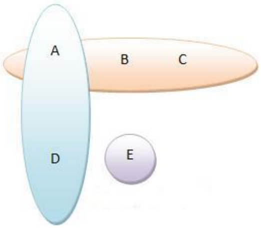

FIGURE 3. Hypergraph $H_1 \cup H_2$

Theorem 3.1. $(\mathbb{H}, \star)$ *is a commutative hypergroup.*

Proof. Let $H_1, H_2, H_3 \in \mathbb{H}$. It is easy to see that $H_1 \star H_2 = H_2 \star H_1$ as $H_1 \cup H_2 = H_2 \cup H_1$. Thus, $\star$ is a commutative hyperoperation.

It is clear that $H_1 \star \mathbb{H} \subseteq \mathbb{H}$. We need to show now that $\mathbb{H} \subseteq H_1 \star \mathbb{H}$. Let $H_2 \in \mathbb{H}$, then $H_2 \in H_1 \star H_2 \subseteq H_1 \star \mathbb{H}$ by Proposition 3.1. Thus, $(\mathbb{H}, \star)$ is a quasihypergroup.

We have that

$$\begin{aligned} H_1 \star (H_2 \star H_3) =& H_1 \star \bigcup \{K : K \text{ is a partial hypergraph of } H_2 \cup H_3\} \\ =& \bigcup \{H_1 \star K : K \text{ is a partial hypergraph of } H_2 \cup H_3\} \\ =& \bigcup \{M : M \text{ is partial hypergraph of } H_1 \cup K, \\ & K \text{ is partial hypergraph of } H_2 \cup H_3\} \\ =& \bigcup \{M : M \text{ is a partial hypergraph of } H_1 \cup H_2 \cup H_3\} \\ =& \text{partial hypergraphs of } H_1 \cup H_2 \cup H_3. \end{aligned}$$

On the other hand, we have that

$$\begin{aligned} (H_1 \star H_2) \star H_3 =& \bigcup \{K : K \text{ is a partial hypergraph of } H_1 \cup H_2\} \star H_3 \\ =& \bigcup \{K \star H_3 : K \text{ is a partial hypergraph of } H_1 \cup H_2\} \\ =& \bigcup \{M : M \text{ is partial hypergraph of } K \cup H_3, \\ & K \text{ is partial hypergraph of } H_1 \cup H_2\} \\ =& \bigcup \{M : M \text{ is a partial hypergraph of } H_1 \cup H_2 \cup H_3\} \\ =& \text{partial hypergraphs of } H_1 \cup H_2 \cup H_3. \end{aligned}$$

Therefore, $(\mathbb{H}, \star)$ is a commutative hypergroup. □

Proposition 3.3. *The only idempotent elements in* $(\mathbb{H}, \star)$ *are hypergraphs with one hyperedge.*

Proof. A hypergraph with exactly one hyperedge has only one partial hypergraph (which is itself) and hence it is idempotent.

If H is an idempotent in $(\mathbb{H}, \star)$, then

$$H \star H = \bigcup \{K : K \text{ is a partial hypergraph of } H\} = H.$$

The latter implies that H has only one partial hypergraph. Thus, H has one hyperedge. □

Proposition 3.4. $(\mathbb{H}, \star)$ *is a regular hypergroup.*

Proof. Proposition 3.1 implies that every element in $\mathbb{H}$ is an identity as $H_1 \in H_1 \star H_2$ for all $H_1, H_2 \in \mathbb{H}$. Let $I(H_1)$ be the set of all inverses of H_1 in $\mathbb{H}$. It is clear that $I(H_1) = \mathbb{H}$. □

Definition 3.2. A nonempty subset M of a hypergroup $(H, \star)$ is linear if $\alpha \star \beta \subseteq M$ and $\alpha/\beta \subseteq M$ for all α, $\beta \in M$. Here, $\alpha/\beta = \{x \in H \mid \alpha \in x \star \beta\}$.

Proposition 3.5. $(\mathbb{H}, \star)$ *has no proper linear subsets.*

Proof. Let M be a linear subset of $(\mathbb{H}, \star)$ and $H_1 \in M$. Having M a linear subset of $(\mathbb{H}, \star)$ implies that $H_1/H_1 \subseteq M$. We have that

$$H_1/H_1 = \{K \in \mathbb{H} : H_1 \in K \star H_1\}.$$

The latter and Proposition 3.1 imply that $H_1/H_1 = \mathbb{H} \subseteq M$. ☐

Proposition 3.6. $(\mathbb{H}, \star)$ *has no proper normal subhypergroups.*

Proof. For contradiction, suppose that N is a proper normal subhypergroup of $(\mathbb{H}, \star)$. Then there exists $k \in \mathbb{H}$ that is not in N. Having that $k \in k \star N$ (by Proposition 3.1) implies that $N \neq k \star N$. ☐

Proposition 3.7. $(\mathbb{H}, \star)$ *is a single power cyclic hypergroup with one generator and period two.*

Proof. Let $\alpha = \bigcup_{H_i \in \mathbb{H}} H_i \in \mathbb{H}$. It is clear that α is a generator of $\mathbb{H}$ of period two. Moreover, $\alpha \in \alpha^2 = \mathbb{H}$. ☐

Proposition 3.8. *Let M be any non-empty set of hypergraphs and*

$$\mathbb{K}_M = \left\{ \lambda : \lambda \textit{ is a partial hypergraph of } \bigcup_{K \in M} K \right\}.$$

Then $(\mathbb{K}_M, \star)$ is a cyclic subhypergroup of $(\mathbb{H}, \star)$.

Proof. The proof is straightforward. ☐

Proposition 3.9. *A subset A of $\mathbb{H}$ is a proper subhypergroup of $(\mathbb{H}, \star)$ if and only if $A = \mathbb{K}_M$ for some non-empty set M of hypergraphs.*

Proof. Let A be a proper subhypergroup of $(\mathbb{H}, \star)$ and suppose, for contradiction, that $A \neq \mathbb{K}_M$. Then there exists K, a partial hypergraph of $\bigcup_{\alpha \in A} \alpha$ that is not in A. The latter implies that K is in the hyperproduct of all elements of A. ☐

Proposition 3.10. $(\mathbb{H}, \star)$ *is a simple hypergroup.*

Proof. Let $\mathbb{I}$ be a proper hyperideal of $(\mathbb{H}, \star)$. Then $\mathbb{I}\mathbb{H} \subseteq \mathbb{I}$ and there exists $H \in \mathbb{H}$ such that H is not an element in $\mathbb{I}$. Having $H \in \mathbb{I}\mathbb{H}$ implies that $H \in \mathbb{I}$ which contradicts our hypothesis that H is not in $\mathbb{I}$. ☐

Corollary 3.1. *The only subhypergroups of $(\mathbb{H}, \star)$ are $(\mathbb{K}_M, \star)$ and they are cyclic.*

Proof. The proof results from Propositions 3.8 and 3.9. ☐

4. FUNDAMENTAL RELATION, AUTOMORPHISM GROUP AND COMPLETE PARTS OF $(\mathbb{H}, \star)$

In this section, we present some results related to fundamental relation, automorphism group, strongly regular relations and complete parts of $(\mathbb{H}, \star)$.

Definition 4.1. Let $(H, \circ)$ be a semihypergroup and R be an equivalence relation on H. If A and B are non-empty subsets of H, then

(a) $A\overline{R}B$ means that for every $a \in A$ there exists $b \in B$ such that aRb and for every $b' \in B$ there exists $a' \in A$ such that $a'Rb'$;

(b) $A\overline{\overline{R}}B$ means that for every $a \in A$ and $b \in B$, we have aRb.

The equivalence relation R is called

(a) regular on the right (on the left) if for all $x \in H$, from aRb, it follows that $(a \circ x)\overline{R}(b \circ x)$ $((x \circ a)\overline{R}(x \circ b)$ respectively);
(b) strongly regular on the right (on the left) if for all $x \in H$, from aRb, it follows that $(a \circ x)\overline{\overline{R}}(b \circ x)$ $((x \circ a)\overline{\overline{R}}(x \circ b)$ respectively);
(c) regular (strongly regular) if it is regular (strongly regular) on the right and on the left.

Theorem 4.1 ([9])**.** *Let $(H, \circ)$ be a hypergroup and R an equivalence relation on H. Then R is strongly regular if and only if $(H/R, \otimes)$, the set of all equivalence classes, is a group. Here, $\overline{x} \otimes \overline{y} = \{\overline{z} : x \in x \circ y\}$ for all $\overline{x}, \overline{y} \in H/R$.*

The fundamental relation has an important role in the study of semihypergroups and especially of hypergroups.

Definition 4.2 ([9])**.** For all $n \geq 1$, we define the relation β_n on a semihypergroup H, as follows: β_1 is the diagonal relation and, if $n > 1$, then

$$a\beta_n b \Leftrightarrow \exists (x_1, \ldots, x_n) \in H^n : \{a, b\} \subseteq \prod_{i=1}^{n} x_i,$$

$\beta = \bigcup_{n \geq 1} \beta_n$ and $\beta^\star$ is the transitive closure of β.

$\beta^\star$ is called the *fundamental equivalence relation* on H and $H/\beta^\star$ is called the *fundamental group.*

$\beta^\star$ is the smallest strongly regular relation on H and if H is a hypergroup then $\beta = \beta^\star$.

Proposition 4.1. *$(\mathbb{H}, \star)$ has trivial fundamental group.*

Proof. Let $H_1, H_2 \in \mathbb{H}$. Proposition 3.1 asserts that $\{H_1, H_2\} \subset H_1 \star H_2$. The latter implies that $H_1 \beta_2 H_2$. We get now that $H_1 \beta H_2$. Since $(\mathbb{H}, \star)$ is a hypergroup, it follows that $\beta = \beta^\star$. Consequently, $\mathbb{H}/\beta^\star$ has only one equivalence class. □

Proposition 4.2. *Let R be an equivalence relation on $\mathbb{H}$. Then R is strongly regular relation on $\mathbb{H}$ if and only if $\mathbb{H}/R$ is the trivial group.*

Proof. Theorem 4.1 asserts that if $\mathbb{H}/R$ is the trivial group then R is strongly regular relation on $\mathbb{H}$.

Let R be a strongly regular relation on $\mathbb{H}$. For all $x \in \mathbb{H}$, if aRb then $(a \star x)\overline{\overline{R}}(b \star x)$. The latter and having $x \in b \star x$, $a \in a \star x$ imply that aRx. Thus, $\mathbb{H}/R$ contains only one equivalence class. □

Definition 4.3. Let $(H, \circ)$ be an H_v- group and A be a nonempty subset of H. A is a complete part of H if for any natural number n and for all hyperproducts $P \in H_H(n)$, the following implication holds:

$$A \cap P \neq \emptyset \Rightarrow P \subseteq A.$$

Proposition 4.3. *The complete part of* $(\mathbb{H},\star)$ *is* $\mathbb{H}$.

Proof. Let A be a complete part of $(\mathbb{H},\star)$ and $a \in A$. Proposition 3.1 asserts that for all $b \in \mathbb{H}$, $a \in A \cap (a \star b) \neq \emptyset$. Having A a complete part of $\mathbb{H}$ implies that $b \in a \star b \subseteq A$. ☐

Proposition 4.4. *Let* $f \in \mathrm{Aut}(\mathbb{H},\star)$ *and* $\alpha \in \mathbb{H}$. *If* λ *is a partial hypergraph of* α, *then* $f(\lambda)$ *is a partial hypergraph of* $f(\alpha)$. *Moreover,* α *and* $f(\alpha)$ *have same number of partial hypergraphs.*

Proof. Let $f \in \mathrm{Aut}(\mathbb{H},\star)$ and $\alpha \in \mathbb{H}$. Having $f(\alpha \star \alpha) = f(\alpha) \star f(\alpha)$ implies that $\{f(\lambda) : \lambda \text{ is partial of } \alpha\} = \{\delta : \delta \text{ is partial of } f(\alpha)\}$. The latter implies that if λ is a partial hypergraph of α then $f(\lambda)$ is a partial hypergraph of $f(\alpha)$. Since f is bijective, it follows that α and $f(\alpha)$ have same number of partial hypergraphs. ☐

Theorem 4.2. *Let* f *be a bijective function. Then* $f \in \mathrm{Aut}(\mathbb{H},\star)$ *if and only if for all* $\alpha, \beta \in \mathbb{H}$ *the following conditions are satisfied:*

1. *if* λ *is a partial hypergraph of* α *then* $f(\lambda)$ *is a partial hypergraph of* $f(\alpha)$, *and*
2. $f(\alpha \star \beta) \subseteq f(\alpha) \star f(\beta)$.

Proof. Let $f \in \mathrm{Aut}(\mathbb{H},\star)$ and $\alpha \in \mathbb{H}$. Then $f(\alpha \star \beta) = f(\alpha) \star f(\beta)$. The latter and Proposition 4.4 imply that conditions 1. and 2. are satisfied.

Let f be any bijective function satisfying conditions 1. and 2. and let $\alpha, \beta \in \mathbb{H}$. Since α, β are partial hypergraphs of $\alpha \cup \beta$, it follows by condition 1. that $f(\alpha)$, $f(\beta)$ are partial hypergraphs of $f(\alpha \cup \beta)$. The latter implies that $f(\alpha) \cup f(\beta)$ is a partial hypergraph of $f(\alpha \cup \beta)$. Moreover, every partial hypergraph of $f(\alpha) \cup f(\beta)$ is a partial hypergraph of $f(\alpha \cup \beta)$. We get now that

$$\begin{aligned} f(\alpha) \star f(\beta) &= \{\delta \in \mathbb{H} : \delta \text{ is partial hypergraph of } f(\alpha) \cup f(\beta)\} \\ &\subseteq \{\lambda \in \mathbb{H} : \lambda \text{ is partial hypergraph of } f(\alpha \cup \beta)\}. \end{aligned}$$

Consequently, we get that $f(\alpha) \star f(\beta) \subseteq f(\alpha \star \beta)$. Thus, f is a good homomorphism by condition 2. ☐

Remark 4.1. It is easy to see that the identity function satisfies conditions 1. and 2. of Theorem 4.2.

Example 4.1. Let $H \in \mathbb{H}$, α be the hypergraph with vertex v_1 having only one hyperedge and β be the hypergraph with vertex v_2 having only one hyperedge. We define $f : (\mathbb{H},\star) \to (\mathbb{H},\star)$ as follows:

$$f(H) = \begin{cases} H, & \text{if } \alpha \cup \beta \text{ is a partial hypergraph of } H; \\ H, & \text{if neither } \alpha \text{ nor } \beta \text{ are partial hypergraphs of } H; \\ \beta \cup (H \setminus \{\alpha\}), & \text{if } \alpha \text{ is a partial hypergraph of } H; \\ \alpha \cup (H \setminus \{\beta\}), & \text{if } \beta \text{ is a partial hypergraph of } H. \end{cases}$$

Then $f \in \mathrm{Aut}(\mathbb{H},\star)$.

It is clear that f is a bijective function. Also, one can easily show that f satisfies condition 1. and 2. of Theorem 4.2.

5. RELATION OF $(\mathbb{H}, \star)$ TO ANOTHER HYPERGROUP $(\mathbb{H}, \circ)$

In this section, we define a new hyperoperation $(\circ)$ on $\mathbb{H}$ and find some relations between $(\mathbb{H}, \star)$, defined in Section 3, and $(\mathbb{H}, \circ)$.

Definition 5.1. Let $\mathbb{H}$ be the set of all hypergraphs and define $(\mathbb{H}, \circ)$ as follows. For all $H_1, H_2 \in \mathbb{H}$

$$H_1 \circ H_2 = \{H_1, H_2, H_1 \cup H_2\}.$$

We present some results on $(\mathbb{H}, \circ)$ in which their proofs are easy.

Theorem 5.1. $(\mathbb{H}, \circ)$ *is a regular commutative hypergroup.*

Proposition 5.1. *Every element in* $(\mathbb{H}, \circ)$ *is idempotent.*

Proposition 5.2. $(\mathbb{H}, \circ)$ *has no nontrivial cyclic subhypergroup.*

Proof. Proposition 5.1 asserts that $\alpha^k = \alpha$ for all $\alpha \in \mathbb{H}$ and $k \in \mathbb{N}$. $\square$

Definition 5.2. Let $(H, \circ)$ and $(H, \star)$ be two hypergroups. We say that $\circ \leq \star$ if there is $f \in \mathrm{Aut}(H, \star)$ such that $\alpha \circ \beta \subseteq f(\alpha) \star f(\beta)$ for all $\alpha, \beta \in H$.

Proposition 5.3. $\circ \leq \star$.

Proof. Let $i : (\mathbb{H}, \star) \to (\mathbb{H}, \star)$ be the identity map defined by: $i(H) = H$ for all $H \in \mathbb{H}$. It is clear that $i \in \mathrm{Aut}(\mathbb{H}, \star)$.

For all $H_1, H_2 \in \mathbb{H}$, we have each element in $H_1 \circ H_2 = \{H_1, H_2, H_1 \cup H_2\}$ is a partial hypergraph of $H_1 \cup H_2$. On the other hand, we have that $i(H_1) \star i(H_2) = H_1 \star H_2$ is the set of all partial hypergraphs of $H_1 \cup H_2$. Thus, $H_1 \circ H_2 \subseteq i(H_1) \star i(H_2)$. $\square$

Definition 5.3. Let R be a nonempty set with two hyperoperations ($+$ and $\cdot$). We say that $(R, +, \cdot)$ is a hyperring if $(R, +)$ is a commutative hypergroup, $(R, \cdot)$ is a semihypergroup and the hyperoperation $\cdot$ is distributive with respect to $+$, i.e., $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in R$.

If the hyperoperation $\cdot$ is weak distributive with respect to $+$, i.e., $x \cdot (y + z) \subseteq x \cdot y + x \cdot z$ for all $x, y, z \in R$, we say $(R, +, \cdot)$ that is a weak hyperring.

Proposition 5.4. $(\mathbb{H}, \star, \circ)$ *is a weak commutative hyperring.*

Proof. Propositions 3.1 and 5.1 imply that $(\mathbb{H}, \circ)$ and $(\mathbb{H}, \star)$ are commutative hypergroups. We need to prove that $(\mathbb{H}, \star, \circ)$ is weak distributive. For all $\alpha, \beta, \gamma \in \mathbb{H}$ we have

$$\begin{aligned}\alpha \circ (\beta \star \gamma) &= \bigcup\{\alpha \circ \lambda : \lambda \text{ is a partial hypergraph of } \beta \cup \gamma\} \\ &= \bigcup\{\alpha, \lambda, \alpha \cup \lambda : \lambda \text{ is a partial hypergraph of } \beta \cup \gamma\}.\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}(\alpha \circ \beta) \star (\alpha \circ \gamma) &= \{\alpha, \beta, \alpha \cup \beta\} \star \{\alpha, \gamma, \alpha \cup \gamma\} \\ &= \text{partial hypergraphs of } \{\alpha, \alpha \cup \gamma, \beta \cup \alpha, \beta \cup \alpha \cup \gamma, \beta \cup \gamma\} \\ &= \text{partial hypergraphs of } \alpha \cup \beta \cup \gamma.\end{aligned}$$

It is easy to see that $\alpha \circ (\beta \star \gamma) \subseteq (\alpha \circ \beta) \star (\alpha \circ \gamma)$. ☐

Proposition 5.5. $(\mathbb{H}, \circ, \star)$ *is a commutative hyperring.*

Proof. Propositions 3.1 and 5.1 imply that $(\mathbb{H}, \circ)$ and $(\mathbb{H}, \star)$ are commutative hypergroups. We need to prove that $(\mathbb{H}, \circ, \star)$ is distributive. For all $\alpha, \beta, \gamma \in \mathbb{H}$ we have

$$\begin{aligned}\alpha \star (\beta \circ \gamma) &= \alpha \star \{\beta, \gamma, \beta \cup \gamma\} \\ &= \text{partial hypergraphs of } \alpha \cup \beta \cup \gamma.\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}(\alpha \star \beta) \circ (\alpha \star \gamma) =& \text{partial hypergraphs of } \alpha \cup \beta \circ \text{ partial hypergraphs of } \alpha \cup \gamma \\ =& \bigcup \{\lambda, \lambda^\star, \lambda \cup \lambda^\star : \lambda \text{ and } \lambda^\star \text{ are partial hypergraphs} \\ & \text{of } \alpha \cup \beta \text{ and } \alpha \cup \gamma \text{ respectively}\} \\ =& \text{partial hypergraphs of } \alpha \cup \beta \cup \gamma.\end{aligned}$$

Thus, $\alpha \star (\beta \circ \gamma) = (\alpha \star \beta) \circ (\alpha \star \gamma)$. ☐

Proposition 5.6. *Let* $f : (\mathbb{H}, \circ) \to (\mathbb{H}, \star)$ *be any function. Then* f *is a weak homomorphism.*

Proof. Let $\alpha, \beta \in \mathbb{H}$. We have that $f(\alpha \circ \beta) = \{f(\alpha), f(\beta), f(\alpha \cup \beta)\}$. Having $f(\alpha)$, $f(\beta)$ partial hypergraphs of $f(\alpha) \cup f(\beta)$ implies that

$$\{f(\alpha), f(\beta)\} \subseteq f(\alpha \circ \beta) \cap f(\alpha) \star f(\beta) \neq \emptyset.$$

☐

Proposition 5.7. *Let* $c : (\mathbb{H}, \circ) \to (\mathbb{H}, \star)$ *be the constant function defined by:* $c(H) = K$*, where* K *is the hypergraph defined on any set of vertices with one hyperedge. Then* c *is a good homomorphism.*

Proof. The proof is straightforward by Proposition 3.3. ☐

Proposition 5.8. *Let* $f : (\mathbb{H}, \circ) \to (\mathbb{H}, \star)$ *be any function that is not equal to that defined in Proposition* 5.7. *Then* f *is not a good homomorphism.*

Proof. Let H be a hypergraph such that $f(H)$ has more than two hyperedges (such an element exists). We have that $f(H \circ H) = f(H)$ and $f(H) \star f(H)$ is the set of all partial hypergraphs of $f(H)$. Since $f(H)$ has more than two hyperedges, it follows that $|f(H) \star f(H)| \geq 2$. Thus, f is not a good homomorphism. ☐

Proposition 5.9. *Let* $f : (\mathbb{H}, \star) \to (\mathbb{H}, \circ)$ *be any function. Then* f *is a weak homomorphism.*

Proof. It is easy to see that $\{f(\alpha), f(\beta)\} \subseteq f(\alpha \star \beta) \cap f(\alpha) \circ f(\beta) \neq \emptyset$. ☐

Proposition 5.10. *Let* $k : (\mathbb{H}, \star) \to (\mathbb{H}, \circ)$ *be the function defined by* $k(\alpha) = H$ *for all* $\alpha \in \mathbb{H}$. *Then* f *is a good homomorphism.*

Proof. The proof is straightforward using Proposition 5.1. ☐

Proposition 5.11. *Let* $f : (\mathbb{H}, \star) \to (\mathbb{H}, \circ)$ *be any function other than that defined in Proposition* 5.10*. Then* f *is not a homomorphism.*

Proof. Since f is a function other than that defined in Proposition 5.10, it follows that there exist $\alpha, \beta \in \mathbb{H}$ such that $f(\alpha) \neq f(\beta)$. Let $\gamma = \alpha \cup \beta \in \mathbb{H}$. We have that $f(\gamma) \circ f(\gamma) = f(\gamma)$ and $f(\gamma \star \gamma) = \{f(\lambda) : \lambda \text{ is a partial hypergraph of } \gamma\}$. Having that $\alpha \neq \beta$ partial hypergraphs of γ and that $f(\alpha) \neq f(\beta)$ imply that $|f(\gamma \star \gamma)| \geq 2$. The latter implies that $f(\gamma \star \gamma)$ is not a subset of $f(\gamma) \circ f(\gamma)$. ☐

6. Conclusion

Hypergraph theory, introduced by Berge, is a generalization of graph theory and it has been considered an important topic in Mathematics due to its applications to numerous fields of Science. Our paper studied a connection between hypergraph theory and hypergroup theory. Here we defined hypergroups and hyperrings on the set of all hypergraphs. Also, we studied the fundamental group and regular relations of the defined hypergroups. Several results were obtained.

For future research, one may consider hyperfields associated to hypergraphs and study their properties.

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