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$\alpha\beta$-WEIGHTED d_g-STATISTICAL CONVERGENCE IN PROBABILITY

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Abstract. In this paper we consider the notion of generalized density, namely, the natural density of weight g was introduced by Balcerzak et al. (Acta Math. Hungar. **147**(1) (2015) 97–115) and the entire investigation is performed in the setting of probability space extending the recent results of Ghosal (Appl. Math. Comput. **249** (2014) 502–509) and Das et al. (Filomat **31**(5) (2017) 1463–1473).

1. Introduction

In the year 1932, Agnew [1] defined the *deferred Cesàrro mean* of sequences of real numbers such as

$$(D_{p,q}x)_n = \frac{1}{q_n - p_n} \sum_{k=p_n+1}^{q_n} x_k,$$

where $\{p_n\}_{n\in\mathbb{N}}$ and $\{q_n\}_{n\in\mathbb{N}}$ are sequences of non-negative integers satisfying

$$p_n < q_n, \quad \text{for all } n \in \mathbb{N}, \quad \text{and} \quad \lim_{n\to\infty} q_n = +\infty.$$

In 2016, the concept of *deferred statistical convergence* (similar concept has been discussed by Aktuğlu [3] in 2014 which was named as *$\alpha\beta$-statistical convergence*) were given by Küçükaslan and Yilmaztürk [21] such as (earlier this concept has been defined by the same authors and submitted as a thesis to Mersin University/Turkey).

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Let $\{p_n\}_{n\in\mathbb{N}}$ and $\{q_n\}_{n\in\mathbb{N}}$ be two sequences as above. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be deferred statistically convergent to L if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} |\{p_n < k \leq q_n : |x_k - L| \geq \varepsilon\}| = 0.$$

After then some work has been carried out with related to this concept [13, 21, 26].

In [4], the notion of natural density [14, 15, 23] (and also the natural density of order α [5, 7]) was further extended as follows: Let $g : \mathbb{N} \to [0, \infty)$ be a function with $\lim_{n\to\infty} g(n) = \infty$. The upper density of weight g was defined in [4] by the formula

$$\bar{d}_g(A) = \limsup_{n\to\infty} \frac{\operatorname{card}(A \cap [1, n])}{g(n)},$$

for $A \subset \mathbb{N}$. Then the family $\mathcal{I}_g = \{A \subset \mathbb{N} : \bar{d}_g(A) = 0\}$ forms an ideal. It was also observed in [4] that $\mathbb{N} \in \mathcal{I}_g$ if and only if $\frac{n}{g(n)} \to 0$. Hence, we additionally assume that $\frac{n}{g(n)} \nrightarrow 0$. So that $\mathbb{N} \notin \mathcal{I}_g$ and it was observed in [4, 10], that $\mathcal{I}_g$ is a proper admissible P-ideal of $\mathbb{N}$. The collection of all functions g of this kind satisfying the above-mentioned property is denoted by G.

A sequence $x = \{x_n\}_{n\in\mathbb{N}}$ in a metric space (X, ρ) is said to be d_g-statistically convergent to $a \in X$ if for any $\varepsilon > 0$ we have $d_g(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{n \in \mathbb{N} : \rho(x_n, a) \geq \varepsilon\}$.

Another generalization of the statistical convergence is known as weighted statistical convergence which was established by Karakaya and Chishti [20] in 2009 and gradually improved by Aizpuru et al. [2], Cinar and Et [6, 12], Das et al. [9], Ghosal [16–18], Işik and Altin [19], Mursaleen et al. [22] and Som [25].

In this paper the idea of four types of convergences of a sequence of random variables, namely,

(a) $\alpha\beta$-weighted d_g-statistically convergent sequence in probability;

(b) $\alpha\beta$-weighted d_g-strongly Cesàrro convergence in probability;

(c) g-weighted $S_{\alpha\beta}$-convergence in probability;

(d) g-weighted $N_{\alpha\beta}$-convergence in probability all have been introduced and the interrelations among them have been investigated. Also their certain basic properties are analyzed.

The main object of this paper is to improve all the existing results in this direction [4, 9, 11, 16, 17] which could be effectively extended. Moreover, we intend to establish the relations among these four summability notions. It is important to note that the methods of proofs and in particular the examples are not analogous to the real case.

2. Definitions and Notations

The following definitions and notions will be needed in sequel.

Definition 2.1 (see [3]). Let $\alpha = \{\alpha_n\}_{n\in\mathbb{N}}$ and $\beta = \{\beta_n\}_{n\in\mathbb{N}}$ be two sequences of positive real numbers such that

(i) α and β are both non-decreasing;

(ii) $\beta_n \geq \alpha_n$ for all $n \in \mathbb{N}$;
(iii) $(\beta_n - \alpha_n) \to \infty$ as $n \to \infty$.

Then the sequence of real numbers $\{x_n\}_{n\in\mathbb{N}}$ is said to be $\alpha\beta$-statistically convergent of order γ (where $0 < \gamma \leq 1$) to a real number x if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{(\beta_n - \alpha_n + 1)^\gamma} \left|\{k \in [\alpha_n, \beta_n] : |x_k - x| \geq \varepsilon\}\right| = 0.$$

In this case we write $x_n \xrightarrow{S^\gamma_{\alpha\beta}} x$ and the set of all sequences which are $\alpha\beta$-statistically convergent of order γ is denoted by $S^\gamma_{\alpha\beta}$.

Definition 2.2 (see [9])**.** A sequence of real numbers $\{t_n\}_{n\in\mathbb{N}}$ is said to be a weighted sequence if there exists a positive real number δ such that $t_n > \delta$ for all $n \in \mathbb{N}$.

Definition 2.3 (see [17])**.** Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $\liminf\limits_{n\to\infty} t_n > 0$ and $T_{\alpha\beta(n)} = \sum_{k\in[\alpha_n,\beta_n]} t_k$ for all $n \in \mathbb{N}$. A sequence of real numbers $\{x_n\}_{n\in\mathbb{N}}$ is said to be weighted $\alpha\beta$-statistically convergent of order γ (where $0 < \gamma \leq 1$) to x if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{T^\gamma_{\alpha\beta(n)}} \left|\{k \leq T_{\alpha\beta(n)} : t_k|x_k - x| \geq \varepsilon\}\right| = 0.$$

In this case we write $x_n \xrightarrow{(S^\gamma_{\alpha\beta}, t_n)} x$. The class of all weighted $\alpha\beta$-statistically convergent sequences of order γ is denoted by $(S^\gamma_{\alpha\beta}, t_n)$.

Definition 2.4 (see [17])**.** Let ϕ be a modulus function and $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $\liminf_{n\to\infty} t_n > 0$ and $T_{\alpha\beta(n)} = \sum_{k\in[\alpha_n,\beta_n]} t_k$ for all $n \in \mathbb{N}$. A sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is said to be weighted modulus $\alpha\beta$-statistical convergence of order γ (where $0 < \gamma \leq 1$) in probability to a random variable X (where $X : \mathcal{W} \to \mathbb{R}$) if for any $\varepsilon, \delta > 0$

$$\lim_{n\to\infty} \frac{1}{T^\gamma_{\alpha\beta(n)}} \left|\{k \leq T_{\alpha\beta(n)} : t_k\phi(P(|X_k - X| \geq \varepsilon)) \geq \delta\}\right| = 0.$$

In this case, $X_n \xrightarrow{(S^\gamma_{\alpha\beta}, P^\phi, t_n)} X$ and the class of all weighted modulus statistically convergent sequences of order γ in probability is denoted by $(S^\gamma_{\alpha\beta}, P^\phi, t_n)$.

Definition 2.5 (see [17])**.** Let ϕ be a modulus function and $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers such that $t_1 > 0$ and $T_{\alpha\beta(n)} = \sum_{k\in[\alpha_n,\beta_n]} t_k \to \infty$ as $n \to \infty$. A sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is said to be weighted modulus $\alpha\beta$-strongly Cesàrro convergent of order γ (where $0 < \gamma \leq 1$) in probability to a random variable X if for any $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{T^\gamma_{\alpha\beta(n)}} \sum_{k\in[\alpha_n,\beta_n]} t_k\phi(P(|X_k - X| \geq \varepsilon)) = 0.$$

In this case, $X_n \xrightarrow{(N^{\gamma}_{\alpha\beta}, P^{\phi}, t_n)} X$ and the class of all sequences of random variables which are weighted modulus $\alpha\beta$-strong Cesàrro convergent of order γ in probability, is denoted by $(N^{\gamma}_{\alpha\beta}, P^{\phi}, t_n)$.

Definition 2.6 (see [17]). Let ϕ be a modulus function and $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $\liminf_{n\to\infty} t_n > 0$ and $T_{\alpha\beta(n)} = \sum_{k\in[\alpha_n,\beta_n]} t_k$ for all $n \in \mathbb{N}$. A sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is said to be weighted modulus $S_{\alpha\beta}$-convergence of order γ in probability (where $0 < \gamma \le 1$) to a random variable X if for every $\varepsilon, \delta > 0$

$$\lim_{n\to\infty} \frac{1}{T^{\gamma}_{\alpha\beta(n)}} \left|\{k \in I_{\alpha\beta(n)} : t_k\phi(P(|X_k - X| \ge \varepsilon)) \ge \delta\}\right| = 0,$$

where $I_{\alpha\beta(n)} = (T_{[\alpha(n)]}, T_{[\beta(n)]}]$ and $[x]$ denotes the greatest integer not grater than x. In this case we write $X_n \xrightarrow{(WS^{\gamma}_{\alpha\beta}, P^{\phi}, t_n)} X$. The class of all weighted modulus $S_{\alpha\beta}$-convergent sequences of order γ in probability is denoted by $(WS^{\gamma}_{\alpha\beta}, P^{\phi}, t_n)$.

Definition 2.7 ([17]). Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers such that $t_1 > 0$ and $T_{\alpha\beta(n)} = \sum_{k\in[\alpha_n,\beta_n]} t_k \to \infty$, as $n \to \infty$ and ϕ be a modulus function. The sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is said to be weighted modulus $N_{\alpha\beta}$-convergence of order γ in probability (where $0 < \gamma \le 1$) to a random variable X if for any $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{T^{\gamma}_{\alpha\beta(n)}} \sum_{k\in I_{\alpha\beta(n)}} t_k\phi(P(|X_k - X| \ge \varepsilon)) = 0.$$

In this case, $X_n \xrightarrow{(WN^{\gamma}_{\alpha\beta}, P^{\phi}, t_n)} X$ and the class of all weighted modulus $N_{\alpha\beta}$-convergent sequences of order γ in probability is denoted by $(WN^{\gamma}_{\alpha\beta}, P^{\phi}, t_n)$.

3. Main Results

First we introduce the definition of $\alpha\beta$-weighted d_g-statistical convergence in probability of random variables as follows.

Definition 3.1. Let $\{t_n\}_{n\in\mathbb{N}}$ be a weighted sequence and $T_{\alpha\beta(n)} = \sum_{k\in[\alpha_n,\beta_n]} t_k$ for all $n \in \mathbb{N}$. Then the sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is said to be $\alpha\beta$-weighted d_g-statistically convergent in probability to a random variable X (where $X : \mathcal{W} \to \mathbb{R}$) if for any $\varepsilon, \delta > 0$

$$\lim_{n\to\infty} \frac{1}{g(T_{\alpha\beta(n)})} \left|\{k \le T_{\alpha\beta(n)} : t_k P(|X_k - X| \ge \varepsilon) \ge \delta\}\right| = 0.$$

Hence, we assume that $g : (0,\infty) \to (0,\infty)$ is a continuous function such that $\lim_{x\to\infty} g(x) = \infty$ and $\lim_{n\to\infty} \frac{T_{\alpha\beta(n)}}{g(T_{\alpha\beta(n)})} \ne 0$ and we write $X_n \xrightarrow{\alpha\beta WS^{p}_{d_g}} X$ and the class of all $\alpha\beta$-weighted d_g-statistically convergent sequences in probability is denoted by $\alpha\beta WS^{p}_{d_g}$.

Throughout the paper we assume that $g:(0,\infty)\to(0,\infty)$ is a continuous function such that $\lim_{x\to\infty} g(x)=\infty$ and $\lim_{n\to\infty}\frac{T_{\alpha\beta(n)}}{g(T_{\alpha\beta(n)})}\neq 0$.

Theorem 3.1. *If* $X_n \xrightarrow{\alpha\beta WS^p_{d_g}} X$ *and* $X_n \xrightarrow{\alpha\beta WS^p_{d_g}} Y$*, then* $P\{X=Y\}=1$.

Proof. If possible let $P\{X=Y\}\neq 1$. Then there exist two positive real numbers ε,δ such that $P(|X-Y|\geq\varepsilon)>\delta$ and $t_n>\delta$ for all $n\in\mathbb{N}$. Then

$$\begin{aligned}\frac{T_{\alpha\beta(n)}}{g(T_{\alpha\beta(n)})} =& \frac{1}{g(T_{\alpha\beta(n)})}\ |\{k\leq T_{\alpha\beta(n)}: t_kP(|X-Y|\geq\varepsilon)\geq\delta^2\}|\\ \leq& \frac{1}{g(T_{\alpha\beta(n)})}\ \left|\left\{k\leq T_{\alpha\beta(n)}: t_kP\left(|X_k-X|\geq\frac{\varepsilon}{2}\right)\geq\frac{\delta^2}{2}\right\}\right|\\ &+\frac{1}{g(T_{\alpha\beta(n)})}\ \left|\left\{k\leq T_{\alpha\beta(n)}: t_kP\left(|X_k-Y|\geq\frac{\varepsilon}{2}\right)\geq\frac{\delta^2}{2}\right\}\right|,\end{aligned}$$

which is impossible because the right hand side tends to zero as $n\to\infty$ but not the left hand side. Hence, the result follows. □

The following example shows that weighted $\alpha\beta$-statistical convergence in probability [17] and $\alpha\beta$-weighted d_g-statistical convergence in probability are totally different.

Example 3.1. Let the sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is defined by,

$$X_n\in\begin{cases}\{-1,1\}\text{ with p.m.f } P(X_n=-1)=P(X_n=0), & \text{if } n\in\{1^2,2^2,3^2,\dots\},\\ \{0,1\}\text{ with p.m.f } P(X_n=0)=1-\frac{1}{n^4}, & \\ P(X_n=1)=\frac{1}{n^4}, & \text{otherwise.}\end{cases}$$

Let $t_n=2n$, $\alpha_n=n$, $\beta_n=n^2$ for all $n\in\mathbb{N}$ and $g(x)=\sqrt[4]{x}$ for all $x\in(0,\infty)$. Then $T_{\alpha\beta(n)}=n^4+n$ for all $n\in\mathbb{N}$ and $\frac{T_{\alpha\beta(n)}}{g(T_{\alpha\beta(n)})}\nrightarrow 0$ as $n\to\infty$.

For $0<\varepsilon<1$, we get

$$P(|X_n-0|\geq\varepsilon)=\begin{cases}1, & \text{if } n=m^2,\ \text{where } m\in\mathbb{N},\\ \frac{1}{n^4}, & \text{if } n\neq m^2,\ \text{where } m\in\mathbb{N}.\end{cases}$$

Now, let $0<\delta<1$. Then

$$\frac{1}{T_{\alpha\beta(n)}}|\{k\leq T_{\alpha\beta(n)}: t_kP(|X_k-0|\geq\varepsilon)\geq\delta\}|\leq\frac{2}{n^2}$$

and

$$\frac{1}{g(T_{\alpha\beta(n)})}|\{k\leq T_{\alpha\beta(n)}: t_kP(|X_k-0|\geq\varepsilon)\geq\delta\}|\geq\frac{\sqrt{n^4+n}-1}{\sqrt[4]{n^4+n}}\geq n.$$

This shows that $\{X_n\}_{n\in\mathbb{N}}$ is weighted $\alpha\beta$-statistically convergent in probability to a random variable 0 but it is not $\alpha\beta$-weighted d_g-statistically convergent in probability to 0.

Therefore we come to a conclusion that Definition 3.1 is the non-trivial extension of the notions obtained by different authors in the past, because if we take $g(x) = x^\gamma$ for all $x \in (0, \infty)$ and $0 < \gamma \leq 1$ then Definition 3.1 reduces to the Definition 2.1 [9] and Definition 2.1 [17].

The proof of the following two theorems are straightforward, so we choose to state these results without proof.

Theorem 3.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function on $\mathbb{R}$. If $X_n \xrightarrow{\alpha\beta WS^p_{d_g}} X$ and $P(|X| \geq \alpha) = 0$ for some positive real number α, then $f(X_n) \xrightarrow{\alpha\beta WS^p_{d_g}} f(X)$.*

Theorem 3.3. *Let $X_n \xrightarrow{\alpha\beta WS^p_{d_g}} x$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, then $f(X_n) \xrightarrow{\alpha\beta WS^p_{d_g}} f(x)$.*

Definition 3.2. Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers such that $t_1 > 0$ and $T_{\alpha\beta(n)} = \sum_{k\in[\alpha_n,\beta_n]} t_k \to \infty$ as $n \to \infty$. The sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is said to be $\alpha\beta$-weighted d_g-strongly Cesàrro convergent in probability to a random variable X if for any $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{g(T_{\alpha\beta(n)})} \sum_{k\in[\alpha_n,\beta_n]} t_k P(|X_k - X| \geq \varepsilon) = 0.$$

In this case, $X_n \xrightarrow{\alpha\beta WN^p_{d_g}} X$ and the class of all $\alpha\beta$-weighted d_g-strongly Cesàrro convergent sequences in probability is denoted by $\alpha\beta WN^p_{d_g}$.

In the following, the relationship between $\alpha\beta WS^p_{d_g}$ and $\alpha\beta WN^p_{d_g}$ is investigated.

Theorem 3.4. *Let ζ be a positive real number such that $t_n > \zeta$ for all $n \in \mathbb{N}$. If $\{\alpha_n\}_{n\in\mathbb{N}}$ be a bounded sequence and $\liminf_{n\to\infty} \frac{\beta_n}{T_{\alpha\beta(n)}} \geq 1$, then $\alpha\beta WN^p_{d_g} \subset \alpha\beta WS^p_{d_g}$.*

Proof. Let $X_n \xrightarrow{\alpha\beta WN^p_g} X$ and $\varepsilon > 0$. Then

$$\begin{aligned} &\frac{1}{g(T_{\alpha\beta(n)})} \sum_{k\in[\alpha_n,\beta_n]} t_k P(|X_k - X| \geq \varepsilon) \\ \geq &\frac{\zeta}{g(T_{\alpha\beta(n)})} |\{k \leq T_{\alpha\beta(n)} : t_k P(|X_k - X| \geq \varepsilon) \geq \delta\}|. \end{aligned}$$

Hence, the result follows. □

The following example shows that, the sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ in $\alpha\beta WS^p_{d_g}$ converges to X but not in $\alpha\beta WN^p_{d_g}$ converges to X.

Example 3.2. Let $t_n = n$, $\alpha_n = 1$, $\beta_n = n$ for all $n \in \mathbb{N}$ and $g(x) = \sqrt[4]{x}$ for all $x \in (0, \infty)$. Then $T_{\alpha\beta(n)} = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$ and $\frac{T_{\alpha\beta(n)}}{g(T_{\alpha\beta(n)})} \nrightarrow 0$ as $n \to \infty$.

Consider the sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is defined by,

$$X_n \in \begin{cases} \{-1,1\} \text{ with probability } \frac{1}{2}, & \text{if } n = \{T_m\}^{T_m} \text{ for any } m \in \mathbb{N}, \\ \{0,1\} \text{ with p.m.f } P(X_n = 0) = 1 - \frac{1}{n^{\frac{3}{2}}}, & \\ P(X_n = 1) = \frac{1}{n^{\frac{3}{2}}}, & \text{if } n \neq \{T_m\}^{T_m} \text{ for any } m \in \mathbb{N}. \end{cases}$$

Let $0 < \varepsilon < 1$, then,

$$P(|X_n - 0| \geq \varepsilon) = \begin{cases} 1, & \text{if } n = \{T_m\}^{T_m} \text{ for any } m \in \mathbb{N}, \\ \frac{1}{n^{\frac{3}{2}}}, & \text{if } n \neq \{T_m\}^{T_m} \text{ for any } m \in \mathbb{N}. \end{cases}$$

This implies $X_n \xrightarrow{\alpha\beta WS^p_{d_g}} 0$.

Let $H = \{n \in \mathbb{N} : \ n \neq \{T_m\}^{T_m} \text{ where } m \in \mathbb{N}\}$.

Now we have the inequality

$$\begin{aligned} \sum_{k\in[\alpha_n,\beta_n]} t_k P(|X_k - 0| \geq \varepsilon) &= \sum_{\substack{k\in[\alpha_n,\beta_n] \\ k\in H}} t_k P(|X_k - 0| \geq \varepsilon) + \sum_{\substack{k\in[\alpha_n,\beta_n] \\ k\notin H}} t_k P(|X_k - 0| \geq \varepsilon) \\ &> \sum_{\substack{k\in[\alpha_n,\beta_n] \\ k\in H}} \frac{1}{\sqrt{k}} + \sum_{\substack{k\in[\alpha_n,\beta_n] \\ k\notin H}} \\ &> \sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sqrt{n} \quad \left(\text{since } \sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sqrt{n} \text{ for all } n \geq 2\right) \end{aligned}$$

This implies $\frac{1}{g(T_{\alpha\beta(n)})} \sum_{k=1}^{n} t_k P(|X_k - 0| \geq \varepsilon) > \frac{\sqrt{n}}{\sqrt[4]{\frac{n(n+1)}{2}}} \geq 1$. This inequality shows that $\{X_n\}_{n\in\mathbb{N}}$ is not $\alpha\beta WN^p_{d_g}$ summable to 0.

Theorem 3.5. *Let the weighted sequence $\{t_n\}_{n\in\mathbb{N}}$ be bounded such that*

$$\limsup_{n\to\infty} \frac{\beta_n - \alpha_n}{g(T_{\alpha\beta(n)})} < \infty.$$

Then $\alpha\beta WS^p_{d_g} \subset \alpha\beta WN^p_{d_g}$.

Proof. Let $X_n \xrightarrow{\alpha\beta WS^p_{d_g}} X$ and $t_n \leq M_1$ for all $n \in \mathbb{N}$ and $\limsup_{n\to\infty} \frac{\beta_n - \alpha_n}{g(T_{\alpha\beta(n)})} < M_2$, where M_1 and M_2 are positive real numbers. For any $\varepsilon, \delta > 0$ setting $H = \{k \leq T_{\alpha\beta(n)} : t_k P(|X_k - X| \geq \varepsilon) \geq \delta\}$. Then

$$\begin{aligned} &\frac{1}{g(T_{\alpha\beta(n)})} \sum_{k\in[\alpha_n,\beta_n]} t_k P(|X_k - X| \geq \varepsilon) \\ =&\frac{1}{g(T_{\alpha\beta(n)})} \sum_{k\in[\alpha_n,\beta_n]\cap H} t_k P(|X_k - X| \geq \varepsilon) + \frac{1}{g(T_{\alpha\beta(n)})} \sum_{k\in[\alpha_n,\beta_n]\cap H^c} t_k P(|X_k - X| \geq \varepsilon) \\ \leq&\frac{M_1}{g(T_{\alpha\beta(n)})} |\{k \leq T_{\alpha\beta(n)} : t_k P(|X_k - X| \geq \varepsilon) \geq \delta\}| + M_2\delta. \end{aligned}$$

Since δ is arbitrary, the result follows. □

The following example shows that the sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ in $\alpha\beta WN^p_{d_g}$ converges to X but not in $\alpha\beta WS^p_{d_g}$ converges to X.

Example 3.3. Let $c \in (0,1)$, $\gamma \in (2c, 4c) \cap \mathbb{Q}$ and a sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is defined by

$$X_n \in \begin{cases} \{-1, 0\}, \text{ with p.m.f } P(X_n = -1) = \frac{1}{n}, & \\ P(X_n = 0) = 1 - \frac{1}{n}, & \text{if } n = [m^{\frac{1}{c}}], \text{ where } m \in \mathbb{N}, \\ \{0, 1\}, \text{ with p.m.f } P(X_n = 0) = 1 - \frac{1}{n^8}, & \\ P(X_n = 1) = \frac{1}{n^8}, & \text{if } n \neq [m^{\frac{1}{c}}], \text{ where } m \in \mathbb{N}. \end{cases}$$

Let $t_n = 2n$, $\alpha_n = n$, $\beta_n = n^2$ for all $n \in \mathbb{N}$ and $g(x) = x^{\frac{\gamma}{4}}$ for all $x \in (0, \infty)$. Then $T_{\alpha\beta(n)} = n^4 + n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \frac{T_{\alpha\beta(n)}}{g(T_{\alpha\beta(n)})} \neq 0$.

For $0 < \varepsilon$, $\delta < 1$, we get

$$\begin{aligned} \frac{1}{g(T_{\alpha\beta(n)})} \sum_{k\in[\alpha_n,\beta_n]} t_k P(|X_k - 0| \geq \varepsilon) \leq & \frac{2}{n^\gamma} \left\{ (n^{2c} - n^c + 1) + \left(\frac{1}{1^3} + \frac{1}{2^3} + \dots + \frac{1}{(n^2)^3} \right) \right\} \\ \leq & \frac{M}{n^{\gamma - 2c}} \quad \text{(where } M \text{ is a positive constant)} \end{aligned}$$

and

$$\frac{1}{g(T_{\alpha\beta(n)})} |\{k \leq T_{\alpha\beta(n)} : t_k P(|X_k - 0| \geq \epsilon) \geq \delta\}| \geq \frac{(n^4 + n)^c}{n^\gamma} > \frac{1}{2} n^{4c - \gamma}.$$

So, $\{X_n\}_{n\in\mathbb{N}} \in \alpha\beta WN^p_{d_g}$ but not in $\alpha\beta WS^p_{d_g}$.

Now we would like to introduce the definitions of g-weighted $S_{\alpha\beta}$-convergence in probability and g-weighted $N_{\alpha\beta}$-convergence in probability for a sequence of random variables as follows.

Definition 3.3. Let $\{t_n\}_{n\in\mathbb{N}}$ be a weighted sequence and $T_{\alpha\beta(n)} = \sum_{k\in[\alpha_n,\beta_n]} t_k$ for all $n \in \mathbb{N}$. Then the sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is said to be g-weighted $S_{\alpha\beta}$-convergence in probability to X if for every $\varepsilon, \delta > 0$

$$\lim_{n\to\infty} \frac{1}{g(T_{\alpha\beta(n)})} |\{k \in I_{\alpha\beta(n)} : t_k P(|X_k - X| \geq \varepsilon) \geq \delta\}| = 0,$$

where $I_{\alpha\beta(n)} = (T_{[\alpha(n)]}, T_{[\beta(n)]}]$ and $[x]$ denotes the greatest integer not greater than x. In this case we write $X_n \xrightarrow{WS^{d_g}_{\alpha\beta}} X$. The class of all g-weighted $S_{\alpha\beta}$-convergent sequences in probability is denoted by $WS^{d_g}_{\alpha\beta}$.

It is very obvious that if $X_n \xrightarrow{WS^{d_g}_{\alpha\beta}} X$ and $X_n \xrightarrow{WS^{d_g}_{\alpha\beta}} Y$, then $P\{X = Y\} = 1$.

Definition 3.4. Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers such that $t_1 > 0$ and $T_{\alpha\beta(n)} = \sum_{k\in[\alpha_n,\beta_n]} t_k \to \infty$ as $n \to \infty$. The sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is said to be g-weighted $N_{\alpha\beta}$-convergence in probability to a random variable X if for any $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{g(T_{\alpha\beta(n)})} \sum_{k\in I_{\alpha\beta(n)}} t_k P(|X_k - X| \geq \varepsilon) = 0.$$

In this case $X_n \xrightarrow{WN_{\alpha\beta}^{d_g}} X$ and the class of all g-weighted $N_{\alpha\beta}$-convergent sequences in probability is denoted by $WN_{\alpha\beta}^{d_g}$.

In the following, the relationship between $WS_{\alpha\beta}^{d_g}$ and $WN_{\alpha\beta}^{d_g}$ is investigated.

Theorem 3.6. *Let $\{t_n\}_{n\in\mathbb{N}}$ be a weighted sequence. Then $WN_{\alpha\beta}^{d_g} \subset WS_{\alpha\beta}^{d_g}$ and this inclusion is strict.*

Proof. For the first part of this theorem, let $\varepsilon, \delta > 0$, then

$$\begin{aligned}
&\sum_{k\in I_{\alpha\beta(n)}} t_k P(|X_k - X| \geq \varepsilon) \\
=& \sum_{k\in I_{\alpha\beta(n)}, t_k P(|X_k - X|\geq\varepsilon)\geq\delta} t_k P(|X_k - X| \geq \varepsilon) + \sum_{k\in I_{\alpha\beta(n)}, t_k P(|X_k - X|\geq\varepsilon)<\delta} t_k P(|X_k - X| \geq \varepsilon) \\
\geq& \delta |\{k \in I_{\alpha\beta(n)} : t_k P(|X_k - X| \geq \varepsilon) \geq \delta\}|.
\end{aligned}$$

For the second part we will give an example. Let $t_n = n$, $\alpha(n) = n!$, $\beta(n) = (n+1)!$ for all $n \in \mathbb{N}$ and $g(x) = \sqrt{x}$ for all $x \in (0, \infty)$ and a sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is defined by,

$$X_n \in \begin{cases} \{-1, 1\}, \text{ with p.m.f } P(X_n = 1) = P(X_n = -1), \quad \text{if } n \text{ is the first} \\ [\sqrt[4]{(T_{[\beta(n)]} - T_{[\alpha(n)]})}] \text{ integer in the interval } (T_{[\alpha(n)]}, T_{[\beta(n)]}], \\ \{0, 1\}, \text{ with p.m.f } P(X_n = 0) = 1 - \frac{1}{n^3}, P(X_n = 1) = \frac{1}{n^3}, \quad \text{otherwise.} \end{cases}$$

For $0 < \varepsilon, \delta < 1$, we get

$$\frac{1}{g(T_{\alpha\beta(n)})} |\{k \in I_{\alpha\beta(n)} : t_k P(|X_k - 0| \geq \varepsilon) \geq \delta\}| \leq \frac{1}{\sqrt[4]{(T_{[\beta(n)]} - T_{[\alpha(n)]})}} \to 0, \quad \text{as } n \to 0.$$

For next

$$\begin{aligned}
&\frac{1}{g(T_{\alpha\beta(n)})} \sum_{k\in I_{\alpha\beta(n)}} t_k P(|X_k - 0| \geq \varepsilon) \\
\geq& \frac{[\sqrt[4]{(T_{[\beta(n)]} - T_{[\alpha(n)]})}]\{[\sqrt[4]{(T_{[\beta(n)]} - T_{[\alpha(n)]})}] + 1\}}{2\sqrt{(T_{[\beta(n)]} - T_{[\alpha(n)]})}} > \frac{1}{3} > 0.
\end{aligned}$$

Hence, the result. □

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