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GENERALIZED AVERAGED GAUSSIAN FORMULAS FOR CERTAIN WEIGHT FUNCTIONS

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Abstract. In this paper we analyze the generalized averaged Gaussian quadrature formulas and the simplest truncated variant for one of them for some weight functions on the interval $[0,1]$ considered by Milovanović in [10]. We shall investigate internality of these formulas for the equivalents of the Jacobi polynomials on this interval and, in some special cases, show the existence of the Gauss-Kronrod quadrature formula. We also include some examples showing the corresponding error estimates for some non-classical orthogonal polynomials.

1. Introduction

Consider the l-point Gauss quadrature formula

$$Q_l^G(f)=\sum_{i=1}^{l} w_i^{(l)} f(x_i^{(l)})$$

on the interval $[a,b]$ with respect to a weight function w for the integral

$$I(f)=\int_a^b f(x)w(x)dx.$$

It has the highest possible degree of exactness, $2l-1$, and

$$Q_l^G(p)=I(p), \quad p\in \mathcal{P}^{2l-1},$$

where $\mathcal{P}^m$ denotes the space of polynomials of degree up to m.

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To estimate the error $(I-Q_l^G)(f)$, one can use the difference $(A-Q_l^G)(f)$, where A is some quadrature formula of degree greater than $2l-1$. Any such quadrature formula A requires at least $l+1$ additional nodes, so it will have at least $2l+1$ nodes. One classical way for constructing a $(2l+1)$-node formula A for certain weight functions is Gauss-Kronrod quadrature formula with degree of exactness at least $3l+1$. The Gauss-Kronrod formulas are of optimal degree, given that the nodes of $G_w^{(l)}$ are included. For some weight functions on compact intervals, such as the Legendre weight function $w(x)=1$ on $[-1,1]$, the Gauss-Kronrod formulas have real zeros inside the interval that interlace with the nodes of the Gauss formula and have positive weights. The polynomials of degree $l+1$ that vanish in the $l+1$ additional nodes are called Stieltjes polynomials. However, a real Gauss-Kronrod extension of a Gauss formula may not exist in general. This happens e.g. for the Gauss-Laguerre and Gauss-Hermite cases (see [6]), as well as for the Jacobi weights $w^{\alpha,\beta}(t)=(1-t)^\alpha(1+t)^\beta$ for $\min\{\alpha,\beta\}\geqslant 0$ and $\max\{\alpha,\beta\}>5/2$ if l is large enough (see [13]).

Another approach (see [7, 8, 11]) is to construct a new quadrature formula Q_{l+1} for the functional

$$I_\theta(f)=\int_a^b f(x)w(x)dx-\theta Q_l^G(f),$$

for a given $\theta\in\mathbb{R}$, and then use the stratified quadrature formulas $Q_{2l+1}=\theta Q_n^G+Q_{l+1}$ to estimate the error Q_n^G. As a special case, Laurie in [8] introduced the anti-Gaussian quadrature formula Q_{l+1}^A

$$(I-Q_{l+1}^A)(p)=-(I-Q_l^G)(p),\quad p\in\mathcal{P}^{2l+1}.$$

The averaged formula

$$Q_{2l+1}^L=\frac{1}{2}(Q_l^G+Q_{l+1}^A),$$

also introduced in [8], is of the stratified type and has the degree of exactness at least $2l+1$. In the case of the Laguerre and Hermite weight functions, more general averaged formulas $\frac{1}{2+\gamma}((1+\gamma)Q_n^G+Q_{l+1}^A)$ with $\gamma>-1$ were considered in [4]. Here γ is chosen so that the degree of exactness is as large as possible. These modified formulas, denoted by Q_{2l+1}^{GF}, are also stratified extensions. Moreover, among all stratified extensions, these are the unique formulas with the maximum degree of exactness.

Recently, by following the results in [12] which characterize positive quadrature formulas, Spalević [16] introduced a new $(2l+1)$-node quadrature formula, called generalized averaged Gaussian quadrature formula. Here we denote it by Q_{2l+1}^S. In the cases of Laguerre and Hermite weight functions, this formula turns out to coincide with Q_{2l+1}^{GF}. The generalized averaged Gaussian formula has a degree of exactness at least $2l+2$, but for one class of weight functions the degree of exactness is $3n+1$ and hence the formula coincides with Gauss-Kronrod formula (see [18]). Further, the truncated generalized averaged Gauss formulas $Q_{2l-r+1}^{(l-r)}$ are introduced in [14], where $l\geqslant 2$ and $r=1,2,\dots,l-1$. These formulas have fewer nodes and the same degree of exactness as the generalized averaged Gauss formulas. Hence, the truncated generalized averaged Gauss formulas can be useful as substitutes when (real) Gauss-Kronrod formula do not exist.

According to [8, 16] and [1], the generalized averaged Gaussian formulas and truncated variant for one of them have real nodes with positive weights, and only the two outermost nodes may be exterior. Thus it remains to analyze when these formulas are internal, i.e., all nodes are interior. This property is important when the integrand f is defined only on the interval $[a, b]$ and has also been investigated in [1, 2] and [3].

In this paper, we are analyzing mentioned averaged formulas for some weight functions recently considered by Milovanović in [10]. In two of these cases the orthogonal polynomials can be expressed in terms of the Jacobi polynomials on $[0, 1]$. For these, we will consider internality of the averaged formulas. In some simple cases of these polynomials, the generalized averaged Gaussian formulas coincide with the Gauss-Kronrod formula. The other two cases yield non-classical polynomials on $[0, 1]$, and in these cases we will give examples showing the error estimates for the Gauss formula.

2. The Extraction of Orthogonal Polynomials from Generating Function for Reciprocal of Odd Numbers

Let $\{\pi_l(x)\}_{l=0}^{\infty}$ be a sequence of monic polynomials orthogonal on $[a, b]$ with respect to the weight function $w(x)$. These polynomials satisfy the three-term recurrence relation

$$\pi_{l+1}(x) = (x - \alpha_l)\pi_l(x) - \beta_l \pi_{l-1}(x), \quad l = 0, 1, \dots, \tag{2.1}$$

with $\pi_0(x) = 1$ and $\pi_{-1}(x) = 0$. Here α_l and β_l are the recurrence coefficients and it is convenient to define $\beta_0 = \int_a^b w(x)dx$. The same recurrence coefficients occur in the Jacobi continued fraction associated with the weight function $w(x)$,

$$F(x) = \int_a^b \frac{w(t)}{x - t} dt \sim \frac{\beta_0}{x - \alpha_0 -} \frac{\beta_1}{x - \alpha_1 -} \cdots,$$

which is known as the Stieltjes transform of the weight function $w(x)$. The l-th convergent of this continued fraction is

$$\frac{\beta_0}{x - \alpha_0 -} \frac{\beta_1}{x - \alpha_1 -} \cdots \frac{\beta_{l-1}}{x - \alpha_{l-1}} = \frac{\sigma_l(x)}{\pi_l(x)},$$

where $\sigma_l(x)$ are the associated polynomials,

$$\sigma_l(x) = \int_a^b \frac{\pi_l(x) - \pi_l(t)}{x - t} w(t) dt, \quad l \geqslant 0.$$

These polynomials satisfy the same recurrence relation (2.1), where $\sigma_0 = 0$ and $\sigma_{-1} = -1$ (see [9, pp. 111–114]).

Recently Shashikala [15] considered the series

$$T(x) = 1 + \frac{1}{3}x + \frac{1}{5}x^2 + \cdots + \frac{1}{2l+1}x^l + \cdots.$$

Using the regular continued fraction,

$$T(x) = \frac{1}{1+}\,\frac{-\frac{1}{3}x}{1+}\,\frac{-\frac{4}{15}x}{1+}\cdots\frac{-\frac{l^2}{4l^2-1}x}{1+}\cdots, \tag{2.2}$$

and taking even and odd convergents, he obtained four sequences of monic orthogonal polynomials $\{Q_l^{(\nu)}(x)\}_{l=0}^{\infty}$, $\nu = 1,2,3,4$. These polynomials satisfy the three-term recurrence relation (2.1), with $Q_0^{(\nu)}(x) = 1$ and $Q_1^{(1)}(x) = x - \frac{1}{3}$, $Q_1^{(2)}(x) = x - \frac{3}{5}$, $Q_1^{(3)}(x) = x - \frac{4}{15}$, $Q_1^{(4)}(x) = x - \frac{11}{21}$. The first two polynomials, extracted from the denominators of (2.2), are classical orthogonal polynomials (cf. [9, pp. 121–146]), whereas the other two, extracted from the numerators, are non-classical polynomials.

Let us consider the polynomials $p_l^{(1)}(x)$ and $p_l^{(2)}(x)$ orthogonal on $[0,1]$ with respect to the weight functions

$$w^{(1)}(x) = (1-x)^{\lambda-1/2}/\sqrt{x} \quad \text{and} \quad w^{(2)}(x) = \sqrt{x}(1-x)^{\lambda-1/2}, \quad \lambda > -1/2. \tag{2.3}$$

These polynomials satisfy the relation (2.1) with the recurrence coefficients (see [10])

$$\begin{aligned} a_0^{(1)} &= \frac{1}{2(\lambda+1)}, & a_l^{(1)} &= \frac{4l^2+4\lambda l+\lambda-1}{2(\lambda+2l-1)(\lambda+2l+1)},\\ b_0^{(1)} &= \frac{\sqrt{\pi}\,\Gamma(\lambda+1/2)}{\Gamma(\lambda+1)}, & b_l^{(1)} &= \frac{l(2l-1)(\lambda+l-1)(2\lambda+2l-1)}{4(\lambda+2l-2)(\lambda+2l-1)^2(\lambda+2l)}, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} a_0^{(2)} &= \frac{3}{2(\lambda+2)}, & a_l^{(2)} &= \frac{3\lambda+4l^2+4(\lambda+1)l}{2(\lambda+2l)(\lambda+2l+2)},\\ b_0^{(2)} &= \frac{\sqrt{\pi}\,\Gamma(\lambda+1/2)}{2\Gamma(\lambda+2)}, & b_l^{(2)} &= \frac{l(2l+1)(\lambda+l)(2\lambda+2l-1)}{4(\lambda+2l-1)(\lambda+2l)^2(\lambda+2l+1)}. \end{aligned} \tag{2.5}$$

Actually, these polynomials are the (monic) Jacobi polynomials transformed to the interval $[0,1]$, with parameters $(\lambda-1/2, \mp 1/2)$, i.e.,

$$p_l^{(1)}(x) = \frac{1}{2^l}p_l^{(\lambda-1/2,-1/2)}(2x-1), \quad p_l^{(2)}(x) = \frac{1}{2^l}p_l^{(\lambda-1/2,1/2)}(2x-1), \tag{2.6}$$

where $p_l^{(\alpha,\beta)}$ are the monic Jacobi polynomials with respect to the weight function $(1-x)^\alpha(1+x)^\beta$ on the interval $[-1,1]$ (see [9, pp. 131–140]).

Milovanović in [10] showed that, for $\lambda = 1/2$, the coefficients (2.4) and (2.5) reduce to the ones for the polynomials $Q^{(1)}(x)$ and $Q^{(2)}(x)$, respectively.

Let $a_l^{(\alpha,\beta)}$ and $b_l^{(\alpha,\beta)}$ be the recurrence coefficients for the monic Jacobi polynomials $p_l^{(\alpha,\beta)}$. It is easy to see that for $l \geqslant 1$ we have

$$a_l^{(1)} = \frac{a_l^{(\lambda-1/2,-1/2)}+1}{2}, \quad b_l^{(1)} = \frac{b_l^{(\lambda-1/2,-1/2)}}{4}, \tag{2.7}$$

$$a_l^{(2)} = \frac{a_l^{(\lambda-1/2,1/2)}+1}{2}, \quad b_l^{(2)} = \frac{b_l^{(\lambda-1/2,1/2)}}{4}. \tag{2.8}$$

We may also be interested in the cases $\lambda = 0$ and $\lambda = 1$. Let $T_l(x)$, $U_l(x)$, $V_l(x)$ and $W_l(x)$ be the Chebyshev polynomials of first, second, third and fourth kinds, respectively. For $\lambda = 0$ we get $p_l^{(1)}(x) = \frac{1}{2^l}T_l(2x-1)$ and $p_l^{(2)}(x) = \frac{1}{2^l}V_l(2x-1)$. Similarly, for $\lambda = 1$ we obtain $p_l^{(1)}(x) = \frac{1}{2^l}W_l(2x-1)$ and $p_l^{(2)}(x) = \frac{1}{2^l}U_l(2x-1)$. In each of these cases, the generalized averaged Gaussian quadrature formula coincides with the Gauss-Kronrod quadrature formula.

It was also proved in [10] that the polynomials $Q^{(3)}(x)$ and $Q^{(4)}(x)$ are orthogonal on $[0,1]$ with respect to the weight functions

$$w^{(3)}(x) = \frac{2/\sqrt{x}}{4(\tanh^{-1}\sqrt{x})^2 + \pi^2} \quad \text{and} \quad w^{(4)}(x) = \frac{2\sqrt{x}}{4(\tanh^{-1}\sqrt{x})^2 + \pi^2}, \tag{2.9}$$

respectively. The corresponding orthogonal polynomials are non-classical on $[0,1]$ and their respective recurrence coefficients are

$$a_0^{(3)} = \frac{4}{15}, \quad a_l^{(3)} = \frac{8l^2+12l+3}{(4l+1)(4l+5)}, \quad b_l^{(3)} = \frac{(2l)^2(2l+1)^2}{(4l-1)(4l+1)^2(4l+3)},$$

and

$$a_0^{(4)} = \frac{11}{21}, \quad a_l^{(4)} = \frac{8l^2+20l+11}{(4l+3)(4l+7)}, \quad b_l^{(4)} = \frac{(2l+1)^2(2l+2)^2}{(4l+1)(4l+3)^2(4l+5)}.$$

Later on we will present some examples showing the error estimates for the Gauss quadrature with respect to these weights using the mentioned averaged formulas.

3. The Generalized Averaged Gaussian Formula Q_{2l+1}^L

The generalized averaged Gaussian formula Q_{2l+1}^L, introduced in [8], is internal if the smallest zero x_1^π and the largest zero x_{l+1}^π of the polynomial

$$\pi_{l+1}(x) = p_{l+1}(x) - \beta_l p_{l-1}(x)$$

belong to the interval $[0,1]$ (see [8]). Here p_j, $j = 0,1,\ldots$, are the orthogonal polynomials and β_j, $j = 1,2,\ldots$, the recurrence coefficients corresponding to the original weight function. The largest zero x_{l+1}^π belongs to $[0,1]$ if and only if

$$\frac{p_{l+1}(1)}{\beta_l p_{l-1}(1)} \geqslant 1.$$

Similarly, the smallest zero x_1^π belongs to $[0,1]$ if and only if

$$\frac{p_{l+1}(0)}{\beta_l p_{l-1}(0)} \geqslant 1.$$

Obviously, the previous conditions are equivalent to the conditions for the Jacobi polynomials with the same parameters. Indeed, using (2.6)–(2.8), these conditions reduce to

$$\frac{p_{l+1}^{(\lambda-1/2,\mp1/2)}(x)}{\beta_l^{(\lambda-1/2,\mp1/2)}p_{l-1}^{(\lambda-1/2,\mp1/2)}(x)} \geqslant 1,$$

where $x \in \{-1, 1\}$. Hence, Theorem 3 from [7] can be applied.

For the weight function $w^{(1)}(x)$, the conditions (18) and (19) from [7] reduce to

$$2\lambda^3 + (8l-1)\lambda^2 + (8l^2-1)\lambda \geqslant 0 \quad \text{and} \quad \lambda^2 - \lambda \geqslant 0,$$

respectively. The first condition obviously holds for $\lambda \geqslant 0$, but not for $\lambda \in (-1/2, 0)$ and sufficiently large l (the leading coefficient in l in the latter case is negative). The second condition holds for $\lambda \in (-1/2, 0] \cup [1, \infty)$.

Similarly, for the weight function $w^{(2)}(x)$, the conditions (18) and (19) from [7] reduce to

$$2\lambda^3 + (8l+3)\lambda^2 + (8l^2+8l+1)\lambda \geqslant 0 \quad \text{and} \quad 8l^2 + (8\lambda+8)l + 3\lambda^2 + 3\lambda \geqslant 0,$$

respectively. The first condition holds for $\lambda \geqslant 0$, but not for $\lambda \in (-1/2, 0)$ and sufficiently large l. The second condition holds for $\lambda > -1/2$.

Thus we have the following result.

Theorem 3.1. *The generalized averaged Gaussian formula Q^L_{2l+1} for the weight functions $w^{(1)}(x)$ and $w^{(2)}(x)$ is internal when $\lambda \geqslant 1$ and $\lambda \geqslant 0$, respectively.*

4. The Generalized Averaged Gaussian Formula Q^S_{2l+1}

Consider the generalized averaged formula Q^S_{2l+1} introduced in [16]. This formula is internal if the smallest zero x^F_1 and the largest zero x^F_{l+1} of the polynomial

$$F_{l+1}(x) = p_{l+1}(x) - \beta_{l+1}p_{l-1}(x)$$

belong to the interval $[0, 1]$ (see [16]). Here p_j, $j = 0, 1, \ldots$, are the orthogonal polynomials and β_j, $j = 2, 3, \ldots$, the recurrence coefficients corresponding to the original weight function. The largest zero x^F_{l+1} belongs to $[0, 1]$ if and only if

$$\frac{p_{l+1}(1)}{\beta_{l+1}p_{l-1}(1)} \geqslant 1.$$

Similarly, the smallest zero x^F_1 belongs to $[0, 1]$ if and only if

$$\frac{p_{l+1}(0)}{\beta_{l+1}p_{l-1}(0)} \geqslant 1.$$

As for the formula Q^L_{2l+1}, the previous conditions reduce to ones for the corresponding Jacobi polynomials. So we use Theorem 3.1 from [17].

For the weight function $w^{(1)}(x)$, the conditions (3.5) and (3.6) from [17] reduce to

$$2\lambda^3 + (8l+3)\lambda^2 + (8l^2-5)\lambda \geqslant 0 \quad \text{and} \quad \lambda - \lambda^2 \geqslant 0.$$

The first condition holds for $\lambda \geqslant 0$, but not for $\lambda \in (-1/2, 0)$ and sufficiently large l. On the other hand, the second condition holds for $\lambda \in [0, 1]$.

For the weight function $w^{(2)}(x)$, the conditions (3.5) and (3.6) from [17] reduce to

$$2\lambda^3 + (8l+7)\lambda^2 + (8l^2+8l-3)\lambda \geqslant 0 \quad \text{and} \quad 8l^2 + (8\lambda+8)l + 7\lambda - \lambda^2 \geqslant 0.$$

The first condition obviously holds for $\lambda \geqslant 0$, but not for $\lambda \in (-1/2, 0)$ and sufficiently large l. The second condition holds for $\lambda \in (-1/2, 7)$, whereas for $\lambda \geqslant 7$ we have

$$8l^2 + (8\lambda + 8)l + 7\lambda - \lambda^2 > 8l^2 + 8\lambda l - \lambda^2 \geqslant 0, \quad \text{for } l \geqslant \frac{\sqrt{6} - 2}{4}\lambda.$$

Hence, we have the following result.

Theorem 4.1. *The generalized averaged Gaussian formula* Q^S_{2l+1} *for the weight function* $w^{(1)}(x)$ *is internal when* $\lambda \in [0, 1]$. *In the case of the weight function* $w^{(2)}(x)$, *that formula is internal when* $\lambda \in [0, 7)$. *For* $\lambda \geqslant 7$, *internality occurs when* $l \geqslant \frac{\sqrt{6}-2}{4}\lambda$.

Now let us consider the cases $\lambda = 0$ and $\lambda = 1$, i.e., the polynomials $\frac{1}{2^l}T_l(2x - 1)$, $\frac{1}{2^l}V_l(2x - 1)$, $\frac{1}{2^l}W_l(2x - 1)$ and $\frac{1}{2^l}U_l(2x - 1)$. We have $\alpha_l = \alpha$ and $\beta_l = \beta > 0$ for $l \geqslant r$, where $r = 2$ for the polynomial $\frac{1}{2^l}T_l(2x - 1)$ and $r = 1$ for the polynomials $\frac{1}{2^l}V_l(2x - 1)$, $\frac{1}{2^l}W_l(2x - 1)$ and $\frac{1}{2^l}U_l(2x - 1)$. Hence, Theorem 3.1 from [18] can be applied and we have the following result.

Theorem 4.2. *For the weight function* $w^{(1)}(x)$ *with* $\lambda = 0$ *and* $l \geqslant 3$, *the quadrature formulas* Q^L_{2l+1} *and* Q^S_{2l+1} *have the algebraic degree of exactness at least* $3l + 1$. *Hence, these formulas coincide with the corresponding Gauss-Kronrod quadrature formula and the monic polynomials* $\pi_{l+1} \equiv F_{l+1}$ *coincide with the corresponding monic Stieltjes polynomials. The same results hold for the the weight function* $w^{(1)}(x)$, *when* $\lambda = 1$ *and weight function* $w^{(2)}(x)$ *when* $\lambda \in \{0, 1\}$ *and* $l \geqslant 1$.

Using the previous fact, one has a simple method to compute the Gauss-Kronrod quadrature formula. The computation of the latter formula is more complicated in general (see [5]).

5. Truncated Generalized Averaged Gaussian Formulas

Let us consider the truncated generalized averaged Gaussian formulas $Q^{(l-r)}_{2l-r+1}$ ($l \geqslant 2$) introduced in [14] for $r = l - 1$. This formula is internal if the smallest zero τ_1 and the largest zero τ_{l+2} of the polynomial

$$t_{l+2}(x) = (x - \alpha_{l-1})p_{l+1}(x) - \beta_{l+1}p_l(x) \tag{5.1}$$

belong to the interval $[0, 1]$ (see [1]). Here p_j, $j = 2, 3, \ldots$, are the orthogonal polynomials and α_j, $j = 1, 2, \ldots$, and β_j, $j = 3, 4, \ldots$, the recurrence coefficients corresponding to the original weight function.

Obviously, in the case of the weight functions given in (2.3), the polynomials (5.1) have two outermost zeros inside the interval $[0, 1]$ if and only if the corresponding polynomials for the Jacobi weight functions with the same parameters have two outermost zeros inside the interval $[-1, 1]$. Using Theorem 3.4 from [1], we have that internality holds for $l \geqslant 3$.

Let $l = 2$. In the case of the weight function $w^{(1)}(x)$, the conditions (3.12) and (3.13) from [1] reduce to

$$-\lambda^3 + 19\lambda^2 + 105\lambda + 45 \geqslant 0 \quad \text{and} \quad 2\lambda^4 + 25\lambda^3 + 81\lambda + 63\lambda + 45 \geqslant 0.$$

The first condition holds for $\lambda \in [\lambda_1, \lambda_2]$, where $\lambda_1 \approx -0.46943$ and $\lambda_2 \approx 23.54142$ are the largest two zeros of the polynomial $-x^3 + 19x^2 + 105x + 45$. The second condition holds for $\lambda > -1/2$.

Similarly, for the weight function $w^{(2)}(x)$, this formula is internal if and only if

$$\lambda^3 + 48\lambda^2 + 260\lambda + 216 \geqslant 0 \quad \text{and} \quad 2\lambda^4 + 31\lambda^3 + 136\lambda^2 + 188\lambda + 168 \geqslant 0.$$

These conditions hold for $\lambda > -1/2$.

Theorem 5.1. *The truncated generalized averaged Gaussian formula for the weight function $w^{(1)}(x)$ is internal when $\lambda > -1/2$ and $l \geqslant 3$. For $l = 2$ internality holds when $\lambda \in [\lambda_1, \lambda_2]$, where $\lambda_1 \approx -0.46943$ and $\lambda_2 \approx 23.54142$ are the largest two zeros of the polynomial $-x^3+19x^2+105x+45$. For the weight function $w^{(2)}(x)$ this formula is internal when $\lambda > -1/2$.*

6. Numerical Results

Example 6.1. We illustrate Theorems 3.1, 4.1 and 5.1 through some computations in the case of the weight function $w^{(2)}$ for some values of l and λ. In the considered cases, the corresponding averaged formulas are internal.

Table 1 displays the values of the nodes x_1^π and x_{l+1}^π for the formula Q_{2l+1}^L.

Table 2 displays the values of the nodes x_1^F and x_{l+1}^F for the formula Q_{2l+1}^S. Note that for $\lambda = 1$ this formula coincides with the previous one, and also with the Gauss-Kronrod quadrature formula (see Theorem 4.2).

Table 3 displays the values of the nodes τ_1 and τ_{l+2} for the formula $Q_{l+2}^{(1)}$.

Table 1: The values of x_1^π and x_{l+1}^π for $w^{(2)}$ and some l and λ.

λ	l	x_1^π	x_{l+1}^π
0.5	5	1.84918630347802(−2)	9.93315648803352(−1)
	10	5.32426071493249(−3)	9.98085997371715(−1)
	15	2.48373203616388(−3)	9.99108179903793(−1)
	20	1.43168514326074(−3)	9.99486155846300(−1)
1	5	1.70370868554659(−2)	9.82962913144534(−1)
	10	5.08927905953363(−3)	9.94910720940466(−1)
	15	2.40763666390156(−3)	9.97592363336098(−1)
	20	1.39810140940993(−3)	9.98601898590590(−1)

Table 2: The values of x_1^F and x_{l+1}^F for $w^{(2)}$ and some l and λ.

λ	l	x_1^F	x_{l+1}^F
0.5	5	1.85485046684558(−2)	9.93270563061661(−1)
	10	5.32892821283948(−3)	9.98082336550544(−1)
	15	2.48474645373049(−3)	9.99107386760496(−1)
	20	1.43202203935648(−3)	9.99485892741121(−1)
1	5	1.70370868554659(−2)	9.82962913144534(−1)
	10	5.08927905953363(−3)	9.94910720940466(−1)
	15	2.40763666390156(−3)	9.97592363336098(−1)
	20	1.39810140940993(−3)	9.98601898590590(−1)

Table 3: The values of τ_1 and τ_{l+2} for $w^{(2)}$ and some l and λ.

λ	l	τ_1	τ_{l+2}
0.5	5	4.05074383379349(−2)	9.76146311190531(−1)
	10	1.50966909367400(−2)	9.91134246875255(−1)
	15	7.80960712033176(−3)	9.95418464436467(−1)
	20	4.75922686471797(−3)	9.97209253940011(−1)
1	5	3.80602337443566(−2)	9.61939766255643(−1)
	10	1.45290912869740(−2)	9.85470908713026(−1)
	15	7.59612349389597(−3)	9.92403876506104(−1)
	20	4.65702698183462(−3)	9.95342973018165(−1)

Example 6.2. We find the outermost nodes in the case of the weight function $w^{(1)}$ for the formula Q_{2l+1}^L with $\lambda = 0.5$ (Table 4) and for the formula Q_{2l+1}^S with $\lambda = -0.2$ (Table 5) for some l. Here these formulas have exterior node(s).

Table 4: The values of x_1^π and x_{l+1}^π for $w^{(1)}$, $\lambda = 0.5$ and some l.

λ	l	x_1^π	x_{l+1}^π
0.5	5	−1.03583467673738(−5)	9.91983668229218(−1)
	10	−7.09110640371522(−7)	9.97894782375997(−1)
	15	−1.44570778097492(−7)	9.99048751274800(−1)
	20	−4.64835853269242(−8)	9.99460470025489(−1)

Table 5: The values of x_1^F and x_{l+1}^F for $w^{(2)}$, $\lambda = -0.2$ and some l.

λ	l	x_1^F	x_{l+1}^F
−0.2	5	−4.13229856738924(−5)	1.00140197341566
	10	−2.37471751038235(−6)	1.00033417984287
	15	−4.59266799101858(−7)	1.00014681665572
	20	−1.43959966526914(−7)	1.00008217031089

Example 6.3. Consider the integral

$$I(f) = \int_0^1 f(t)w(t)dt,$$

where $f(t) = 999.1^{\log_{10}(\varepsilon+t)}$, $\varepsilon = 10^{-6}$ and $w(t) = w^{(2)}(t)$. In Table 6, the estimation of the errors $|I(f) - Q_l^G(f)|$ for Gauss quadrature formula are obtained by means of the quantities $E_{LG} = \left|Q_{2l+1}^L(f) - Q_l^G(f)\right|$, $E_{SG} = |Q_{2l+1}^S(f) - Q_l^G(f)|$ and $E_{TSG} = |Q_{l+2}^{(1)}(f) - Q_l^G(f)|$, for some l and λ. As in the previous example, $Q_{2l+1}^L \equiv Q_{2l+1}^S$ for $\lambda = 1$. The sharp errors are denoted by Error.

Table 6: The estimates E_{LG}, E_{SG}, E_{TSG} and the sharp errors Error for some l and λ.

λ	l	E_{LG}	E_{SG}	E_{TSG}	Error
0.5	5	1.5198(−10)	1.5192(−10)	1.4323(−10)	1.5219(−10)
	10	4.3114(−13)	4.3106(−13)	3.4123(−13)	4.3190(−13)
	15	1.3219(−14)	1.3218(−14)	8.7665(−15)	1.3244(−14)
	20	1.0866(−15)	1.0865(−15)	6.1493(−16)	1.0886(−15)
1	5	1.1092(−10)	1.1092(−10)	1.0410(−10)	1.1108(−10)
	10	3.5846(−13)	3.5846(−13)	2.8175(−13)	3.5911(−13)
	15	1.1599(−14)	1.1599(−14)	7.6384(−15)	1.1621(−14)
	20	9.8190(−16)	9.8190(−16)	5.5211(−16)	9.8378(−16)

Note that the integrand in the previous example is not defined for some nodes in Example 6.2.

Example 6.4. The next table displays the same estimations as in the previous example for the integrand $f(t) = e^{3t}\sin 10t$ and the weight function $w(t) = w^{(3)}(t)$ from (2.9). Note that for the weight functions given in (2.9), the corresponding orthogonal polynomials are non-classical. Thus there is no analytical expression for the orthogonal polynomials. Consequently, there is no general claim for internality of the averaged formulas.

Table 7: The estimates E_{LG}, E_{SG}, E_{TSG} and the sharp errors Error for some l.

l	E_{LG}	E_{SG}	E_{TSG}	Error
5	3.4273(−3)	3.4276(−3)	3.4209(−3)	3.4276(−3)
10	8.4359(−11)	8.4359(−11)	8.4340(−11)	8.4359(−11)
15	9.6941(−21)	9.6941(−21)	9.6934(−21)	9.6941(−21)
20	3.1798(−32)	3.1798(−32)	3.1797(−32)	3.1798(−32)

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