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COEFFICIENT ESTIMATES FOR SUBCLASS OF m-FOLD SYMMETRIC BI-UNIVALENT FUNCTIONS

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Abstract. In the present paper, a general subclass $\mathcal{M}_{\Sigma_m}^{h,p}(\lambda,\gamma)$ of the m-Fold symmetric bi-univalent functions is defined. Also, the estimates of the Taylor-Maclaurin coefficients $|a_{m+1}|$, $|a_{2m+1}|$ and Fekete-Szegö problems are obtained for functions in this new subclass. The results presented in this paper would generalize and improve some recent works of several earlier authors.

1. Introduction

Let $\mathcal{A}$ be a class of analytic functions in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$ of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n. \tag{1.1}$$

Denote by $\mathcal{S}$ the class of all functions in the normalized analytic function class $\mathcal{A}$ which are univalent in $\mathbb{U}$ (see details in [2, 3]).

Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk $\mathbb{U}$. In fact, the Koebe one-quarter theorem [3] ensures that the image of $\mathbb{U}$ under every univalent function $f\in\mathcal{S}$ contains a disk of radius $1/4$. Therefore, every function $f\in\mathcal{S}$ has an inverse f^{-1} satisfying $f^{-1}(f(z))=z$ $(z\in\mathbb{U})$ and

$$f(f^{-1}(w))=w \quad \left(|w|<r_0(f),\, r_0(f)\geq\frac{1}{4}\right).$$

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In fact, the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$, if both f and f^{-1} are univalent in $\mathbb{U}$ (see [10]). We denote $\sigma_{\mathcal{B}}$ the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). For examples the functions $\frac{z}{1-z}$ and $-\log(1-z)$ belong to the class $\sigma_{\mathcal{B}}$.

The first time in 1967, Lewin [4] introduced the class $\sigma_{\mathcal{B}}$ and proved that the bound for the second coefficients of every $f \in \sigma_{\mathcal{B}}$ satisfies the inequality $|a_2| < 1.51$. Also, Smith [5] showed that $|a_2| < 2/\sqrt{27}$ and $|a_3| < 4/27$ for bi-univalent polynomial $f(z) = z + a_2 z^2 + a_3 z^3$ with real coefficients.

Recently many researchers introduced subclasses of bi-univalent functions and obtain non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. For example, we refer the reader to Srivastava et al. [6, 8, 10] and others [13, 14]. The coefficient estimate problem, i.e., bound of $|a_n|$ $(n \in \mathbb{N} - \{2, 3\})$ for each $f \in \sigma_{\mathcal{B}}$, is still an open problem.

Let m be a positive integer. A domain E is known as m-Fold symmetric if a rotation of E around origin with an angle $2\pi/$ maps E on itself. A function $f(z)$ analytic in $\mathbb{U}$ is said to be m-Fold symmetric if

$$f\left(e^{i\frac{2\pi}{m}} z\right) = e^{i\frac{2\pi}{m}} f(z).$$

For each function $f \in \mathcal{S}$, function

$$h(z) = \sqrt[m]{f(z^m)} \tag{1.3}$$

is univalent and maps unit disk $\mathbb{U}$ into a region with m-Fold symmetry.

We denote by $\mathcal{S}_m$ the class of m-Fold symmetric univalent functions in $\mathbb{U}$ and clearly $\mathcal{S}_1 = \mathcal{S}$. Every $f \in \mathcal{S}_m$ has a series expansion of the form

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \quad (z \in \mathbb{U},\ m \in \mathbb{N}). \tag{1.4}$$

Srivastava et al. [11], introduced a natural extensions of m-Fold symmetric univalent functions and defined the class Σ_m of symmetric bi-univalent functions. They obtained the series expansion for $g = f^{-1}$ as:

$$\begin{aligned} f^{-1}(w) =& w - a_{m+1} w^{m+1} + [(m+1)a_{m+1}^2 - a_{2m+1}]w^{2m+1} \\ & - \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right] w^{3m+1} + \cdots . \end{aligned} \tag{1.5}$$

For $m = 1$ formula (1.5) coincides with formula (1.2) of the class $\sigma_{\mathcal{B}}$.

In fact, this widely-cited work by Srivastava et al. [7] actually revived the study of m-Fold bi-univalent functions in recent years and that it has led to a flood of papers on the subject by (for example) Srivastava et al. [7, 9, 11, 12].

The aim of the this paper is to introduce new subclass $\mathcal{M}_{\Sigma_m}^{h,p}(\lambda, \gamma)$ of the m-Fold symmetric bi-univalent functions class Σ_m. Moreover, we obtain estimates on initial coefficients $|a_{m+1}|$, $|a_{2m+1}|$ and Fekete-Szegö problems for functions in this subclass.

The results presented in this paper would generalize and improve some recent works of Altinkaya et al. [1] and Li et al. [13].

2. Subclass $\mathcal{M}_{\Sigma_m}^{h,p}(\lambda,\gamma)$

In this section, we introduce and consider the subclass $\mathcal{M}_{\Sigma_m}^{h,p}(\lambda,\gamma)$.

Definition 2.1. Assume that $h : \mathbb{U} \to \mathbb{C}$ and $p : \mathbb{U} \to \mathbb{C}$, are analytic functions of the form

$$\begin{aligned} h(z) =& 1 + h_m z^m + h_{2m} z^{2m} + h_{3m} z^{3m} + \cdots ,\\ p(w) =& 1 + p_m w^m + p_{2m} w^{2m} + p_{3m} w^{3m} + \cdots , \end{aligned}$$

such that

$$\min\{\mathrm{Re}((h(z)), \mathrm{Re}\,(p(z))\} > 0 \quad (z \in \mathbb{U}).$$

Let $\lambda \geq 0$ and $\gamma \in \mathbb{C} - \{0\}$. We say that a function f given by (1.4) is in the subclass $\mathcal{M}_{\Sigma_m}^{h,p}(\lambda,\gamma)$, if the following conditions are satisfied:

$$1 + \frac{1}{\gamma}\left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1\right] \in h(\mathbb{U}) \quad (z \in \mathbb{U}) \tag{2.1}$$

and

$$1 + \frac{1}{\gamma}\left[(1-\lambda)\frac{wg'(w)}{g(w)} + \lambda\left(1 + \frac{wg''(w)}{g'(w)}\right) - 1\right] \in p(\mathbb{U}) \quad (w \in \mathbb{U}), \tag{2.2}$$

where g is the extension of f^{-1} to $\mathbb{U}$.

Definition 2.2. A function $f \in \Sigma_m$ given by (1.4) is said to be in the subclass $\mathcal{C}_{\Sigma_m}(\beta)$ $(0 \leq \beta < 1)$, if two following conditions are satisfied:

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta \quad \text{and} \quad \mathrm{Re}\left(1 + \frac{wg''(w)}{g'(w)}\right) > \beta \quad (z, w \in \mathbb{U}),$$

where g is the extension of f^{-1} to $\mathbb{U}$.

Remark 2.1. There are many selections of the functions $h(z)$ and $p(z)$ which would provide interesting classes of m-Fold symmetric bi-univalent functions Σ_m. For example, if we let

$$h(z) = p(z) = \left(\frac{1+z^m}{1-z^m}\right)^{\alpha} = 1 + 2\alpha z^m + 2\alpha^2 z^{2m} + \cdots \quad (0 < \alpha \leq 1),$$

it is easy to verify that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 2.1. If $f \in \mathcal{M}_{\Sigma_m}^{h,p}(\lambda,\gamma)$, then

$$\left|\arg\left(1 + \frac{1}{\gamma}\left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1\right]\right)\right| < \frac{\alpha\pi}{2}$$

and

$$\left|\arg\left(1 + \frac{1}{\gamma}\left[(1-\lambda)\frac{wg'(w)}{g(w)} + \lambda\left(1 + \frac{wg''(w)}{g'(w)}\right) - 1\right]\right)\right| < \frac{\alpha\pi}{2}.$$

In this case we say that f belongs to the subclass $\mathcal{M}_{\Sigma_m}(\alpha,\lambda,\gamma)$.

Also, for $h(z)=p(z)=\left(\frac{1+z^m}{1-z^m}\right)^{\alpha}$, $\gamma=1$ and $\lambda=0$, the subclass $\mathcal{M}^{h,p}_{\Sigma_m}(\lambda,\gamma)$ reduces to the subclass $\mathcal{S}^{\alpha}_{\Sigma_m}$ which was considered by Altinkaya and Yalcin [1].

If we let

$$h(z)=p(z)=\frac{1+(1-2\beta)z^m}{1-z^m}=1+2(1-\beta)z^m+2(1-\beta)z^{2m}+\cdots \quad (0\le\beta<1),$$

it is easy to verify that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 2.1. If $f\in\mathcal{M}^{h,p}_{\Sigma_m}(\lambda,\gamma)$, then

$$\operatorname{Re}\left(1+\frac{1}{\gamma}\left[(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)-1\right]\right)>\beta$$

and

$$\operatorname{Re}\left(1+\frac{1}{\gamma}\left[(1-\lambda)\frac{wg'(w)}{g(w)}+\lambda\left(1+\frac{wg''(w)}{g'(w)}\right)-1\right]\right)>\beta.$$

In this case we say that f belongs to the subclass $\mathcal{M}_{\Sigma_m}(\beta,\lambda,\gamma)$.

Also, for $h(z)=p(z)=\frac{1+(1-2\beta)z^m}{1-z^m}$, $\gamma=1$ and $\lambda=0$, the subclass $\mathcal{M}^{h,p}_{\Sigma_m}(\lambda,\gamma)$ reduces to the subclass $\mathcal{S}^{\beta}_{\Sigma_m}$ considered by Altinkaya and Yalcin [1].

Furthermore, for $h(z)=p(z)=\frac{1+(1-2\beta)z^m}{1-z^m}$, $\gamma=1$ and $\lambda=1$, the subclass $\mathcal{M}^{h,p}_{\Sigma_m}(\lambda,\gamma)$ reduces to Definition 2.2.

Remark 2.2. For one-fold symmetric bi-univalent functions, we denote the subclass $\mathcal{M}^{h,p}_{\Sigma_1}(\lambda,\gamma)=\mathcal{M}^{h,p}_{\Sigma}(\lambda,\gamma)$. Special cases of this subclass are illustrated below.

(i) By putting $h(z)=p(z)=\left(\frac{1+z}{1-z}\right)^{\alpha}$ and $\gamma=1$, the subclass $\mathcal{M}^{h,p}_{\Sigma}(\lambda,\gamma)$ reduces to the subclass $M_{\Sigma}(\alpha,\lambda)$ studied by Li and Wang [13].

(ii) By putting $h(z)=p(z)=\left(\frac{1+z}{1-z}\right)^{\alpha}$, $\gamma=1$ and $\lambda=0$, the subclass $\mathcal{M}^{h,p}_{\Sigma}(\lambda,\gamma)$ reduces to the subclass $\mathcal{S}^{\alpha}_{\sigma_{\mathrm{B}}}$ of strongly bi-starlike functions of order α $(0<\alpha\le1)$.

(iii) By putting $h(z)=p(z)=\frac{1+(1-2\beta)z}{1-z}$ and $\gamma=1$, the subclass $\mathcal{M}^{h,p}_{\Sigma}(\lambda,\gamma)$ reduces to the subclass $B_{\Sigma}(\beta,\lambda)$ studied by Li and Wang [13].

(iv) By putting $h(z)=p(z)=\frac{1+(1-2\beta)z}{1-z}$, $\gamma=1$ and $\lambda=0$, the subclass $\mathcal{M}^{h,p}_{\Sigma}(\lambda,\gamma)$ reduces to the subclass $\mathcal{S}_{\sigma_{\mathrm{B}}}(\beta)$ of bi-starlike functions of order β $(0\le\beta<1)$.

(v) By putting $h(z)=p(z)=\frac{1+(1-2\beta)z}{1-z}$ and $\lambda=\gamma=1$, the subclass $\mathcal{M}^{h,p}_{\Sigma}(\lambda,\gamma)$ reduces to the subclass $\mathcal{C}_{\sigma_{\mathrm{B}}}(\beta)$ of bi-convev functions of order β $(0\le\beta<1)$.

Theorem 2.1. *Let f given by* (1.4) *be in the subclass* $\mathcal{M}^{h,p}_{\Sigma_m}(\lambda,\gamma)$ $(\lambda\ge0,\ \gamma\in\mathbb{C}-\{0\})$. *Then*

$$|a_{m+1}|\le\min\left\{\frac{|\gamma||h_m|}{m(1+\lambda m)},\sqrt{\frac{|\gamma|(|h_{2m}|+|p_{2m}|)}{2m^2(1+\lambda m)}}\right\}$$

and

$$|a_{2m+1}| \leq \min\left\{\frac{|\gamma|(|h_{2m}|+|p_{2m}|)}{4m(1+2\lambda m)}+\frac{(m+1)|\gamma|^2(|h_m|^2+|p_m|^2)}{4m^2(1+\lambda m)^2},\right.$$
$$\left.\frac{(3\lambda m^2+2\lambda m+2m+1)|\gamma||h_{2m}|+(\lambda m^2+2\lambda m+1)|\gamma||p_{2m}|}{4m^2(1+2\lambda m)(1+\lambda m)}\right\}.$$

Proof. The main idea in the proof of Theorem 2.1 is to get the desired bounds for the coefficient $|a_{m+1}|$ and $|a_{2m+1}|$. Indeed, by considering the relations (2.1) and (2.2), we have

$$1+\frac{1}{\gamma}\left[(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)-1\right]=h(z) \quad (z\in\mathbb{U}) \tag{2.3}$$

and

$$1+\frac{1}{\gamma}\left[(1-\lambda)\frac{wg'(w)}{g(w)}+\lambda\left(1+\frac{wg''(w)}{g'(w)}\right)-1\right]=p(w) \quad (w\in\mathbb{U}), \tag{2.4}$$

where each of the functions h and p satisfies the conditions of Definition 2.1. For precise comparison of the coefficients of the above equations, in the following we obtain Taylor-Maclaurin series expansions each side of the equations

(2.5)
$$\begin{aligned}&1+\frac{1}{\gamma}\left[(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)-1\right]\\ =&1+\frac{m(1+\lambda m)}{\gamma}a_{m+1}z^m+\left\{\frac{2m(1+2\lambda m)}{\gamma}a_{2m+1}-\frac{m(1+2\lambda m+\lambda m^2)}{\gamma}a_{m+1}^2\right\}z^{2m}\\ &+\cdots,\end{aligned}$$

and

$$\begin{aligned}&1+\frac{1}{\gamma}\left[(1-\lambda)\frac{wg'(w)}{g(w)}+\lambda\left(1+\frac{wg''(w)}{g'(w)}\right)-1\right]\\ =&1-\frac{m(1+\lambda m)}{\gamma}a_{m+1}w^m+\left\{-\frac{2m(1+2\lambda m)}{\gamma}a_{2m+1}\right.\\ &\left.+\frac{m(1+2m+2\lambda m+3\lambda m^2)}{\gamma}a_{m+1}^2\right\}w^{2m}+\cdots.\end{aligned} \tag{2.6}$$

Also from the Definition 2.1, the analytic functions h and p have the following Taylor-Maclaurin series expansions

$$h(z)=1+h_mz^m+h_{2m}z^{2m}+h_{3m}z^{3m}+\cdots \tag{2.7}$$

and

$$p(w)=1+p_mw^m+p_{2m}w^{2m}+p_{3m}w^{3m}+\cdots. \tag{2.8}$$

By comparing the coefficients of the equations (2.5), (2.7), (2.6) and (2.8), respectively, we get

$$\frac{m(1+\lambda m)}{\gamma}a_{m+1} = h_m, \tag{2.9}$$

$$\frac{2m(1+2\lambda m)}{\gamma}a_{2m+1} - \frac{m(1+2\lambda m+\lambda m^2)}{\gamma}a_{m+1}^2 = h_{2m}, \tag{2.10}$$

$$-\frac{m(1+\lambda m)}{\gamma}a_{m+1} = p_m \tag{2.11}$$

and

$$-\frac{2m(1+2\lambda m)}{\gamma}a_{2m+1} + \frac{m(1+2m+2\lambda m+3\lambda m^2)}{\gamma}a_{m+1}^2 = p_{2m}. \tag{2.12}$$

From (2.9) and (2.11), we get

$$h_m = -p_m \tag{2.13}$$

and

$$a_{m+1}^2 = \frac{\gamma^2(h_m^2+p_m^2)}{2m^2(1+\lambda m)^2}. \tag{2.14}$$

Adding (2.10) and (2.12), we get

$$a_{m+1}^2 = \frac{\gamma(h_{2m}+p_{2m})}{2m^2(1+\lambda m)}. \tag{2.15}$$

Therefore, we find from the equations (2.13), (2.14) and (2.15) that

$$|a_{m+1}| \leq \frac{|\gamma||h_m|}{m(1+\lambda m)} \quad \text{and} \quad |a_{m+1}| \leq \sqrt{\frac{|\gamma|(|h_{2m}|+|p_{2m}|)}{2m^2(1+\lambda m)}},$$

respectively. So, we get the desired estimate on the coefficient $|a_{m+1}|$.

The proof is completed by finding the bound on the coefficient $|a_{2m+1}|$. Upon subtracting (2.12) from (2.10), we get

$$a_{2m+1} = \frac{\gamma(h_{2m}-p_{2m})}{4m(1+2\lambda m)} + \frac{(m+1)}{2}a_{m+1}^2. \tag{2.16}$$

Putting the value of a_{m+1}^2 from (2.14) into (2.16), it follows that

$$a_{2m+1} = \frac{\gamma(h_{2m}-p_{2m})}{4m(1+2\lambda m)} + \frac{(m+1)\gamma^2(h_m^2+p_m^2)}{4m^2(1+\lambda m)^2}. \tag{2.17}$$

By substituting the value of a_{m+1}^2 from (2.15) into (2.16), we obtain

$$a_{2m+1} = \frac{\gamma(h_{2m}-p_{2m})}{4m(1+2\lambda m)} + \frac{(m+1)\gamma(h_{2m}+p_{2m})}{4m^2(1+\lambda m)}. \tag{2.18}$$

Therefore, from the equations (2.17) and (2.18), we get

$$|a_{2m+1}| \leq \frac{|\gamma|(|h_{2m}|+|p_{2m}|)}{4m(1+2\lambda m)} + \frac{(m+1)|\gamma|^2(|h_m|^2+|p_m|^2)}{4m^2(1+\lambda m)^2}$$

and

$$|a_{2m+1}| \leq \frac{(3\lambda m^2+2\lambda m+2m+1)|\gamma||h_{2m}|+(\lambda m^2+2\lambda m+1)|\gamma||p_{2m}|}{4m^2(1+2\lambda m)(1+\lambda m)}. \qquad \square$$

Theorem 2.2. *Let f given by (1.4) be in the subclass $\mathcal{M}_{\Sigma_m}^{h,p}(\lambda,\gamma)$ ($\lambda \geq 0$, $\gamma \in \mathbb{C}-\{0\}$). Also let ρ be real number. Then*

$$|a_{2m+1}-\rho a_{m+1}^2| \leq \begin{cases} \dfrac{|\gamma|}{4m(1+2\lambda m)}\left\{(1+T(\rho))\,|h_{2m}|+(1-T(\rho))\,|p_{2m}|\right\}, & |T(\rho)| \leq 1, \\ \dfrac{|\gamma|}{4m(1+2\lambda m)}\left\{\Big|1+T(\rho)\Big||h_{2m}|+\Big|T(\rho)-1\Big||p_{2m}|\right\}, & |T(\rho)| \geq 1, \end{cases}$$

where

$$T(\rho) = \frac{(m-2\rho+1)(1+2\lambda m)}{m(1+\lambda m)}.$$

Proof. From the equation (2.16), we get

$$a_{2m+1}-\rho a_{m+1}^2 = \frac{\gamma(h_{2m}-p_{2m})}{4m(1+2\lambda m)} + \frac{m-2\rho+1}{2}a_{m+1}^2. \tag{2.19}$$

From the equation (2.15) and (2.19), we have

$$\begin{aligned} a_{2m+1}-\rho a_{m+1}^2 =& \frac{|\gamma|}{4m(1+2\lambda m)}\left\{\left[1+\frac{(m-2\rho+1)(1+2\lambda m)}{m\,(1+\lambda m)}\right]h_{2m}\right. \\ &\left. + \left[\frac{(m-2\rho+1)(1+2\lambda m)}{m(1+\lambda m)}-1\right]p_{2m}\right\}. \end{aligned}$$

Next, taking the absolute values we obtain

$$\begin{aligned} |a_{2m+1}-\rho a_{m+1}^2| \leq& \frac{|\gamma|}{4m(1+2\lambda m)}\left\{\left|1+\frac{(m-2\rho+1)(1+2\lambda m)}{m\,(1+\lambda m)}\right||h_{2m}|\right. \\ &\left. + \left|\frac{(m-2\rho+1)(1+2\lambda m)}{m(1+\lambda m)}-1\right||p_{2m}|\right\}. \end{aligned}$$

Then, we conclude that

$$|a_{2m+1}-\rho a_{m+1}^2| \leq \begin{cases} \dfrac{|\gamma|}{4m(1+2\lambda m)}\left\{(1+T(\rho))\,|h_{2m}|+(1-T(\rho))\,|p_{2m}|\right\}, & |T(\rho)| \leq 1, \\ \dfrac{|\gamma|}{4m(1+2\lambda m)}\left\{\Big|1+T(\rho)\Big||h_{2m}|+\Big|T(\rho)-1\Big||p_{2m}|\right\}, & |T(\rho)| \geq 1. \end{cases}$$

$\square$

3. Corollaries and Consequences

By setting

$$h(z)=p(z)=\left(\frac{1+z^m}{1-z^m}\right)^{\alpha}=1+2\alpha z^m+2\alpha^2 z^{2m}+\cdots \quad (0<\alpha\leq 1,\ z\in\mathbb{U}),$$

in Theorem 2.1, we conclude the following result.

Corollary 3.1. *Let f given by* (1.4) *be in the subclass* $\mathcal{M}_{\Sigma_m}(\alpha,\lambda,\gamma)$ $(0<\alpha\leq 1,$ $\lambda\geq 0,\ \gamma\in\mathbb{C}-\{0\})$. *Then*

$$|a_{m+1}|\leq \min\left\{\frac{2\alpha|\gamma|}{m(1+\lambda m)},\frac{\alpha}{m}\sqrt{\frac{2|\gamma|}{1+\lambda m}}\right\}$$

and

$$|a_{2m+1}|\leq \min\left\{\frac{\alpha^2|\gamma|}{m(1+2\lambda m)}+\frac{2\alpha^2(m+1)|\gamma|^2}{m^2(1+\lambda m)^2},\frac{\alpha^2|\gamma|(m+1)}{m^2(1+\lambda m)}\right\}.$$

By setting $h(z)=p(z)=\left(\frac{1+z^m}{1-z^m}\right)^{\alpha}$ $(0<\alpha\leq 1)$ in Theorem 2.2, we conclude the following result.

Corollary 3.2. *Let f given by* (1.4) *be in the subclass* $\mathcal{M}_{\Sigma_m}(\alpha,\lambda,\gamma)$ $(0<\alpha\leq 1,$ $\lambda\geq 0,\ \gamma\in\mathbb{C}-\{0\})$. *Also let ρ be real number. Then*

$$|a_{2m+1}-\rho a_{m+1}^2|\leq \begin{cases}\dfrac{\alpha^2|\gamma|}{m(1+2\lambda m)}, & |T(\rho)|\leq 1,\\[2mm] \dfrac{\alpha^2|T(\rho)||\gamma|}{m(1+2\lambda m)}, & |T(\rho)|\geq 1,\end{cases}$$

where

$$T(\rho)=\frac{(m-2\rho+1)(1+2\lambda m)}{m(1+\lambda m)}.$$

By setting $\gamma=1$ and $\lambda=0$ in Corollary 3.1, we conclude the following result.

Corollary 3.3. *Let f given by* (1.4) *be in the subclass* $\mathcal{S}^{\alpha}_{\Sigma_m}$ $(0<\alpha\leq 1)$. *Then*

$$|a_{m+1}|\leq \frac{\sqrt{2}\alpha}{m}$$

and

$$|a_{2m+1}|\leq \min\left\{\frac{\alpha^2}{m}+\frac{2(m+1)\alpha^2}{m^2},\frac{(m+1)\alpha^2}{m^2}\right\}=\frac{(m+1)\alpha^2}{m^2}.$$

Remark 3.1. The bounds on $|a_{m+1}|$ and $|a_{2m+1}|$ given in Corollary 3.3 are better than those given in [1, Corolary 6], because of

$$\frac{\sqrt{2}\alpha}{m}\leq \frac{2\alpha}{m\sqrt{\alpha+1}}$$

and

$$\frac{(m+1)\alpha^2}{m^2}\leq \frac{\alpha^2}{m}+\frac{2(m+1)\alpha^2}{m^2}\leq \frac{\alpha}{m}+\frac{2(m+1)\alpha^2}{m^2}.$$

By setting $m = 1$ and $\gamma = 1$ in Corollary 3.1, we conclude the following result.

Corollary 3.4. *Let f given by* (1.1) *be in the subclass $M_\Sigma(\alpha, \lambda)$* $(0 < \alpha \leq 1,\ \lambda \geq 0)$*. Then*

$$|a_2| \leq \begin{cases} \alpha\sqrt{\dfrac{2}{1+\lambda}}, & 0 \leq \lambda \leq 1, \\ \dfrac{2\alpha}{1+\lambda}, & \lambda \geq 1, \end{cases}$$

and

$$|a_3| \leq \begin{cases} \dfrac{2\alpha^2}{1+\lambda}, & 0 \leq \lambda \leq \dfrac{2+\sqrt{13}}{3}, \\ \dfrac{\alpha^2}{1+2\lambda} + \dfrac{4\alpha^2}{(1+\lambda)^2}, & \lambda \geq \dfrac{2+\sqrt{13}}{3}. \end{cases}$$

Remark 3.2. The bounds on $|a_2|$ and $|a_3|$ given in Corollary 3.4 are better than those given in [13, Theorem 2.2].

By setting $m = 1$ in Corollary 3.3, we conclude the following result.

Corollary 3.5. *Let f given by* (1.1) *be in the subclass $\mathcal{S}^{\alpha}_{\sigma_{\mathsf{B}}}$ of strongly bi-starlike functions of order α* $(0 < \alpha \leq 1)$*. Then*

$$|a_2| \leq \sqrt{2}\alpha \quad \textit{and} \quad |a_3| \leq 2\alpha^2.$$

By setting

$$\begin{aligned} h(z) =& p(z) = \frac{1+(1-2\beta)z^m}{1-z^m} \\ =& 1 + 2(1-\beta)z^m + 2(1-\beta)z^{2m} + \cdots \quad (0 \leq \beta < 1, z \in \mathbb{U}), \end{aligned}$$

in Theorem 2.1, we conclude the following result.

Corollary 3.6. *Let f given by* (1.4) *be in the subclass $\mathcal{M}_{\Sigma_m}(\beta, \lambda, \gamma)$* $(0 \leq \beta < 1,$ $\lambda \geq 0,\ \gamma \in \mathbb{C} - \{0\})$*. Then*

$$|a_{m+1}| \leq \min\left\{\frac{2(1-\beta)|\gamma|}{m(1+\lambda m)}, \sqrt{\frac{2(1-\beta)|\gamma|}{m^2(1+\lambda m)}}\right\}$$

and

$$|a_{2m+1}| \leq \min\left\{\frac{(1-\beta)|\gamma|}{m(1+2\lambda m)} + \frac{2(1-\beta)^2(m+1)|\gamma|^2}{m^2(1+\lambda m)^2}, \frac{(1-\beta)(m+1)|\gamma|}{m^2(1+\lambda m)}\right\}.$$

By setting $h(z) = p(z) = \frac{1+(1-2\beta)z^m}{1-z^m}$ $(0 \leq \beta < 1)$ in Theorem 2.2, we conclude the following result.

Corollary 3.7. *Let f given by (1.4) be in the subclass $\mathcal{M}_{\Sigma_m}(\beta,\lambda,\gamma)$ $(0 \le \beta < 1$, $\lambda \ge 0$, $\gamma \in \mathbb{C} - \{0\})$. Also let ρ be real number. Then*

$$|a_{2m+1} - \rho a_{m+1}^2| \le \begin{cases} \dfrac{(1-\beta)|\gamma|}{m(1+2\lambda m)}, & |T(\rho)| \le 1, \\ \dfrac{(1-\beta)|\gamma||T(\rho)|}{m(1+2\lambda m)}, & |T(\rho)| \ge 1, \end{cases}$$

where

$$T(\rho) = \frac{(m-2\rho+1)(1+2\lambda m)}{m(1+\lambda m)}.$$

By setting $\gamma = 1$ and $\lambda = 0$ in Corollary 3.6, we conclude the following result.

Corollary 3.8. *Let f given by (1.4) be in the subclass $\mathcal{S}^{\beta}_{\Sigma_m}$ $(0 \le \beta < 1)$. Then*

$$|a_{m+1}| \le \begin{cases} \dfrac{\sqrt{2(1-\beta)}}{m}, & 0 \le \beta \le \dfrac{1}{2}, \\ \dfrac{2(1-\beta)}{m}, & \dfrac{1}{2} \le \beta < 1, \end{cases}$$

and

$$|a_{2m+1}| \le \begin{cases} \dfrac{(m+1)(1-\beta)}{m^2}, & 0 \le \beta \le \dfrac{1+2m}{2(1+m)}, \\ \dfrac{2(m+1)(1-\beta)^2}{m^2} + \dfrac{1-\beta}{m}, & \dfrac{1+2m}{2(1+m)} \le \beta < 1. \end{cases}$$

Remark 3.3. The bounds on $|a_{m+1}|$ and $|a_{2m+1}|$ given in Corollary 3.8 are better than those given in [1, Corolary 7].

By setting $\gamma = 1$ and $\lambda = 1$ in Corollary 3.6, we conclude the following result.

Corollary 3.9. *Let f given by (1.4) be in the subclass $\mathcal{C}_{\Sigma_m}(\beta)$ $(0 \le \beta < 1)$. Then*

$$|a_{m+1}| \le \begin{cases} \dfrac{1}{m}\sqrt{\frac{2(1-\beta)}{(1+m)}}, & 2\beta + m \le 1, \\ \dfrac{2(1-\beta)}{m(1+m)}, & 2\beta + m \ge 1, \end{cases}$$

and

$$|a_{2m+1}| \le \begin{cases} \dfrac{1-\beta}{m^2}, & 0 \le \beta \le \dfrac{1+2m-m^2}{2(1+2m)}, \\ \dfrac{1-\beta}{m(1+2m)} + \dfrac{2(1-\beta)^2}{m^2(1+m)}, & \dfrac{1+2m-m^2}{2(1+2m)} \le \beta < 1. \end{cases}$$

By setting $m = 1$ and $\gamma = 1$ in Corollary 3.6, we conclude the following result.

Corollary 3.10. *Let f given by* (1.1) *be in the subclass* $B_\Sigma(\beta,\lambda)$ $(0 \le \beta < 1,\ \lambda \ge 0)$. *Then*

$$|a_2| \le \begin{cases} \sqrt{\dfrac{2(1-\beta)}{1+\lambda}}, & \lambda + 2\beta \le 1, \\ \dfrac{2(1-\beta)}{1+\lambda}, & \lambda + 2\beta \ge 1, \end{cases}$$

and

$$|a_3| \le \begin{cases} \dfrac{2(1-\beta)}{1+\lambda}, & 0 \le \beta \le \dfrac{3+4\lambda-3\lambda^2}{4(1+2\lambda)}, \\ \dfrac{1-\beta}{1+2\lambda} + \dfrac{4(1-\beta)^2}{(1+\lambda)^2}, & \dfrac{3+4\lambda-3\lambda^2}{4(1+2\lambda)} \le \beta < 1. \end{cases}$$

Remark 3.4. The bounds on $|a_2|$ and $|a_3|$ given in Corollary 3.10 are better than those given in [13, Theorem 3.2].

By setting $m = 1$ in Corollary 3.8, we conclude the following result.

Corollary 3.11. *Let f given by* (1.1) *be in the subclass* $\mathcal{S}_{\sigma_{\mathrm{B}}}(\beta)$ *of bi-starlike functions of order* β $(0 \le \beta < 1)$. *Then*

$$|a_2| \le \begin{cases} \sqrt{2(1-\beta)}, & 0 \le \beta \le \dfrac{1}{2}, \\ 2(1-\beta), & \dfrac{1}{2} \le \beta < 1, \end{cases}$$

and

$$|a_3| \le \begin{cases} 2(1-\beta), & 0 \le \beta \le \dfrac{3}{4}, \\ 4(1-\beta)^2 + (1-\beta), & \dfrac{3}{4} \le \beta < 1. \end{cases}$$

By setting $m = 1$ in Corollary 3.9, we conclude the following result.

Corollary 3.12. *Let f given by* (1.1) *be in the subclass* $\mathcal{C}_{\sigma_{\mathrm{B}}}(\beta)$ *of bi-convex functions of order* β $(0 \le \beta < 1)$. *Then*

$$|a_2| \le 1-\beta \quad \text{and} \quad |a_3| \le \begin{cases} 1-\beta, & 0 \le \beta \le \dfrac{1}{3}, \\ \dfrac{1-\beta}{3} + (1-\beta)^2, & \dfrac{1}{3} \le \beta < 1. \end{cases}$$

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