Kragujevac Journal of Mathematics
Volume 46(2) (2022), Pages 267–282.

WELL-POSEDNESS AND GENERAL DECAY OF SOLUTIONS FOR THE HEAT EQUATION WITH A TIME VARYING DELAY TERM

ABDELKADER BRAIK[1], ABDERRAHMANE BENIANI[2], AND YAMINA MILOUDI[3]

Abstract. We consider the nonlinear heat equation in a bounded domain with a time varying delay term

$$u_t + \Delta^2 u - J(t)\int_0^t g(t-s)\Delta^2 u(s)ds + \alpha K(t)u + \beta K(t)u\left(t-\tau(t)\right) = 0,$$

with initial conditions. By introducing suitable energy and Lyapunov functionals, under some assumptions, we then prove a general decay result of the energy associated of this system under some conditions.

1. Introduction and Statement

Let us consider the following problem

$$\begin{cases} u_t + \Delta^2 u - J(t)\displaystyle\int_0^t g(t-s)\Delta^2 u(s)ds + \alpha K(t)u & \\ \quad + \beta K(t)u\left(t-\tau(t)\right) = 0, & \text{in } \Omega\times\,]0,+\infty[\,, \\ u = 0, & \text{on } \partial\Omega\times\,]0,+\infty[\,, \\ u(0) = u_0, & \text{in } \Omega, \\ u\left(t-\tau(0)\right) = h_0\left(t-\tau(0)\right), & \text{in } \Omega\times\,]0,\tau(0)[\,, \end{cases} \tag{1.1}$$

where $\Delta^2 u = \Delta(\Delta u)$, Ω be a bounded open domain in $\mathbb{R}^n, n\in\mathbb{N}^*$ of regular boundary $\partial\Omega$, the function $\tau :]0,+\infty[\longrightarrow]0,+\infty[$, $\tau(t)$ is a time varying delay, α and β are positive real numbers, and the initial data (u_0,h_0) belongs to a suitable function space.

Key words and phrases. Heat equation, time varying delay, energy decay, Lyapunov functional, global existence, viscoelastic term.
2010 *Mathematics Subject Classification.* Primary: 35L05, 35L15. Secondary: 35L70, 35B05.
DOI 10.46793/KgJMat2202.267B
Received: April 22, 2019.
Accepted: November 10, 2019.

Time delay is the property of a physical system by which the response to an applied force is delayed in its effect (see [12]). Whenever material, information or energy is physically transmitted from one place to another, there is a delay associated with the transmission. In, physical, chemical, biological, electrical, mechanical and economic phenomena.

in recent years, the stability of partial differential equations with time-varying delays has been studied in [8, 15, 20] via the Lyapunov method.

In the constant delay case the exponential stability was proved in [11, 18] by using the observability inequality which can not be applicable in the time-varying case (since the system is not invariant by translation).

In recent years, the control of PDEs with time delay effects has become an active area of research, see for example [1, 19, 23] and the references therein. In the case of distributed parameter systems, even arbitrarily small delays in the feedback may destabilize the system (see, e.g., [5, 9, 11, 15]). Hence the stability issue of systems with delay is of theoretical and practical importance.

There are more works on the Lyapunov-based technique for delayed PDEs. Most of these studies analyze the case of constant delays. Thus, the conditions of stability and the exponential limits have been derived for some heat equations and scalar waves with constant delays and boundary conditions of Dirichlet without delay in [21].

S. Bernard, J. Belair and M. C. Mackey [16] studied the stability of the following linear differential equation

$$x' = -\alpha x(t) - \beta \int_0^{+\infty} x(t-s) f(s) ds,$$

where α and β are constants.

Chengming Huang and Stefan Vandewalle [2] considered a more general equation,

$$y'(t) = \alpha y(t) + \beta y(t-\tau) + \gamma \int_{t-\tau}^{t} y(s) ds, \tag{1.2}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ and $u(t) = \phi(t)$ on $[-\tau, 0]$, and proved that the repeated trapezium rule retains the asymptotic stability of (1.2). Wu and Gan in [22] further extended the above study to the case of neutral equations.

In Section 3, page 16, Chengming Huang and Stefan Vandewalle [3] considered the asymptotic stability of multi-dimensional equations of the form

$$y'(t) = Ly(t) + My(t-\tau) + K \int_{t-\tau}^{t} y(\nu) d\nu, \quad t > 0, \tag{1.3}$$

where $L, M, K \in C^{d\times d}$ and $y(t) = \phi(t)$ on $[-\tau, 0]$. The characteristic equation equals

$$\det \left[\lambda I_d - L - M e^{-\tau\lambda} - K \int_{-\tau}^{0} e^{-\tau\nu} d\nu \right] = 0, \tag{1.4}$$

where I_d is the $d \times d$ identity matrix. The zero solution of (1.3) is asymptotically stable if and only if all the roots λ of (1.4) have negative real parts.

Recently the stability of PDEs with time-varying delays was analyzed in [8, 20].

Later, Mohamed Ferhat and Ali Hakem in [7] studied the decay properties of solutions of the folowing system for the initial boundary value problem of a nonlinear wave equation

$$\begin{cases} (|u'|^{\gamma-2}u')' - \Delta_x u - \int_0^t g(t-s)\Delta u(s)ds + \mu_1\Psi(u'(x,t)) & \\ +\mu_2\Psi(u'(x,t-\tau(t))) = 0, & \textit{in } \Omega\times(0,+\infty), \\ u = 0, & \textit{on } \Gamma\times(0,+\infty), \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), & \textit{in } \Omega, \\ u'(t-\tau(0)) = f_0(t-\tau(0)), & \textit{on } \Omega\times(0,\tau(0)), \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$, with a smooth boundary $\partial\Omega$, $\tau(t) > 0$ is a time varying delay, μ_1 and μ_2 are positive real numbers.

Recently, the case of time-varying delay has been studied in [13, 18]. For example, in Nicaise et al. [18] in one space dimension studied

$$\begin{cases} u' - au_{xx} = 0, & 0 < x < \pi,\ t > 0, \\ u(0,t) = 0, & t > 0, \\ u_x(\pi,t) = \mu_0 u(\pi,t) - \mu_1 u(\pi, t-\tau(t)), & t > 0, \\ u(x,0) = u_0(x), & 0 < x < \pi, \\ u(\pi, t-\tau(0)) = f_0(t-\tau(0)), & 0 < t < \tau(0), \end{cases}$$

where $\mu_0, \mu_1 \geq 0$ and $a > 0$. They proved the exponential stability result under the conditions

$$\begin{aligned} &\tau' < 1, \quad \text{for all } t > 0, \\ &\text{exists } M > 0, \quad 0 < \tau_0 \leq \tau \leq M, \quad \text{for all } t > 0, \\ &\tau \in W^{2,\infty}([0,T]), \quad \text{for all } T > 0. \end{aligned}$$

And in 2011 S. Nicaise and C. Pignotti in [13] considered an problem of the form

$$\begin{cases} u'' - \Delta u - a\Delta u' = 0, & \text{in } \Omega\times(0,+\infty), \\ u = 0, & \text{on } \Gamma\times(0,+\infty), \\ \mu u'' = \dfrac{\partial(u+au')}{\partial\nu} - ku'(t-\tau(t)), & \text{on } \Gamma_1\times(0,+\infty), \\ u(x,0) = u_0(x), u'(x,0) = u_1(x), & \text{in } \Omega, \\ u' = f_0, & \text{on } \Gamma_1\times(-\tau(0),0). \end{cases}$$

We also recall the result by Xu, Yung and Li [4], where the authors proved a result similar to the one in [11] for the one-space dimension by adopting the spectral analysis approach. The case of time-varying delay in the wave equation has been studied recently by Nicaise, Valein and Fridman [18]) in one-space dimension. They proved an exponential stability result under the condition $\mu_2 \leq \sqrt{1-d}\mu_1$, where the fuction τ satisfies $\tau'(t) \leq d < 1$ for all $t > 0$.

In [14], Nicaise, Pignotti and Valein extended the above result to higher space dimensions and established an exponential decay.

The paper is organized as follows. In Section 2 we present some assumptions and state the main result. The general decay result is proved in Sections 3.

We use the ideas given by G. Li, B. Zhu and Wenjun Liu in [10], and the multiplier technique to prove our result.

2. Preliminaries and Main Results

Firstly we assume the following hypotheses.

(H1) $k : \mathbb{R}_+ \longrightarrow]0, +\infty[$ is a non-increasing function of class $C^1(\mathbb{R}_+)$ satisfying

$$k'(t) \leq -ck(t), \quad \text{for all } t \geq 0, \tag{2.1}$$

where c is a positive constant.

(H2) $J, g, \psi : \mathbb{R}_+ \longrightarrow]0, +\infty[$ are non-increasing differentiable functions satisfying

$$\int_0^{+\infty} g(s)ds < +\infty, \quad 1 - J(t) \int_0^t g(s)ds \geq l > 0, \tag{2.2}$$

and

$$g'(t) < -\psi(t)g(t), \quad \text{for all } t \geq 0, \quad \lim_{t \longrightarrow +\infty} \frac{J'(t)}{\psi(t)J(t)} = 0. \tag{2.3}$$

(H3) For the time-varying delay τ, it is varying betwin two positive constants τ_0, τ_1, and

$$\tau \in W^{2,\infty}([0, T]), \quad \text{for all } T > 0, \tag{2.4}$$

$$0 < \tau_0 \leq \tau(t) \leq \tau_1, \quad \text{for all } t > 0, \tag{2.5}$$

$$\tau'(t) \leq d < 1, \quad \text{for all } t > 0. \tag{2.6}$$

(H4) α, β and δ are three positive constants satisfy,

$$\alpha \geq \beta\delta \tag{2.7}$$

and

$$\beta \leq \frac{1}{2\delta k(0)}, \tag{2.8}$$

for some $\delta > 0$.

We now state some lemmas needed later.

Lemma 2.1 (Sobolev-Poincare's inequality)**.** *There exists a constant $C_p = C(\Omega)$ such that*

$$\int_\Omega |w|^2 \, dx \leq C_p \int_\Omega |\Delta w|^2 \, dx, \quad \text{for all } w \in H_0^1(\Omega). \tag{2.9}$$

We introduce, as in [11], the new variable

$$z(x,\rho,t)=u(x,t-\rho\tau(t)),\quad (x,\rho,t)\in\Omega\times(0,1)\times(0,+\infty). \tag{2.10}$$

Then, we have

$$\tau(t)z'(x,\rho,t)=(\tau'(t)\rho-1)z_\rho(x,\rho,t),\ \text{ in }\Omega\times(0,1)\times(0,+\infty), \tag{2.11}$$

where $z':=\frac{\partial z}{\partial t}$ and $z_\rho:=\frac{\partial z}{\partial\rho}$. Then problem (1.1) may be rewritten as

$$\begin{cases} u_t+\Delta^2u-J(t)\int_0^t g(t-s)\Delta^2u(s)ds+\alpha k(t)u & \\ +\beta k(t)z\,(1,t)=0, & \text{in }\Omega\times(0,+\infty),\\ \tau(t)z'(x,\rho,t)=(\rho\tau'(t)-1)z_\rho(x,\rho,t), & \text{in }\Omega\times(0,1)\times(0,+\infty),\\ u=0, & \text{on }\partial\Omega\times]0,+\infty[,\\ u\,(0)=u_0, & \text{in }\Omega,\\ z\,(0,t)=u(t), & \text{in }\Omega(0,+\infty),\\ z\,(\rho,0)=h_0\,(-\rho\tau(0))\,, & \text{in }\Omega\times(0,1). \end{cases} \tag{2.12}$$

We define the energy of solution of problem (2.12) by

$$\begin{aligned} E(t)=&\frac{1}{2}\Big[\alpha k(t)\,\|u\|_2^2+\Big(1-J(t)\int_0^t g(s)ds\Big)\,\|\Delta u\|_2^2+J(t)\,(g\circ\Delta u)\,(t)\\ &+\xi k(t)\tau(t)\int_\Omega\int_0^1|z(\rho,t)|^2\,d\rho dx\Big], \end{aligned} \tag{2.13}$$

where ξ is a positive constant, and

$$(g\circ\Delta w)\,(t)=\int_0^t g(t-\nu)\,\|\Delta w(t)-\Delta w(\nu)\|^2\,d\nu.$$

Now we will establish a general decay rate estimate for the energy.

3. Decay of Solutions

We firstly give the global existence of solutions of the system, which has been proved in [10].

Proposition 3.1. ([10, Lemma 2.1]). *Let* **(H1)-(H4)** *hold. Then given* $u_0\in H_1^0(\Omega)$, $h_0\in L^2(\Omega\times(0,1))$ *and* $T>0$, *there exists a unique weak solution* (u,z) *of the problem* (2.12) *on* $(0,T)$ *such that*

$$u\in C(0,T;H_1^0(\Omega))\cap C_1(0,T;L^2(\Omega)).$$

Lemma 3.1. *Let* (2.6) *and* (2.7) *be satisfied,* ξ *be a positive constant and* δ *sufficiently small such that*

$$\frac{\beta}{2\delta\,(1-d)}\le\xi\le\alpha c, \tag{3.1}$$

and (u,z) *the solution of the problem* (2.12). *Then, the energy functional defined by* (2.13) *it may be non-increasing function and satisfies*

$$\begin{aligned} E'(t) \leq & \frac{1}{2} J(t) \left(g' \circ \Delta u\right)(t) - \frac{1}{2} J'(t) \left(\int_0^t g(s)ds\right) \|\Delta u\|_2^2 \\ & + \frac{\xi}{2} k'(t)\tau(t) \int_\Omega \int_0^1 |z(\rho,t)|^2 \, d\rho dx \\ \leq & \frac{1}{2} J(t) \left(g' \circ \Delta u\right)(t) - \frac{1}{2} J'(t) \left(\int_0^t g(s)ds\right) \|\Delta u\|_2^2 . \end{aligned} \tag{3.2}$$

Proof. At first, multiplying the first equation in (2.12) by u', integrating over Ω and using integration by parts, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|\Delta u\|_2^2 + \alpha k(t) \|u\|_2^2\right) + \|u'\|_2^2 - \frac{1}{2} \alpha k'(t) \|u\|_2^2 + \beta k(t) \int_\Omega u' z(1,t) dx \\ & - J(t) \int_0^t g(t-s) \int_\Omega \Delta u' \Delta u(s) dx ds = 0. \end{aligned} \tag{3.3}$$

We denote by $I_1(t)$ to the last term on the left side of (3.3) for $I_1(t)$ we have

$$\begin{aligned} I_1(t) = & J(t) \int_0^t g(t-s) \int_\Omega \Delta u'(t) \left(\Delta u(t) - \Delta u(s)\right) dx ds \\ & - J(t) \int_0^t g(t-s) \int_\Omega \Delta u'(t) \Delta u(t) dx ds \\ = & \frac{1}{2} \frac{d}{dt} \left[\int_0^t J(t) g(t-s) \int_\Omega |\Delta u(t) - \Delta u(s)|^2 \, dx ds \right. \\ & \left. - J(t) \int_0^t g(s) \int_\Omega |\Delta u(t)|^2 \, dx ds\right] + \frac{1}{2} \left(J(t) \int_0^t g(s) ds\right)' \int_\Omega |\Delta u(t)|^2 \, dx ds \\ & - \frac{1}{2} \left(\int_0^t \left(J(t) g(t-s)\right)' \int_\Omega |\Delta u(t) - \Delta u(s)|^2\right) dx ds \\ = & \frac{1}{2} \frac{d}{dt} \left[\int_0^t J(t) g(t-s) \int_\Omega |\Delta u(t) - \Delta u(s)|^2 \, dx ds \right. \\ & \left. - J(t) \int_0^t g(s) ds \int_\Omega |\Delta u(t)|^2 \, dx\right] + \frac{1}{2} J(t) g(t) \|\Delta u\|_2^2 \\ & + \frac{1}{2} J'(t) \left(\int_0^t g(s) ds\right) \|\Delta u\|_2^2 - \frac{1}{2} J'(t) \left(g \circ \Delta u\right)(t) - \frac{1}{2} J(t) \left(g' \circ \Delta u\right)(t). \end{aligned} \tag{3.4}$$

Inserting (3.4) into (3.3) and using Young's inequality, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\alpha k(t) \|u\|_2^2 + \left(1 - J(t) \int_0^t g(s) ds\right) \|\Delta u\|_2^2 + J(t) \left(g \circ \Delta u\right)(t)\right) \\ \leq & \frac{1}{2} \alpha k'(t) \|u\|_2^2 - \left(1 - \delta \beta k(t)\right) \|u'\|_2^2 + \frac{\beta k(t)}{4\delta} \|z(1,t)\|_2^2 + \frac{1}{2} J'(t) \left(g \circ \Delta u\right)(t) \\ & + \frac{1}{2} J(t) \left(g' \circ \Delta u\right)(t) - \frac{1}{2} \left(J'(t) \left(\int_0^t g(s) ds\right) + J(t) g(t)\right) \|\Delta u\|_2^2 . \end{aligned} \tag{3.5}$$

Secondly, we multiply the second equation in (2.12) by $\xi k(t)z(x,\rho,t)$ and integrate over $\Omega\times(0,1)$, to get

$$\frac{\xi}{2}k(t)\tau(t)\int_\Omega\int_0^1\frac{d}{dt}\left|z(\rho,t)\right|^2d\rho dx=-\frac{\xi}{2}k(t)\int_\Omega\int_0^1(1-\rho\tau'(t))\frac{\partial}{\partial\rho}\left|z(\rho,t)\right|^2d\rho dx.$$

And from there we find

$$\begin{aligned}\frac{d}{dt}\left(\frac{\xi}{2}k(t)\tau(t)\int_\Omega\int_0^1\left|z(\rho,t)\right|^2d\rho dx\right)=&\frac{\xi}{2}\left(k(t)\tau(t)\right)'\int_\Omega\int_0^1\left|z(\rho,t)\right|^2d\rho dx\\&-\frac{\xi}{2}k(t)\int_\Omega\left[\left(1-\rho\tau'(t)\right)\left|z(\rho,t)\right|^2\right]_0^1dx\\&-\frac{\xi}{2}k(t)\tau'(t)\int_\Omega\int_0^1\left|z(\rho,t)\right|^2d\rho dx.\end{aligned}\tag{3.6}$$

Taking the sum of (3.5) and (3.6), we obtain that

(3.7)

$$\begin{aligned}E'(t)\le&\frac{1}{2}\left(\alpha k'(t)+\xi k(t)\right)\|u\|_2^2-\left(1-\delta\beta k(t)\right)\|u'\|_2^2+\frac{1}{2}J'(t)\left(g\circ\Delta u\right)(t)\\&-\frac{k(t)}{2}\left(\xi\left(1-\tau'(t)\right)-\frac{\beta}{2\delta}\right)\|z(1,t)\|_2^2+\frac{1}{2}J(t)\left(g'\circ\Delta u\right)(t)\\&-\frac{1}{2}\left(J'(t)\left(\int_0^t g(s)ds\right)+J(t)g(t)\right)\|\Delta u\|_2^2+\frac{\xi}{2}k'(t)\tau(t)\int_\Omega\int_0^1\left|z(\rho,t)\right|^2d\rho dx.\end{aligned}$$

Combining (3.1), (3.7) and hypotheses (**H1**)-(**H4**), the proof of Lemma 3.1 is complete. □

Theorem 3.1. *Assume* **(H1)-(H4)**. *Then there exist positive constants C and K_0 such that for any solution of problem* (2.12), *the energy satisfies the following estimate*

$$E(t)\le Ce^{-K_0\int_0^t\psi(t)J(t)dt},\tag{3.8}$$

for every $t\ge0$.

Now, we define the functional $F(t)$ as follows

$$F(t)=\frac{1}{2}\int_\Omega u^2dx.\tag{3.9}$$

Lemma 3.2. *The functional F satisfies the following estimate*

$$\begin{aligned}F'(t)\le&\left[\delta-1+\left(\int_0^t g(s)ds\right)J(t)\right]\|\Delta u\|_2^2+\frac{1-l}{4\delta}J(t)\left(g\circ u\right)(t)\\&+\left(\delta\beta-\alpha\right)k(t)\|u\|_2^2+\frac{\beta}{4\delta}\|z(t,1)\|_2^2.\end{aligned}\tag{3.10}$$

Proof. Differentiating and integrating by parts, we get

(3.11)

$$F'(t)=-\|\Delta u\|_2^2+J(t)\int_\Omega\int_0^t g(t-s)\Delta u(t)\Delta u(s)dsdx-k(t)\int_\Omega\left(\alpha u^2+\beta uz(t,1)\right)dx.$$

We denote by $F_1(t)$ the second term on the right-hand side of above equality. By using Young's and Cauchy-Schwarz inequalities, we have

(3.12)
$$\begin{aligned} F_1(t) =& J(t)\int_\Omega\int_0^t g(t-s)\Delta u(t)\left[\Delta u(s)-\Delta u(t)\right]dsdx + J(t)\left(\int_0^t g(s)ds\right)\|\Delta u\|_2^2 \\ \leq & J(t)\int_\Omega\int_0^t g(t-s)\left|\Delta u(t)\right|\left|\Delta u(s)-\Delta u(t)\right|dsdx + J(t)\left(\int_0^t g(s)ds\right)\|\Delta u\|_2^2 \\ \leq & \delta\|\Delta u\|_2^2 + \frac{J^2(t)}{4\delta}\int_\Omega\left(\int_0^t g(t-s)\left|\Delta u(s)-\Delta u(t)\right|ds\right)^2 dx \\ & + J(t)\left(\int_0^t g(s)ds\right)\|\Delta u\|_2^2 \\ \leq & \left(\int_0^t g(s)ds\right)\frac{J^2(t)}{4\delta}\int_\Omega\int_0^t g(t-s)\left|\Delta u(s)-\Delta u(t)\right|^2 dsdx \\ & + \left(\delta + J(t)\left(\int_0^t g(s)ds\right)\right)\|\Delta u\|_2^2 \\ \leq & \left(\delta + J(t)\left(\int_0^t g(s)ds\right)\right)\|\Delta u\|_2^2 + \frac{1-l}{4\delta}J(t)\left(g\circ\Delta u\right)(t). \end{aligned}$$

Inserting (3.12) into (3.11), we obtain the required proof. □

Lemma 3.3. *Let $G(t)$ be the function defined by*

$$G(t) = \int_\Omega\int_0^t g(t-s)u(t)\left[u(s)-u(t)\right]dsdx. \tag{3.13}$$

satisfies the estimate

(3.14)
$$\begin{aligned} G'(t) \leq & \left[\delta + 2\delta\left(1-l\right)^2 + \left(1-l\right)\left(\delta_1 - \left(\int_0^t g(s)ds\right)\right)\right]\|\Delta u\|_2^2 \\ & + \left[2\delta + \left(\alpha k(0)+\delta\beta\right)\left(\int_0^t g(s)ds\right)\right]\|u\|_2^2 \\ & + \left(\int_0^t g(s)ds\right)\left[\frac{1}{2\delta} + 2\delta J^2(0) + \frac{\alpha^2+\beta^2}{4\delta}k^2(0) + \left(\frac{1-l}{4\delta_1}\right)\right]\left(g\circ\Delta u\right)(t) \\ & - \frac{g(0)}{4\delta}C_p^2\left(g'\circ\Delta u\right)(t) + \left(\frac{\beta}{4\delta}k(0)+\delta\right)\|z(t,1)\|_2^2. \end{aligned}$$

Proof. We take the derivative of $G(t)$ to get,

$$\begin{aligned} G'(t) =& \int_\Omega\int_0^t g(t-s)u'(t)\left[u(s)-u(t)\right]dsdx - \left(\int_0^t g(s)ds\right)\int_\Omega u.u'dx \\ & + \int_\Omega\int_0^t g'(t-s)u(t)\left[u(s)-u(t)\right]dsdx, \end{aligned} \tag{3.15}$$

using the problem (2.12) we obtain

$$
\begin{aligned}
G'(t) = & \int_\Omega \int_0^t g(t-s)\left[u(s)-u(t)\right]ds \Big[-\Delta^2 u + J(t)\int_0^t g(t-s)\Delta^2 u(s)ds \\
& -\alpha K(t)u - \beta K(t) z\left(1,t\right)\Big] dx + \int_\Omega \int_0^t g'(t-s)u(t)\left[u(s)-u(t)\right]dsdx \\
& - \left(\int_0^t g(s)ds\right)\int_\Omega u \Big[-\Delta^2 u + J(t)\int_0^t g(t-s)\Delta^2 u(s)ds \\
& -\alpha K(t)u - \beta K(t) z\left(1,t\right)\Big] dx \\
= & -\int_\Omega \Delta u \int_0^t g(t-s)\left[\Delta u(s) - \Delta u(t)\right]dsdx \\
& + J(t)\int_\Omega \int_0^t g(t-s)\Delta u(s)ds \int_0^t g(t-s)\left[\Delta u(s)-\Delta u(t)\right]dsdx \\
& + \int_\Omega \int_0^t g'(t-s)u(t)\left[u(s)-u(t)\right]dsdx + \left(\int_0^t g(s)ds\right)\int_\Omega \left|\Delta u\right|^2 dx \\
& - \alpha K(t)\int_\Omega u \int_0^t g(t-s)\left[u(s)-u(t)\right]dsdx \\
& - \beta K(t)\int_\Omega z\left(t,1\right)\int_0^t g(t-s)\left[u(s)-u(t)\right]dsdx \\
& + \left(\int_0^t g(s)ds\right)\int_\Omega u\left(\alpha K(t)u + \beta K(t) z\left(t,1\right)\right)dx \\
& - J(t)\left(\int_0^t g(s)ds\right)\int_\Omega \Delta u \int_0^t g(t-s)\Delta u(s)dsdx \\
= & \sum_{i=1}^{8} G_i(t),
\end{aligned} \tag{3.16}
$$

where $G_i(t)$, $i = \overline{1,8}$, denote the terms on the right side of the above equality in order. $G_1(t)$, $G_2(t)$ and $G_3(t)$ can be estimated as in [17] as follows, for any $\delta > 0$. By Young's and Cauchy-Schwartz, we obtain

$$
G_1(t) \le \delta \left\|\Delta u\right\|_2^2 + \frac{1}{4\delta}\left(\int_0^t g(s)ds\right)\left(g \circ \Delta u\right)(t) \tag{3.17}
$$

and

$$
\begin{aligned}
G_2(t) \le & \delta J^2(t)\int_\Omega \left(\int_0^t g(t-s)\left(\left|\Delta u(t)\right| + \left|\Delta u(s)-\Delta u(t)\right|\right)ds\right)^2 dx \\
& + \frac{1}{4\delta}\int_\Omega \left(\int_0^t g(t-s)\left(\left|\Delta u(s)-\Delta u(t)\right|\right)ds\right)^2 dx \\
\le & \delta J^2(t)\left(\int_0^t g(s)ds\right)\Big[2\int_\Omega \int_0^t g(t-s)\left|\Delta u(s)-\Delta u(t)\right|^2 dsdx \\
& + 2\int_\Omega \int_0^t g(t-s)\left|\Delta u(t)\right|^2 dsdx\Big] + \frac{1}{4\delta}\left(\int_0^t g(s)ds\right)\left(g \circ \Delta u\right)(t) \\
\le & \left(2\delta J^2(t) + \frac{1}{4\delta}\right)\left(\int_0^t g(s)ds\right)\left(g \circ \Delta u\right)(t) + 2\delta\left(1-l\right)^2\left\|\Delta u\right\|_2^2.
\end{aligned} \tag{3.18}
$$

For $G_3(t)$ and $G_5(t)$, we use Cauchy-Schwartz, Young's and Poincare's inequalities, we get

(3.19)
$$\begin{aligned}
G_3(t) &\leq \left(\int_\Omega u^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega \left(\int_0^t g'(t-s)\left(u(s)-u(t)\right) ds\right)^2 dx\right)^{\frac{1}{2}}\\
&\leq \delta \left\|u\right\|_2^2 + \frac{1}{4\delta}\int_\Omega \left(\int_0^t g'(t-s)\left(u(s)-u(t)\right) ds\right)^2 dx\\
&\leq \delta \left\|u\right\|_2^2 + \frac{1}{4\delta}\int_\Omega \left(\int_0^t -g'(t-s) ds\right)\left(\int_0^t -g'(t-s)\left|u(s)-u(t)\right|^2 ds\right) dx\\
&\leq \delta \left\|u\right\|_2^2 + \frac{1}{4\delta}C_p^2 \left(\int_0^t -g'(t-s) ds\right)\left(\int_0^t -g'(t-s)\left|\Delta u(s)-\Delta u(t)\right|^2 ds\right) dx\\
&\leq \delta \left\|u\right\|_2^2 - \frac{1}{4\delta}C_p^2 \left(\int_0^t -g'(t-s) ds\right)\left(g'\circ \Delta u\right)(t)\\
&\leq \delta \left\|u\right\|_2^2 - \frac{g(0)}{4\delta}C_p^2 \left(g'\circ \Delta u\right)(t)
\end{aligned}$$

and

(3.20)
$$\begin{aligned}
G_5(t) &\leq \delta \left\|u\right\|_2^2 + \frac{\alpha^2 k(^2 t)}{4\delta}\int_\Omega \left(\int_0^t g(t-s)\left(u(s)-u(t)\right) ds\right)^2 dx\\
&\leq \delta \left\|u\right\|_2^2 + \frac{\alpha^2 k^2(t)}{4\delta}\left(\int_0^t g(s) ds\right)\int_\Omega\int_0^t g(t-s)\left(u(s)-u(t)\right)^2 ds dx\\
&\leq \delta \left\|u\right\|_2^2 + \frac{\alpha^2 k^2(t)}{4\delta}\left(\int_0^t g(s) ds\right) C_p^2 \int_\Omega\int_0^t g(t-s)\left(\Delta u(s)-\Delta u(t)\right)^2 ds dx\\
&\leq \delta \left\|u\right\|_2^2 + \frac{\alpha^2 k^2(0)}{4\delta}C_p^2\left(\int_0^t g(s) ds\right)\left(g\circ \Delta u\right)(t).
\end{aligned}$$

Similarly, we have

$$G_6(t) \leq \delta \left\|z(t,1)\right\|_2^2 + \frac{\alpha^2 k^2(0)}{4\delta}C_p^2 \left(\int_0^t g(s) ds\right)\left(g\circ \Delta u\right)(t), \tag{3.21}$$

$$G_7(t) \leq \left(\int_0^t g(s) ds\right)\left[\left(\alpha k(0)+\delta\beta\right)\left\|u\right\|_2^2 + \frac{\beta}{4\delta}k^2(0)\left\|z(t,1)\right\|_2^2\right] \tag{3.22}$$

and

$$\begin{aligned}
G_8(t) \leq &-\left(\int_0^t g(s) ds\right) J(t)\left[\int_\Omega \Delta u \int_0^t g(t-s)\left(\Delta u(s)-\Delta u(t)\right) ds dx\right.\\
&\left.+\left(\int_0^t g(s) ds\right)\left\|\Delta u\right\|_2^2\right]\\
\leq &-\left(\int_0^t g(s) ds\right) J(t)\int_\Omega \left|\Delta u\right|\int_0^t g(t-s)\left|\Delta u(s)-\Delta u(t)\right| ds dx\\
&-\left(\int_0^t g(s) ds\right)^2 J(t)\left\|\Delta u\right\|_2^2
\end{aligned}$$

$$\le \left(\int_0^t g(s)ds\right) J(t) \left[\delta_1 \|\Delta u\|_2^2 + \frac{1}{4\delta_1}\int_\Omega \left(\int_0^t g(t-s)\,|\Delta u(s)-\Delta u(t)|\,ds\right)^2 dx\right] - \left(\int_0^t g(s)ds\right)^2 J(t)\,\|\Delta u\|_2^2,$$

(3.23)

$$\begin{aligned} G_8(t) \le& \left(\int_0^t g(s)ds\right) J(t) \left[\delta_1 \|\Delta u\|_2^2 + \frac{\left(\int_0^t g(s)ds\right)}{4\delta_1} (g\circ \Delta u)(t)\right] \\ &- \left(\int_0^t g(s)ds\right)^2 J(t)\,\|\Delta u\|_2^2 \\ \le& \left(\int_0^t g(s)ds\right) J(t) \left[\left(\delta_1 - \int_0^t g(s)ds\right)\|\Delta u\|_2^2 + \frac{\left(\int_0^t g(s)ds\right)}{4\delta_1}(g\circ\Delta u)\right] \\ \le& \left(\int_0^t g(s)ds\right)\left(\delta_1 - \int_0^t g(s)ds\right) J(t)\,\|\Delta u\|_2^2 + \frac{1-l}{4\delta_1}\left(\int_0^t g(s)ds\right)(g\circ\Delta u)\,. \end{aligned}$$

Summarizing these estimates with (3.16), we get (3.14). □

Lemma 3.4. *Now, as in* [7, Lemma 3.4]*, we introduce the folowing functional*

$$\Phi(t) = \int_0^1 e^{-2\rho\tau(t)} \int_\Omega z^2(t,\rho)dxd\rho. \tag{3.24}$$

Then

$$\Phi'(t) \le \frac{d-1}{\tau_1} e^{-2\tau_1}\|z(t,1)\|_2^2 + \frac{1}{\tau_0}\|u\|_2^2 - \left(\frac{\tau'(t)}{\tau_1} + 2\right) e^{-2\tau_1}\int_0^1 \|z(t,\rho)\|_2^2\,d\rho. \tag{3.25}$$

Proof. By differentiating, using the second equation in (2.12) and integrating by parts over $(0,1)$, we get

(3.26)

$$\begin{aligned} \Phi'(t) =& -2\tau'(t)\int_0^1 \rho e^{-2\rho\tau(t)}\int_\Omega z^2(t,\rho)dxd\rho + 2\int_0^1 e^{-2\rho\tau(t)}\int_\Omega z'(t,\rho)z(t,\rho)dxd\rho \\ =& -2\tau'(t)\int_0^1 \rho e^{-2\rho\tau(t)}\int_\Omega z^2(t,\rho)dxd\rho \\ &+ 2\int_0^1 e^{-2\rho\tau(t)}\int_\Omega \frac{\rho\tau'(t)-1}{\tau(t)} z_\rho(t,\rho)z(t,\rho)dxd\rho. \end{aligned}$$

We denote by $\Phi_1(t)$ the last term in the right-hand side of the equality above

$$\begin{aligned} \Phi_1(t) =& \int_0^1 e^{-2\rho\tau(t)}\int_\Omega \frac{\rho\tau'(t)-1}{\tau(t)}\frac{d}{d\rho}z^2(t,\rho)dxd\rho \\ =& \left[e^{-2\rho\tau(t)}\int_\Omega \frac{\rho\tau'(t)-1}{\tau(t)} z^2(t,\rho)dx\right]_0^1 \end{aligned}$$

$$- \int_\Omega \int_0^1 z^2(t,\rho) \frac{d}{d\rho} \left(e^{-2\rho\tau(t)} \frac{\rho\tau'(t)-1}{\tau(t)} \right) d\rho dx$$

(3.27)

$$\begin{aligned}\Phi_1(t) = & \left[e^{-2\rho\tau(t)} \int_\Omega \frac{\rho\tau'(t)-1}{\tau(t)} z^2(t,\rho) dx \right]_0^1 + 2\tau'(t) \int_\Omega \int_0^1 \rho e^{-2\rho\tau(t)} z^2(t,\rho) d\rho dx \\ & - \left(\frac{\tau'(t)}{\tau(t)} + 2 \right) \int_\Omega \int_0^1 e^{-2\rho\tau(t)} z^2(t,\rho) d\rho dx \\ \leq & \frac{\tau'(t)-1}{\tau(t)} e^{-2\tau(t)} \left\| z^2(t,1) \right\|_2^2 + \frac{1}{\tau(t)} \left\| z^2(t,0) \right\|_2^2 \\ & + 2\tau'(t) \int_\Omega \int_0^1 \rho e^{-2\rho\tau(t)} z^2(t,\rho) d\rho dx - \left(\frac{\tau'(t)}{\tau(t)} + 2 \right) e^{-2\tau(t)} \int_\Omega \int_0^1 z^2(t,\rho) d\rho dx.\end{aligned}$$

Since

$$e^{-2\tau_1} \leq e^{-2\tau(t)} \leq e^{-2\rho\tau(t)} \leq 1, \quad \text{for all } \rho \in (0,1),\ t > 0,$$

inserting (3.27) in (3.26), we obtain (3.25). □

Now, we are ready to prove the general decay result. For this, we define the Lyapunov functional $\mathcal{L}$ by

$$\mathcal{L}(t) = NE(t) + J(t) \left(\epsilon F(t) + \epsilon_1 G(t) + \epsilon_2 \Phi(t) \right).$$

Taking the derivative of $\mathcal{L}(t)$ with respect to t we have

(3.28)

$$\mathcal{L}'(t) = NE'(t) + J(t) \left(\epsilon F'(t) + \epsilon_1 G'(t) + \epsilon_2 \Phi'(t) \right) + J'(t) \left(\epsilon F(t) + \epsilon_1 G(t) + \epsilon_2 \Phi(t) \right).$$

By using (3.9), (3.13), (3.24), Young's and Poincare's inequalities, we obtain

(3.29)

$$\begin{aligned} J'(t) \left[\epsilon F(t) + \epsilon_1 G(t) + \epsilon_2 \Phi(t) \right] \leq & \left(\epsilon - \frac{\epsilon_1}{2} \right) J'(t) \|u\|_2^2 + \epsilon_2 J'(t) e^{-2\tau_0} \int_0^1 \|z(t,\rho)\|_2^2 d\rho \\ & - \frac{\epsilon_1}{2} \left(\int_0^t g(s) ds \right) C_p^2 J'(t) \left(g \circ \Delta u \right)(t). \end{aligned}$$

Exploiting (3.29) in (3.28) and using (3.7), (3.10), (3.14) and **(H2)**, we arrive at

(3.30)

$$\begin{aligned} \mathcal{L}'(t) \leq & - J(t) \left[\left(\epsilon - \frac{\epsilon_1}{2} \right) \frac{J'(t)}{J(t)} + \epsilon \left(\alpha - \beta\delta_0' \right) M \right. \\ & \left. - \epsilon_1 \left(2\delta + (k(0)\alpha + \delta\beta) \left(\int_0^t g(s) ds \right) J(t) \right) - \frac{\epsilon_2}{\tau_0} \right] \|u\|_2^2 \\ & - \left[N \frac{M}{2} \left(\xi (1-d) - \frac{\beta}{2\delta} \right) - J(0) \left(\frac{\epsilon\beta}{4\delta} + \epsilon_1 \left(\frac{\beta}{4\delta} k^2(0) + \delta \right) \right) \right] \|z(1,t)\|_2^2 \\ & + J(t) \left[\frac{N}{2} - \epsilon_1 \frac{g(0)}{4\delta} C_p^2 \right] \left(g' \circ \Delta u \right)(t) \end{aligned}$$

$$
\begin{aligned}
&- J(t)\left[\frac{N}{2}\left(\int_0^t g(s)ds\right)\frac{J'(t)}{J(t)} + \epsilon(1-\delta) - \epsilon_1\left(\delta + 2\delta(1-l)^2\right)\right.\\
&\left.-\left(\int_0^t g(s)ds\right)J(t)\left((\epsilon+\epsilon_1\delta_1) - \epsilon_1\left(\int_0^t g(s)ds\right)\right)\right]\|\Delta u\|_2^2\\
&- J(t)\left[\epsilon_2 e^{-2\tau_1} + \frac{J'(t)}{J(t)}e^{-2\tau_0}\right]\int_0^1 \|z(\rho,t)\|_2^2\, d\rho\\
&+\left[\epsilon_1\left(\int_0^t g(s)ds\right)\left(\frac{1}{2\delta} + 2\delta J^2(0) + \frac{\alpha^2+\beta^2}{4\delta}k^2(0) + \frac{1-l}{4\delta_1}\right)\right.\\
&\left.+\epsilon\frac{1-l}{4\delta}J(0) - \frac{\epsilon_1}{2}\left(\int_0^t g(s)ds\right)C_p^2\frac{J'(t)}{J(t)}\right]J(t)\left(g\circ\Delta u\right)(t).
\end{aligned}
$$

At this point, choose ϵ_1, ϵ_2 small enough such that $0<\epsilon_2<\epsilon_1<\epsilon$ and δ_1 sufficiently small such that

$$
\epsilon(l-\delta) > \epsilon_1\left(\delta + 2\delta(1-l)^2 - (1-l)\left(\delta_1 - \int_0^t g(s)ds\right)\right) = C(\epsilon_1,\delta) > 0
$$

and

$$
C_0(\epsilon_1,\epsilon_2) = \epsilon(\alpha-\beta\delta)M - \epsilon_1\left(2\delta + (k(0)\alpha+\delta\beta)\left(\int_0^t g(s)ds\right)\right) - \frac{\epsilon_2}{\tau_0} > 0.
$$

Since (3.1), once ϵ_1 and δ are fixed, we want to choose N large enough such that

$$
N\frac{M}{2}\left(\xi(1-d) - \frac{\beta}{2\delta}\right) - J(0)\left(\frac{\epsilon\beta}{4\delta} + \epsilon_1\left(\frac{\beta}{4\delta}k^2(0) + \delta\right)\right) > 0
$$

and

$$
\frac{N}{2} - \epsilon_1\frac{g(0)}{4\delta}C_p^2 > 0.
$$

For this, (3.30) becomes

$$
\begin{aligned}
\mathcal{L}'(t) \le &- J(t)\left[\frac{N}{2}\left(\int_0^t g(s)ds\right)\frac{J'(t)}{J(t)} + \epsilon(1-\delta) - C(\epsilon_1,\delta)\right]\|\Delta u\|_2^2\\
&- J(t)\left[\epsilon_2 e^{-2\tau_1} + \frac{J'(t)}{J(t)}e^{-2\tau_0}\right]\int_0^1\|z(\rho,t)\|_2^2\, d\rho\\
&+\left[C_1 - \frac{\epsilon_1}{2}\left(\int_0^t g(s)ds\right)C_p^2\frac{J'(t)}{J(t)}\right]J(t)\left(g\circ\Delta u\right)(t)\\
&- J(t)\left[\epsilon - C_0(\epsilon_1,\epsilon_2)\right]\|u\|_2^2,
\end{aligned}
\tag{3.31}
$$

where

$$
C_1 = \epsilon_1\left(\int_0^t g(s)ds\right)\left(\frac{1}{4\delta} + \frac{1}{4\delta} + 2\delta J^2(0) + \frac{\alpha^2+\beta^2}{4\delta}k^2(0) + \frac{1-l}{4\delta_1}\right) + \epsilon\frac{1-l}{4\delta}J(0).
\tag{3.32}
$$

We then use **(H2)** and choose $t_1 \geq t_0$ so that there exist two positive constants C_2 and C_3, such that (3.31) takes the form

$$\mathcal{L}'(t) \leq -C_2 J(t)E(t) + C_3 J(t) (g \circ \Delta u)(t), \quad \text{for all } t > t_1. \tag{3.33}$$

On the other hand, as in [6], multiplying (3.33) by $\psi(t)$ and using (3.31) and (3.2), we have

$$\begin{aligned} \psi(t)\mathcal{L}'(t) &\leq -C_2\psi(t)J(t)E(t) + C_3\psi(t)J(t)(g \circ \Delta u)(t) \\ &\leq -C_2\psi(t)J(t)E(t) - C_3J(t)(g' \circ \Delta u)(t) \\ &\leq -C_2\psi(t)J(t)E(t) - 2C_3E'(t) - C_3J'(t)\left(\int_0^t g(s)ds\right)\|\Delta u\|_2^2 . \end{aligned} \tag{3.34}$$

By (2.13), we have

$$\begin{aligned} (\psi(t)\mathcal{L}(t) + 2C_3E(t))' &\leq -C_2\psi(t)J(t)E(t) - C_3J'(t)\left(\int_0^t g(s)ds\right)\|\Delta u\|_2^2 \\ &\leq -\psi(t)J(t)\left[C_2 + \frac{2}{l\psi(t)J(t)}C_3J'(t)\left(\int_0^t g(s)ds\right)\right]E(t). \end{aligned} \tag{3.35}$$

From $\lim_{t\to+\infty}\frac{J'(t)}{\psi(t)J(t)} = 0$, we can choose $t_2 \geq t_1$ and then (3.35) gives

$$(\psi(t)\mathcal{L}(t) + 2C_3E(t))' \leq -\frac{C_2}{2}\psi(t)J(t)E(t), \quad \text{for all } t > t_2. \tag{3.36}$$

We define here, the function $\mathfrak{L}$ by

$$\mathfrak{L}(t) = \psi(t)\mathcal{L}(t) + 2C_3E(t). \tag{3.37}$$

By the definition of the functionals $F(t)$, $G(t)$, $\Phi(t)$ and $E(t)$, since $\psi'(t) \leq 0$, we can prove $\mathfrak{L}(t)$ equivalent to $E(t)$ and there exists a positive constant λ such that

$$\mathfrak{L}'(t) \leq -\lambda\psi(t)J(t)\mathfrak{L}(t), \quad \text{for all } t \geq t_2. \tag{3.38}$$

By simple integration of 3.38 over $[t_2, t]$ and use the equivalence of $\mathfrak{L}(t)$ and $E(t)$ we obtain

$$E(t) \leq Ce^{-K_0\int_{t_2}^t \psi(t)J(t)dt}, \quad \text{for all } t \geq t_2.$$

By the continuity and boundedness of $E(t)$ in the interval $[0, t_2]$, we have

$$E(t) \leq Ce^{-K_0\int_0^t \psi(t)J(t)dt}, \quad \text{for all } t \geq 0.$$

The proof of Theorem 3.1 is completed.

References

[1] C. T. Abdallah, P. D̃orato, J. Benites-Read and R. Byrne, *Delayed positive feedback can stabilize oscillatory system*, American Control Conference, San Francisco, 1993, 3106–3107.

[2] S. V. C. H̃uang, *An analysis of delay-dependent stability for ordinary and partial differential equations with fixed and distributed delays*, SIAM J. Sci. Comput. **25** (2004), 1608–1632.

[3] S. V. C. H̃uang, *Stability of Runge-Kutta-Pouzet methods for Volterra integro-differential equations with delays*, Front. Math. China **4** (2009), 63–87.

[4] S. Y. C. Q. Xu and L. Li, *Stabilization of the wave system with input delay in the boundary control*, ESAIM Control Optim. Calc. Var. **12** (2006), 770–785.

[5] R. Datko, *Not all feedback stabilized hyperbolic systems are robust with respect to small time delays in their feedbacks*, SIAM J. Control Optim. **26** (1988), 697–713.

[6] B. Feng, *General decay for a viscoelastic wave equation with strong time-dependent delay*, Bound. Value Probl. (2017), 1–11.

[7] M. Ferhat and A. Hakem, *On convexity for energy decay rates of a viscoelastic wave equation with a dynamic boundary and nonlinear delay term*, Facta Univ. Ser. Math. Inform. **30** (2015), 67–87.

[8] E. Fridman and Y. Orlov, *Exponential stability of linear distributed parameter systems with time-varying delays*, Automatica **45** (2009), 194–201.

[9] R. R. H. L̃ogemann and G. Weiss, *Conditions for robustness and nonrobustness of the stability of feedback systems with respect to small delays in the feedback loop*, SIAM J. Control Optim. **34** (1996), 572–600.

[10] W. Liu, *General decay rate estimate for the energy of a weak viscoelastic equation with an internal time-varying delay term*, Taiwanese J. Math. **17** (2013), 2101–2115.

[11] S. Nicaise and C. Pignotti, *Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks*, SIAM J. Control Optim. **45** (2006), 1561–1585.

[12] S. Nicaise and C. Pignotti, *Stabilization of the wave equation with boundary or internal distributed delay*, SIAM J. Control Optim. **21** (2008), 9–10.

[13] S. Nicaise and C. Pignotti, *Interior feedback stabilization of wave equations with time dependent delay*, Electron. J. Differential Equations **41** (2011), 1–20.

[14] S. Nicaise, C. Pignotti and J. Valein, *Exponential stability of the wave equation with boundary time-varying delay*, Discrete Contin. Dyn. Syst. Ser. B **21** (2011), 693–722.

[15] R. Rebarber and S. Townley, *Robustness with respect to delays for exponential stability of distributed parameter systems*, SIAM J. Control Optim. **37** (1998), 230–244.

[16] J. B. S. B̃ernard and M. C. Mackey, *Sufficient conditions for stability of linear differential equations with distributed delay*, Discrete Contin. Dyn. Syst. Ser. B **1**(2) (2001), 233–256.

[17] S. A. Messaoudi, *General decay of solutions of a weak viscoelastic equation*, Arab. J. Sci. Eng. **36** (2011), 1569–1579.

[18] J. V. S. Ñicaise and E. Fridman, *Stability of the heat and of the wave equations with boundary time-varying delays*, Discrete Contin. Dyn. Syst. Ser. S (2009), 559–581.

[19] I. Suh and Z. Bien, *Use of time delay action in the controller design*, IEEE Trans. Automat. Control **25** (1980), 600–603.

[20] J. R. T. C̃araballo and L. Shaikhet, *Method of lyapunov functionals construction in stability of delay evolution equations*, J. Math. Anal. Appl. (2007), 1130–1145.

[21] T. X. Wang, *Stability in abstract functional-differential equations. Part II. Applications*, J. Math. Anal. Appl. **186** (1994), 835–861.

[22] S. Wu and S. Gan, *Analytical and numerical stability of neutral delay integro-differential equations and neutral delay partial differential equations*, Comput. Math. Appl. **55** (2008), 2426–2443.

[23] Q. C. Zhong, *Robust control of time-delay systems*, Springer, London, 2006.

[1]Department Mathematics,
University of Hassib Benbouali-Chlef,
Laboratory of Fundamental and Applicable Mathematics of Oran,
University of Hassib Benbouali-Chlef 02000, Algeria.
Email address: braik.aek@gmail.com

[2]Department Mathematics,
Center University of Belhadj Bouchaib,
Laboratory ACEDP, Center University of Belhadj Bouchaib -B.P. 284 RP,
Ain Temouchent 46000, Algeria.
Email address: a.beniani@yahoo.fr

[3]Department Mathematics,
University of Oran1 Ahmed Ben Bella,
Laboratory of Fundamental and Applicable Mathematics of Oran,
University of Oran1 Ahmed Ben Bella, B.P 1524 El M'naouar, Oran 31000, Algeria.
Email address: yamina69@yahoo.fr