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ON THE LIE CENTRALIZERS OF QUATERNION RINGS

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Abstract. In this paper, we investigate the problem of describing the form of Lie centralizers on quaternion rings. We provide some conditions under which a Lie centralizer on a quaternion ring is the sum of a centralizer and a center valued map.

1. Introduction and Preliminaries

Let R be a ring with the center $Z(R)$. For $a, b \in R$ denote the Lie product of a, b by $[a, b] = ab - ba$ and the Jordan product of a, b by $a \circ b = ab + ba$. Let $\phi : R \to R$ be an additive map. Recall that ϕ is said to be a right (left) centralizer map if $\phi(ab) = a\phi(b)(\phi(ab) = \phi(a)b)$ for all $a, b \in R$. It is called a centralizer if ϕ is both a right centralizer and a left centralizer. We say that ϕ is a Jordan centralizer if $\phi(a \circ b) = a \circ \phi(b)$ for all $a, b \in R$. An additive map $\phi : R \to R$ is called a Lie centralizer if

$$\phi[a, b] = [\phi(a), b] \quad (\text{or } \phi[a, b] = [a, \phi(b)]),$$

for each $a, b \in R$. We say that $\phi : R \to R$ is a center valued map if $\phi(R) \subseteq Z(R)$.

In the recently years, the structure of Lie centralizers on rings has been studied by some authors. An important question that naturally arises in this setting is under what conditions on a quaternion ring, a Lie centralizer can be decomposed into the sum of a centralizer and a center valued map. Jing [9] was the first one who introduced the concept of Lie centralizer and showed that every Lie centralizer on some triangular algebras is the sum of a centralizer and a center valued map. The authors [6] proved that a Lie centralizer under some conditions on some trivial extention algebras is the sum of a centralizer and a center valued map. Fošner and Jing [3] studied this result on triangular rings and nest algebras.

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Let S be a ring with identity. Set

$$H(S) = \{s_0 + s_1 i + s_2 j + s_3 k : s_i \in S\} = S \oplus Si \oplus Sj \oplus Sk,$$

where $i^2 = j^2 = k^2 = ijk = -1$ and $ij = -ji$. Then, with the componentwise addition and multiplication subject to the given relations and the conventions that i, j, k commute with S elementwise, $H(S)$ is a ring called the quaternion ring over S.

In this paper, we suppose that S be an unital ring in which 2 is invertible. We describe the Lie centralizers on $H(S)$, we show that if S is commutative or semiprime, then every Lie centralizer on $H(S)$ decomposes into the sum of a centralizer and a center valued map. Among the reasons for studying the mappings on quternion rings, we cite the recently published books and papers ([1,2,8]), in which the authors have considered the important roles of quaternion algebras in other branches of mathematics, such as differential geometry, analysis and quantum fields.

2. Lie Centralizers of Quaternion Rings

Our aim is to study a Lie centralizer map on a quaternion ring. We give conditions under which it is a sum of a centralizer and a center valued map. In the following, we establish a theorem which will be used to prove the fundamental results. From now on, we assume that S is a 2-torsion free ring with identity such that $\frac{1}{2} \in S$ and $R = H(S)$.

Theorem 2.1. *Let $f : R \to R$ be a Lie centralizer. Then there exists a Lie centralizer α on S and a Jordan centralizer β on S such that $f(t) = \alpha(x) + \beta(y)i + \beta(z)j + \beta(w)k$ for every element $t = x + yi + zj + wk \in R$.*

Proof. Assume that $f(i) = a + bi + cj + dk$ and $f(j) = a' + b'i + c'j + d'k$ for some suitable coefficients in S. Since f is a Lie centralizer, we have

$$f(k) = \frac{1}{2} f[i, j] = \frac{1}{2}[f(i), j] = bk - di.$$

Furthermore,

$$a + bi + cj + dk = f(i) = \frac{1}{2} f[j, k] = \frac{1}{2}[f(j), k] = -b'j + c'i.$$

Therefore, we get $a = d = 0$, $b' = -c$ and $c' = b$. Hence, $f(i) = bi + cj$ and $f(k) = bk$. Since f is a Lie centralizer, we have

$$f(j) = \frac{1}{2} f[k, i] = \frac{1}{2}[f(k), i] = bj.$$

After renaming the constants, we obtain

$$f(i) = ai + bj, \quad f(j) = aj, \quad f(k) = ak, \tag{2.1}$$

for suitable $a, b, c \in S$. Now, assume that $f(1) = t = x + yi + zj + wk$. We have

$$0 = f[1, i] = ti - it = 2wj - 2zk.$$

Thus, $w = z = 0$. On the other hand, we have

$$0 = f[1, j] = tj - jt = 2yk - 2wi.$$

Hence, $y = w = 0$. Therefore, we have $f(1) = x \in S$. Let $s \in S$, we have

$$0 = f[1, si] = (xs - sx)i.$$

Therefore, we get $xs = sx$. Hence, $f(1) \in Z(S)$. Let $s \in S$ and set $f(si) = x + yi + zj + wk$. Applying f on $[si, i] = 0$, we get $w = z = 0$ and hence $f(si) = x + yi$. Now, applying f on the identities $sk = \frac{1}{2}[si, j]$, $sj = \frac{1}{2}[sk, i]$ and $si = \frac{1}{2}[sj, k]$, and putting $y = \beta(s)$, we obtain

$$f(si) = \beta(s)i, \quad f(sj) = \beta(s)j, \quad f(sk) = \beta(s)k, \tag{2.2}$$

where $\beta : S \to S$ is an additive map uniquely determined by f.

Our next aim is to find $f(s)$ for arbitrary $s \in S$. Set $f(s) = x + yi + zj + wk$. Applying f on $[s, i] = 0$, we obtain $-2zk + 2wj = 0$. So, $z = w = 0$. Now, applying f on $[s, j] = 0$, we obtain that $y = 0$. Therefore, we have $f(s) = x$. Putting $x = \alpha(s)$, we have

$$f(s) = \alpha(s), \tag{2.3}$$

where $\alpha : S \to S$ is a map determined by f. Since f is a Lie centralizer, (2.3) implies that α is a Lie centralizer on S.

Let $s_1, s_2 \in S$. It is obvious that $[s_1i, s_2j] = (s_1 \circ s_2)k$, $[s_1i, s_2i] = [s_2, s_1]$ and $[s_1, s_2i] = [s_1, s_2]i$. Now, applying f on this identities and using (2.2) and (2.3), we find, respectively, that

$$\beta(s_1 \circ s_2) = \beta(s_1) \circ s_2, \tag{2.4}$$

$$\alpha[s_1, s_2] = [\beta(s_1), s_2], \tag{2.5}$$

$$\beta[s_1, s_2] = [\alpha(s_1), s_2]. \tag{2.6}$$

(2.4) shows that β is a Jordan centralizer on S. Now, let $t = x + yi + zj + wk$ be an arbitrary element in R. By (2.2) and (2.3), we get $f(t) = \alpha(x) + \beta(y)i + \beta(z)j + \beta(w)k$, as desired. □

As a consequence of Theorem 2.1, we have the following results.

Corollary 2.1. *Let S be a 2-torsion free commutative ring with identity such that $\frac{1}{2} \in S$. If $f : H(S) \to H(S)$ be a Lie centralizer, then f is the sum of a centralizer and a center valued map.*

Proof. Since S is 2-torsion free and commutative, the Jordan centralizer β is a centralizer on S. Let $t = x + yi + zj + wk \in H(S)$. Define $\Gamma : H(S) \to H(S)$ by $\Gamma(t) = \beta(x) + \beta(y)i + \beta(z)j + \beta(w)k$. It is easily verified that Γ is a centralizer. By Theorem 2.1, we have $f(t) = \Gamma(t) + \alpha(x) - \beta(x)$. It remains to show that the mapping $\tau : H(S) \to H(S)$ given by $\tau(t) = \alpha(x) - \beta(x)$ is a center valued map. Obviously, τ is a well-defined additive map such that $\tau(H(S)) \subseteq S$. By [4, Lemma 2.1], we have $Z(H(S)) = S$. Therefore, we have $\tau(H(S)) \subseteq Z(H(S))$. This completes the proof. $\square$

Corollary 2.2. *Let S be a 2-torsion free semiprime ring with identity such that $\frac{1}{2} \in S$. If $f : H(S) \to H(S)$ be a Lie centralizer, then f is the sum of a centralizer and a center valued map.*

Proof. Since S is a 2-torsion free semiprime ring, the Jordan centralizer β is a centralizer on S by [10]. Now, let Γ and τ be the mappings defined in Corollary 2.1. It is easily verified that Γ is a centralizer. It remains to show that the mapping τ is a center valued map. Let $s_1, s_2 \in S$. Since β is a centralizer on S, from (2.6), we obtain

$$[\tau(s_1), s_2] = [\alpha(s_1) - \beta(s_1), s_2] = 0. \tag{2.7}$$

Let $t = x + yi + zj + wk$, $t' = x' + y'i + z'j + w'k \in H(S)$. Using (2.7), we have

$$\begin{aligned}
[\tau(t), t'] =&[\alpha(x) - \beta(x), t'] \\
=&[\tau(x), x'] + [\tau(x), y']i + [\tau(x), z']j + [\tau(x), w']k \\
=&0.
\end{aligned}$$

Therefore, we have $\tau(H(S)) \subseteq Z(H(S))$. This completes the proof. $\square$

References

[1] S. L. Adler, *Quaternionic Quantum Mechanics and Quantum Fields,* Oxford University Press, New York, 1995.

[2] O. P. Agrawal, *Hamilton operators and dual-number-quaternions in spatial kinematics*, Mechanism and Machine Theory **22**(6) (1987), 569–575.

[3] A. Fošner and W. Jing, *Lie centralizers on triangular rings and nest algebras,* Adv. Oper. Theory **4** (2019), 342–350.

[4] H. Ghahramani, M. N. Ghosseiri and L. Heidari Zade, *On the Lie derivations and generalized Lie derivations of quaternion rings,* Comm. Algebra **47**(3) (2019), 1215–1221.

[5] H. Ghahramani, M. N. Ghosseiri and S. Safari, *Some questions concerning superderivations on $\mathbb{Z}_2$-graded rings,* Aequationes Math. **91**(4) (2017), 725–738.

[6] F. Ghomanjani and M. A. Bahmani, *A note on Lie centralizer maps,* Palest. J. Math. **7**(2) (2018), 468–471.

[7] B. Hvala, *Generalized Lie derivations in prime rings,* Taiwanese J. Math. **11**(5) (2007), 1425–1430.

[8] M. Jafari and Y. Yayli, *Generalized quaternions and their algebraic properties,* Commun. Fac. Sci. Univ. Ank. Ser. A1. Math. Stat. **64**(1) (2015), 15–27.

[9] W. Jing, *Additivity of Lie centralizers on triangular rings,* Math and Computer Science Working Papers (2011), 1–10.

[10] B. Zalar, *On centralizers of semiprime rings,* Comment. Math. Univ. Carolin. **32** (1991), 609–614.

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