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SOLUTION SET FOR IMPULSIVE FRACTIONAL DIFFERENTIAL INCLUSIONS

MOUSTAFA BEDDANI[1]

ABSTRACT. This paper aims to an initial value problem for an impulsive fractional differential inclusion with the Riemann-Liouville fractional derivative. We apply Covitz and Nadler theorem concerning the study of the fixed point for multivalued maps to obtain the existence results for the given problems. We also obtain some topological properties about the solution set.

1. INTRODUCTION

We study the existence of solutions and determine certain topological properties of the solutions set for the following impulsive fractional differential inclusion:

$$\begin{cases} {}^{RL}D^{\alpha}y(t) \in F(t, y(t)) \quad \text{a.e. } t \in J = (0, T],\ t \neq t_k, \\ \lim\limits_{t \longrightarrow 0^+} t^{1-\alpha} y(t) = c, \\ \Delta^* y\,|_{t_k} = I_k(y(t_k^-)), \end{cases} \tag{1.1}$$

where $k = 1, \dots, m$, $0 < \alpha \leq 1$, ${}^{RL}D^{\alpha}$ is the Riemann-Liouville fractional derivative, $F : J \times \mathbb{R} \to \{X \subset \mathbb{R} : X \neq \emptyset\}$ is a given multivalued function, $c \in \mathbb{R}$, $I_k : \mathbb{R} \to \mathbb{R}$ are continuous functions, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$ and $\Delta^* y\,|_{t_k} = y^*(t_k^+) - y(t_k^-)$, where $y^*(t_k^+) = \lim_{t \to t_k^+} (t - t_k)^{1-\alpha} y(t)$ and $y(t_k^-) = \lim_{t \longrightarrow t_k^-} y(t)$.

More precisely, we present an overall existence result for (1.1) by using Covitz and Nadler fixed-point theorem for multivalued maps. Afterwards, we prove the compactness and acyclicity of the solution set for this problem.

Key words and phrases. Impulsive fractional differential inclusions, Riemann-Liouville fractional derivative, fixed point, solution set, compactness, contractible.

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Since the 1960s, the subject of functional differential inclusions and impulsive ordinary differential inclusions with various conditions have been investigated by many authors [1, 2, 8, 11, 14, 24–26, 32, 35, 39, 40], and has several applications in different areas as engineering, electrical, networks electrochemistry, fluid flow, etc. For more details we refer the reader to see the following references [4, 13, 20, 21, 30, 31, 34, 36].

The topological and geometric properties of the solution set for differential inclusions are examined by many mathematicians (see for example [3, 10, 16, 17, 22, 37]) where the concept of quasi-concavity is extended to multifunctions contractibility, absolutely retract, acyclicity, R_δ-sets properties are given.

This work is structured as follows: in the second section, we recall some definitions and properties that are needed throughout this article. Afterwards, in the third section, we show that the solution sets is contractible to a point. Finally, we give an example which illustrate the principle result of this paper.

2. Preliminary Results

The object of this section is to recall some basic definitions and useful notations in multivalued analysis. Let $C([a,b],\mathbb{R})$ be the Banach space of all continuous functions h from the interval $[a,b]$ into $\mathbb{R}$ with the norm

$$\|h\|_\infty = \sup_{t\in[a,b]} |h(t)|,$$

and $L^1([a,b],\mathbb{R})$ the Banach space of all Lebesgue integrable functions h from the interval $[a,b]$ into $\mathbb{R}$ with the norm

$$\|h\|_{L^1} = \int_a^b |h(t)|\, dt.$$

For a given metric space E, we denote:

- $\mathcal{P}(E) = \{X \subset E : X \neq \emptyset\}$;
- $\mathcal{P}_{cl}(E) = \{X \in \mathcal{P}(E) : X \text{ is closed}\}$;
- $\mathcal{P}_b(E) = \{X \in \mathcal{P}(E) : X \text{ is bounded}\}$;
- $\mathcal{P}_{cp}(E) = \{X \in \mathcal{P}(E) : X \ \text{ is compact}\}$;
- $\mathcal{P}_c(E) = \{X \in \mathcal{P}(E) : X \ \text{ is convex}\}$;
- $\mathcal{P}_{cp,c}(E) = \mathcal{P}_{cp}(E) \cap \mathcal{P}_c(E)$.

If X and Y are two subsets of the metric space E, and x (resp. y) is a point in X (resp. Y), we denote:

$$d(x,Y) = \inf_{y\in Y} d(x,y) \quad \text{and} \quad d(X,y) = \inf_{x\in X} d(x,y).$$

Recall that, the Hausdorff pseudo-metric distance H_d on $\mathcal{P}(E)$ defined by

$$H_d(X,Y) := \max\left\{\sup_{x\in X} d(x,Y), \sup_{y\in Y} d(X,y)\right\}.$$

Form the previous definition, it is well-known that $(\mathcal{P}_{cl,b}(E), H_d)$ (resp. $(\mathcal{P}_{cl}(E), H_d)$) is a metric space (resp. is a generalized metric space).

Definition 2.1. Let $M :\to \mathcal{P}(E)$ be a multivalued map.

(a) We say that M is γ-Lipschitz if there exists a positive real number γ such that

$$H_d(M(x), M(y)) \leq \gamma d(x, y), \quad \text{for all } x, y \in E.$$

(b) The map M is called a contraction if it is γ-Lipschitz for some $0 < \gamma < 1$.

Notice that, if M is γ-Lipschitz on a Banach space E, then for every real number γ' greater than γ, $M(x) \subset M(y) + \gamma' d(x, y)B(0, 1)$, where $B(0, 1)$ is the unit ball of E.

Definition 2.2. Let $G : X \to \mathcal{P}_{cl}(Y)$ be a multivalued map, where X and Y are two metric spaces.

(a) We say that G is closed valued (resp. convex valued) if $G(x)$ is closed (resp. convex) for all $x \in E$.
(b) Every single-valued map $g : X \to Y$ is called a selection of G. We write $g \subset G$ whenever $g(x) \in G(x)$ for all $x \in X$. $G : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$, we define the set of selections of G by

$$S_{G,x} = \{v \in L^1(J, \mathbb{R}) : v(t) \in G(t, x(t)) \text{ a.e. } t \in J\}.$$

Definition 2.3. A multivalued map $G : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is called L^1-Carathéodory if the following are satisfied:

(a) the function $G(\cdot, x)$ is measurable for each $x \in \mathbb{R}$;
(b) the function $G(t, \cdot)$ is upper semi-continuous for almost all $t \in J$;
(c) for every positive real r, there exists a function $f_r \in L^1(J, \mathbb{R}_+)$ such that

$$\|G(t, x)\| = \sup\{|v| : v \in G(t, x)\} \leq f_r(t) \quad \text{a.e. } t \in J \text{ and for all } x \in [-r, r].$$

For more details about multivalued analysis, we refer the reader to see [5–7, 12, 23, 27–29].

Below we present the definition of contractible spaces, and for details about this type of spaces, we recommend [5, 10, 15, 33].

Definition 2.4. A contractible subset of a Banach space X is a nonempty subset A of X for which there exists a continuous homotopy $\psi : A \times [0, 1] \to A$ and $a_0 \in A$ such that for all $a \in A$, $\psi(a, 0) = a$ and $\psi(a, 1) = a_0$.

We give now two basic definitions used frequently in fractional computation theory.

Definition 2.5 ([18, 19])**.** Let $h \in L^1([a, b], \mathbb{R}_+)$. The fractional order integral of h is given by

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\, ds.$$

Definition 2.6 ([19])**.** Let b be a positive real number and h a real function defined on the interval $[0, b]$. The Riemann-Liouville fractional derivative of order $\alpha \in \mathbb{R}_+$ of h is defined as follows:

$$^{RL}D^{\alpha}_{0^+}h(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\left(\int_0^t (t-s)^{n-\alpha-1}h(s)\,ds\right).$$

3. Main Results

Let

$$PC_*([0,T],\mathbb{R}) = \Big\{ y : [0,T] \longrightarrow \mathbb{R} : y_k \in C(t_k, t_{k+1}],\ k = 0,\dots,m, \text{ and there exist } y(t_k^-),\ y_*(t_k^+),\ k = 1,\dots,m, \text{ with } y(t_k) = y(t_k^-)\Big\}.$$

It is known that this set is Banach space with the norm

$$\|y\|_{PC_*} = \max_{k=1,\dots m} \|y_k\|_*,$$

where y_k is the restriction of y to the interval $J_k = (t_k, t_{k+1}]$ for every $k = 0,\dots,m$, and

$$\|y_k\|_* = \sup_{t\in[t_k,t_{k+1}]} |(t-t_k)^{1-\alpha}y_k(t)|, \quad \text{for every } k = 1,\dots,m.$$

When $A \subset PC_*([0,T],\mathbb{R})$, we define $\mathcal{A}_\alpha$ by

$$\mathcal{A}_\alpha = \{y_\alpha : y \in A\},$$

where

$$y_\alpha(t) = \begin{cases} (t-t_k)^{1-\alpha}y(t), & \text{if } t \in (t_k, t_k+1], \\ \lim\limits_{t\to t_k}(t-t_k)^{1-\alpha}y(t), & \text{if } t = t_k. \end{cases}$$

Theorem 3.1. *Let A be a bounded subset in the Banach space $PC_*([0,T],\mathbb{R})$ such that $\mathcal{A}_\alpha$ is equicontinuous on $PC([0,T],\mathbb{R})$. Then the set A is relatively compact in $PC_*([0,T],\mathbb{R})$.*

Proof. We know that, if $\{y_n\}_{n=1}^\infty \subset A$, then $\{(y_\alpha)_n\}_{n=1}^\infty \subset PC([0,T],\mathbb{R})$. From Arzela-Ascoli theorem, the set $K_0 = \{(y_\alpha)_n : n \in \mathbb{N}^*\}$ is relatively compact in the space $PC([0,T],\mathbb{R})$. So, we can find a subsequence of $(y_\alpha)_{n\in\mathbb{N}}$ still denoted by the same indices such that $\lim y_n \in (PC([0,T],\mathbb{R}), \|\cdot\|_{PC})$. Put $y = \lim y_n$. We have

$$\|(y_\alpha)_n - y\|_* = \sup_{t\in[t_k,t_{k+1}]} (t-t_k)^{1-\alpha}|y_\alpha)_n(t) - y(t)| \to 0, \quad n \to +\infty.$$

So, $y_n \to y$, $n \to +\infty$, on $PC_*([0,T],\mathbb{R})$. □

To explain our results, we need the following lemmas.

Lemma 3.1 ([38])**.** *Let α be a positive real number and $n = 1+[\alpha]$. Then the following differential equation ${}^{RL}D^{\alpha}_{a^+}h(t) = 0$, has solutions of the form $h(t) = \sum_{k=1}^{n} c_k(t-a)^{\alpha-k}$ for some real numbers $c_1, c_2, \ldots, c_n$.*

Lemma 3.2 ([38])**.** *Let α be a positive real number and $n = 1+[\alpha]$. Then there exist some real numbers $c_1, c_2, \ldots, c_n$ such that*

$$I^{\alpha RL}D^{\alpha}_{a^+}h(t) = h(t) + \sum_{k=1}^{n} c_k(t-a)^{\alpha-k}.$$

Lemma 3.3. *Let α be a real number in the interval $(0,1)$ and h a continuous function. If y is a solution of following problem*

$$ {}^{RL}D^{\alpha}y(t) = h(t), \quad \text{for all } t \in J \text{ and } t \neq t_m \text{ for all } k = 1, \ldots, m, \tag{3.1}$$

$$\Delta^{*} y\,|_{t_k} = I_k(y(t_k^-)), \quad \text{for all } k = 1, \ldots, m, \tag{3.2}$$

$$\lim_{t\to 0} t^{1-\alpha} y(t) = c. \tag{3.3}$$

Then

$$y(t) = \begin{cases} t^{\alpha-1}c + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\,ds, & \text{if } 0 \le t \le t_1, \\[2ex] (t-t_1)^{\alpha-1}t_1^{\alpha-1}c + \dfrac{(t-t_1)^{\alpha-1}}{\Gamma(\alpha)}\displaystyle\int_0^{t_1}(t_1-s)^{\alpha-1}h(s)\,ds \\ +\dfrac{(t-t_1)^{\alpha-1}}{\Gamma(\alpha)} I_1(y(t_1^-)) + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_1}^t (t-s)^{\alpha-1}h(s)\,ds, & \text{if } t_1 < t \le t_2, \\ (t-t_k)^{\alpha-1}\displaystyle\prod_{i=1}^{k}(t_i - t_{i-1})^{\alpha-1}c \\ +\dfrac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\Bigg[\displaystyle\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}h(s)\,ds + \\ \displaystyle\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}h(s)\,ds\Bigg] \\ +\dfrac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\Bigg[I_k(y(t_k^-)) \\ +\displaystyle\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}I_i(y(t_i^-))\Bigg] \\ +\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_k}^t (t-s)^{\alpha-1}h(s)\,ds, & \text{if } t_k < t \le t_{k+1}, \\ & 2 \le k \le m. \end{cases} \tag{3.4}$$

Proof. Suppose that y satisfies (3.1)–(3.3). It is clear when $0 \le t \le t_1$, we have

$${}^{RL}D^{\alpha}y(t) = h(t).$$

From the previous lemma, we get

$$y(t) = t^{\alpha-1}c_1 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\,ds.$$

Hence, $c_1 = c$. Thus,

$$y(t) = t^{\alpha-1}c + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\,ds.$$

When $t_1 < t \le t_2$, we can obtain, by using the previous lemma, that

$$\begin{aligned}
y(t) =&(t-t_1)^{\alpha-1}y^*(t_1^+) + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}h(s)\,ds\\
=&(t-t_1)^{\alpha-1}\left(I_1(y(t_1^-)+y(t_1^-)\right) + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}h(s)\,ds\\
=&(t-t_1)^{\alpha-1}t_1^{\alpha-1}c + \frac{(t-t_1)^{\alpha-1}}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}h(s)\,ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}h(s)\,ds + (t-t_1)^{\alpha-1}I_1(y(t_1^-)).
\end{aligned}$$

If $t_2 < t \le t_3$, Lemma 3.2 implies

$$\begin{aligned}
y(t) =&(t-t_2)^{\alpha-1}y^*(t_2^+) + \frac{1}{\Gamma(\alpha)}\int_{t_2}^t (t-s)^{\alpha-1}h(s)\,ds,\\
y(t) =&(t-t_2)^{\alpha-1}[y(t_2^-)+I_2(y(t_2^-))] + \frac{1}{\Gamma(\alpha)}\int_{t_2}^t (t-s)^{\alpha-1}h(s)\,ds\\
=&(t-t_2)^{\alpha-1}(t_2-t_1)^{\alpha-1}t_1^{\alpha-1} + \frac{(t-t_2)^{\alpha-1}(t_2-t_1)^{\alpha-1}}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}h(s)\,ds\\
&+\frac{(t-t_2)^{\alpha-1}}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}h(t)\,ds + \frac{1}{\Gamma(\alpha)}\int_{t_2}^t (t-s)^{\alpha-1}h(s)\,ds\\
&+(t-t_2)^{\alpha-1}\left[(t_2-t_1)^{\alpha-1}I_1(y(t_1^-)) + I_2(y_2(t_2^-))\right].
\end{aligned}$$

Finally, when $t_k < t \le t_{k+1}$, we obtain (3.4), by using Lemma 3.2. □

Definition 3.1. Let y be a function in PC_*. We say that y is a mild solution of the problem (1.1) if there exists $v \in L^1(J,\mathbb{R})$ such that $v(t) \in F(t,y(t))$ a.e. on $J \setminus \{t_1,\dots,t_m\}$, $\lim\limits_{t\longrightarrow 0} t^{1-\alpha}y(t) = c$, $\Delta^* y\,|_{t_k} = I_k(t_k^-)$ for all $k = 1,\dots,m$, and ${}^{RL}D^\alpha y(t) = v(t)$ for all $t \in J \setminus \{t_1,\dots,t_m\}$.

Definition 3.2 ([6,23])**.** A single-valued map $f : [0,a]\times X \to Y$ be a single-valued map is said to be measurable locally Lipschitz (mLL) if

(1) $f(\cdot,x)$ is measurable for every $x \in X$, and
(2) for each $x \in X$, there exists a neighborhood V_x of x and an integrable function $L_x : [0,a] \to [0,\infty)$ such that

$$d'(f(t,x_1),f(t,x_2)) \le L_x(t)d(x_1,x_2) \quad \text{a.e. } t \in [0,a] \text{ and } x_1,x_2 \in V_x.$$

Definition 3.3 ([6, 23])**.** A mapping $F : [0, a] \times X \to \mathcal{P}(Y)$ is mLL-selectionable provided there exists a measurable locally-Lipchitzian map $f : [0, a] \times X \to Y$ such that $f \in F$.

Lemma 3.4 ([2,11])**.** *If $N : X \to \mathcal{P}_{cl}(X)$ is a contraction on a complete metric space X, then the fixed point set of N is nonempty.*

Theorem 3.2. *Let $F : J \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ be a mLL-selectionable multivalued map such that the following properties are satisfied:*

(H_1) *there exist $\overline{a}$ and $\overline{b}$ in $\mathbb{R}_+$ such that for every $x \in \mathbb{R}$, we have*

$$\|F(t,x)\|_{\mathcal{P}} \le \overline{a}|x| + \overline{b} \quad a.e.\ t \in J;$$

(H_2) *there exist a_k and $b_k \in \mathbb{R}_+$ such that*

$$|I_k(x)| \le a_k|x| + b_k, \quad for\ x \in \mathbb{R};$$

(H_3) *there exist $p \in C([0,T], \mathbb{R}_+)$ such that*

$$H_d(F(t,z_1), F(t,z_2)) \le p(t)\|z_1 - z_2\|, \quad for\ all\ z_1, z_2 \in \mathbb{R},$$

and $d(0, F(t,0)) \le p(t)$, $t \in J$;

(H_4) *there exist a real number $L \in \mathbb{R}_+$ such that*

$$|I_k(z_1) - I_k(z_2)| \le L\|z_1 - z_2\|, \quad for\ all\ z_1, z_2 \in \mathbb{R}.$$

If

$$\frac{T^{\alpha}\|p\|_{\infty}\Gamma(\alpha)(1 + mT^{\alpha-1})}{\Gamma(2\alpha)} + \frac{mT_0^{\alpha-1}L}{\Gamma(\alpha)} < 1,$$

then (1.1) has a solution. In addition, if $F : J \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is a Carathéodory multivalued map with compact convex values, then the solution set is contractible and compact, and hence it is acyclic.

Proof. **Step 1**. *Existence of solutions.* Let $P : PC_* \longrightarrow \mathcal{P}(PC_*)$ the operator defined by

$$\begin{aligned} P(y) = \Bigg\{ & h \in PC_* : h(t) = (t - t_k)^{\alpha-1} \prod_{i=1}^{k} (t_i - t_{i-1})^{\alpha-1} c \\ & + \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)} \left[\int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} h(s)\, ds \right. \\ & \left. + \sum_{i=1}^{k-1} \prod_{j=1}^{k-i} (t_{k-j+1} - t_{k-j})^{\alpha-1} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} v(s)\, ds \right] + \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)} \Big[I_k(y(t_k^-)) \\ & \left. + \sum_{i=1}^{k-1} \prod_{j=1}^{k-i} (t_{k-j+1} - t_{k-j})^{\alpha-1} I_i(y(t_i^-)) \right] + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} v(s)\, ds \Bigg\}, \end{aligned}$$

where $v \in S_{F,y} = \{v \in L^1(J, \mathbb{R}) : v(t) \in F(t, y(t))$ a.e. $t \in J\}$. Now, we show that the operator F satisfies the hypotheses (H_1), (H_2) and (H_3) of Lemma 3.4. To prove that $P(y) \in \mathcal{P}_{cl}(PC_*([0,T],\mathbb{R}))$ for all $y \in PC_*([0,T],\mathbb{R})$, let $\{u_n\}_{n=0}^{\infty} \in F(y)$ be a sequence converges to u on the space $PC_*([0,T],\mathbb{R})$. Then $u \in PC_*([0,T],\mathbb{R})$ and there exists $v_n \in S_{F,y}$ such that, for each $t \in (0,T]$

$$\begin{aligned}
u_n(t) =&(t-t_k)^{\alpha-1}\prod_{i=1}^{k}(t_i-t_{i-1})^{\alpha-1}c\\
&+\frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}v_n(s)\,ds\right.\\
&\left.+\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}v_n(s)\,ds\right]\\
&+\frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[I_k(y(t_k^-))+\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}I_i(y(t_i^-))\right]\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}v_n(s)\,ds.
\end{aligned}$$

We use the fact that F has compact value and by passing (if necessary) onto a subsequence to obtain that v_n converges to v in $L^1([0,T],\mathbb{R})$, we get $v \in S_{F,y}$, and for each $t \in (0,T]$, we have

$$\begin{aligned}
u_n(t) \to u(t) =&(t-t_k)^{\alpha-1}\prod_{i=1}^{k}(t_i-t_{i-1})^{\alpha-1}c\\
&+\frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}v(s)\,ds\right.\\
&\left.+\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}v(s)\,ds\right]\\
&+\frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[I_k(y(t_k^-))+\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}I_i(y(t_i^-))\right]\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}v(s)\,ds.
\end{aligned}$$

Hence, $u \in F(y)$.

Now, we will prove the existence of a real number $\delta < 1$ for which

$$H_d(F(x),F(y)) \le \delta\|x-y\|_{PC_*}, \quad \text{for all } x,y \in PC_*([0,T],\mathbb{R})$$

For each $x,y \in PC_*([0,T],\mathbb{R})$ and $h_1 \in P(x)$, we can find $v_1(t) \in F(t,x(t))$ such that, for all $t \in (0,T]$,

$$h_1(t) =(t-t_k)^{\alpha-1}\prod_{i=1}^{k}(t_i-t_{i-1})^{\alpha-1}c$$

$$
\begin{aligned}
&+\frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}v_1(s)\,ds\right.\\
&\left.+\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}v_1(s)\,ds\right]\\
&+\frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[I_k(x(t_k^-))+\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}I_i(x(t_i^-))\right]\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}v_1(s)\,ds.
\end{aligned}
$$

From (H_3), we have

$$H_d(F(t,x(t)),F(t,y(t)))\le p(t)|x(t)-y(t)|.$$

Consequently, exists $w\in F(t,y(t))$ such that

$$|v_1(t)-w|\le p(t)|x(t)-y(t)|,\quad t\in(0,T].$$

Consider the map U from $(0,T]$ into $\mathcal{P}(\mathbb{R})$ defined by

$$U(t):=\{w(t)\in\mathbb{R}:|v_1(t)-w|\le p(t)|x(t)-y(t)|\}.$$

From [9, Proposition III. 4], the intersection between $U(t)$ and $F(t,y(t))$ is a measurable set. Therefore, we can find a measurable selection $v_2(\cdot)$ for $U(\cdot)\cap F(\cdot,y(\cdot))$. So, $v_2(t)\in F(t,y(t))$ and

$$|v_1(t)-v_2(t)|\le p(t)|x(t)-y(t)|,\quad \text{for all } 0<t\le T.$$

For every $0<t\le T$, we define

$$
\begin{aligned}
h_1(t)=&(t-t_k)^{\alpha-1}\prod_{i=1}^{k}(t_i-t_{i-1})^{\alpha-1}c\\
&+\frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}v_2(s)\,ds\right.\\
&\left.+\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}v_2(s)\,ds\right]\\
&+\frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[I_k(y(t_k^-))+\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}I_i(y(t_i^-))\right]\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}v_2(s)\,ds.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&|(t-t_k)^{1-\alpha}h_1(t)-(t-t_k)^{1-\alpha}h_2(t)|\\
\le&\frac{1}{\Gamma(\alpha)}\left[\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|v_2(s)-v_1(s)|\,ds\right.
\end{aligned}
$$

$$+\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}|v_2(s)-v_1(s)|\,ds\Bigg]$$
$$+\frac{1}{\Gamma(\alpha)}\left[|I_k(y(t_k^-))-I_k(x(t_k^-))|+\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}|I_i(y(t_i^-))-I_i(x(t_i^-))|\right]$$
$$+\frac{(t-t_k)^{1-\alpha}}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|v_2(s)-v_1(s)|\,ds.$$

Hence,

$$\|h_1-h_2\|_{PC_*}\le\left[\frac{T^{\alpha}\|p\|_{\infty}\Gamma(\alpha)(1+mT^{\alpha-1})}{\Gamma(2\alpha)}+\frac{mT_0^{\alpha-1}L}{\Gamma(\alpha)}\right]\|x-y\|_{PC_*},$$

where $T_0=\min_{i=1,\ldots,m}(t_{i+1}-t_i)$. Interchange x by y in the previous computation, we obtain

$$H_d(P(x),P(y))\le\delta\|x-y\|_{PC_*},$$

where $\delta=\frac{T^{\alpha}\|p\|_{\infty}\Gamma(\alpha)(1+mT^{\alpha-1})}{\Gamma(2\alpha)}+\frac{mT_0^{\alpha-1}L}{\Gamma(\alpha)}<1$. Hence, P is a contraction, and from Lemma 3.4, it has a fixed point y considered as a solution of (1.1).

Step 2. *Structure of the solutions set.* Let

$$S_F(c)=\{y\in PC_*([0,T],\mathbb{R}):y\text{ is solution of (1.1)}\}.$$

We will prove that $S_F(c)$ is compact in $PC_*([0,T],\mathbb{R})$. Let $\{y_n\}_{n\in\mathbb{N}}\in S_F(c)$, then there exists $v_n\in S_{F,y_n}$ and $t\in J$ such that

$$\begin{aligned}y_n(t)=&(t-t_k)^{\alpha-1}\prod_{i=1}^{k}(t_i-t_{i-1})^{\alpha-1}c\\&+\frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\Bigg[\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}v_n(s)\,ds\\&+\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}v_n(s)\,ds\Bigg]\\&+\frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[I_k(y(t_k^-))+\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}I_i(y(t_i^-))\right]\\&+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}v_n(s)\,ds.\end{aligned}$$

From (H_1) and (H_2), there exists $M_1>0$ such that $\|y_n\||_{PC_*}\le M_1$ for every $n\ge1$, and the set $\{y_1,y_2,\ldots,y_n,\ldots\}$ is equicontinuous in $PC_*([0,T],\mathbb{R})$. By using Theorem 3.1, we can find a subsequence of (y_n) (still denoted (y_n)) converges to y in the space $PC_*([0,T],\mathbb{R}$. Now, we will prove the existence of $v(\cdot)\in F(\cdot,y(\cdot))$ and an element

$t \in J$ such that

$$\begin{aligned} y(t) =& (t-t_k)^{\alpha-1}\prod_{i=1}^{k}(t_i-t_{i-1})^{\alpha-1}c \\ &+ \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}v(s)\,ds\right. \\ &+ \left.\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}v(s)\,ds\right] \\ &+ \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[I_k(y(t_k^-))+\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}I_i(y(t_i^-))\right] \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}v(s)\,ds. \end{aligned}$$

Use the fact that $F(\cdot,\cdot)$ is upper semi-continuous, we can show that for every positive real number ε, there exists a positive integer n_0 such that

$$v_n(t) \in F(t, y_n(t)) \subset F(t, y_n(t)) + \varepsilon B(0,1) \quad \text{a.e. } t \in J, \text{ for every } n \geq n_0.$$

Using the compactness of $F(\cdot,\cdot)$ we get the subsequence $v_{nm}(\cdot)$ such that

$$v_{nm}(\cdot) \to v(\cdot) \quad \text{and} \quad v(t) \in F(t, y(t)) \quad \text{a.e. } t \in J.$$

From (H_1), we have

$$v_{nm}(\cdot) \leq \overline{a}t^{\alpha-1}M_1 + \overline{b} \quad \text{a.e. } t \in J.$$

Using Lebesgue's dominated convergence theorem, we obtain that $v \in L^1(J,\mathbb{R})$, so $v \in S_{F,y}$. Therefore, for all $t \in J$

$$\begin{aligned} y(t) =& (t-t_k)^{\alpha-1}\prod_{i=1}^{k}(t_i-t_{i-1})^{\alpha-1}c \\ &+ \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}v(s)\,ds\right. \\ &+ \left.\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}v(s)\,ds\right] \\ &+ \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[I_k(y(t_k^-))+\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}I_i(y(t_i^-))\right] \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}v(s)\,ds. \end{aligned}$$

Then $S_F(c)$ is compact.

Let $f \in F$ be a function which is mLL. Consider the following single-valued problem

$$\begin{cases} {}^{RL}D^{\alpha}y(t) = f(t, y(t)) \quad \text{a.e. } t \in J = (0, T],\ t \neq t_k, \\ \lim\limits_{t \longrightarrow 0^+} t^{1-\alpha}y(t) = c, \\ \Delta^* y\,|_{t_k} = I_k(y(t_k^-)). \end{cases} \tag{3.5}$$

Using Banach fixed point theorem, we can prove easily that the problem (3.5) has a unique solution $\overline{x}$. Consider the homotopy $h : S_F(c) \times [0, 1] \to S_F(c)$ defined by

$$h(y, \lambda) := \begin{cases} y, & \text{for } \lambda = 1 \text{ and } y \in S_F(c), \\ \overline{x}, & \text{for } \lambda = 0. \end{cases}$$

Note that

$$h(y, \lambda)(t) = \begin{cases} y(t), & \text{for } 0 < t \leq \lambda T, \\ \overline{x}(t), & \text{for } \lambda T < t \leq T. \end{cases}$$

We will show that h is a continuous homotopy. Let $(y_n, \lambda_n) \in S_F(c) \times [0, 1]$ such that $(y_n, \lambda_n) \to (y, \lambda)$. We shall show that $h(y_n, \lambda_n) \to h(y, \lambda)$. We have

$$h(y_n, \lambda_n)(t) = \begin{cases} y_n(t), & \text{for } t \in (0, \lambda_n T], \\ \overline{x}(t), & \text{for } (\lambda_n T, T]. \end{cases}$$

(a) If $\lim_{n\to\infty} \lambda_n = 0$, then $H(y, 0)(t) = \overline{x}(t)$, for $t \in (0, T]$, hence

$$\|H(y_n, \lambda_n) - H(y, \lambda)\|_{PC_*} \leq \|y_n - y\|_{PC_*} + \|y_n - \overline{x}\|_{[0,\lambda_n T]},$$

which tends to 0 as $n \to +\infty$. The case $\lim_{n\to\infty} \lambda_n = 1$ can be processed in the same way.

(b) If $\lambda_n \neq 0$ and $0 < \lim_{n\longrightarrow\infty} \lambda_n = \lambda < 1$. We distinguish two sub-cases.

(i) Since $y_n \in S_F(c)$, there exists $v_n \in S_{F,y_n}$ such that for $t \in (0, \lambda_n T]$

$$\begin{aligned} y_n(t) =& (t - t_k)^{\alpha-1} \prod_{i=1}^{k} (t_i - t_{i-1})^{\alpha-1} c \\ &+ \frac{(t - t_k)^{\alpha-1}}{\Gamma(\alpha)} \left[\int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} v_n(s)\, ds \right. \\ &\left. + \sum_{i=1}^{k-1} \prod_{j=1}^{k-i} (t_{k-j+1} - t_{k-j})^{\alpha-1} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} v_n(s)\, ds \right] \\ &+ \frac{(t - t_k)^{\alpha-1}}{\Gamma(\alpha)} \left[I_k(y_n(t_k^-)) + \sum_{i=1}^{k-1} \prod_{j=1}^{k-i} (t_{k-j+1} - t_{k-j})^{\alpha-1} I_i(y_n(t_i^-)) \right] \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} v_n(s)\, ds. \end{aligned}$$

Since $y_n \to y$ as $n \to \infty$, there exists a positive real number $r > 0$ such that $\|y_n\|_{PC_*} \leq r$, and as the function $F(\cdot, \cdot)$ is upper semi-continuous, for every positive real number ε, there exists a positive integer n_0 such that for every $n \geq n_0$, we have

$$v_n(t) \in F(t, y_n(t)) \subset F(t, y_n(t)) + \varepsilon B(0,1) \quad \text{a.e. } t \in J.$$

Using the fact that $F(\cdot,\cdot)$ has compact values, we can obtain the subsequence $v_{nm}(\cdot)$ such that

$$v_{nm}(\cdot) \to v(\cdot) \quad \text{and} \quad v(t) \in F(t, y(t)) \quad \text{a.e. } t \in J.$$

From (H_1), we have

$$v_{nm}(\cdot) \leq \overline{a} t^{\alpha-1} M_1 + \overline{b} \quad \text{a.e. } t \in J.$$

Using Lebesgue's dominated convergence theorem, we get $v \in L^1(J, \mathbb{R})$, so $v \in S_{F,y}$. Since I_k are continuous functions, then, for all $t \in J$, we get

$$\begin{aligned} y(t) =& (t-t_k)^{\alpha-1} \prod_{i=1}^{k} (t_i - t_{i-1})^{\alpha-1} c \\ &+ \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)} \left[\int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} v(s)\, ds \right. \\ &+ \left. \sum_{i=1}^{k-1} \prod_{j=1}^{k-i} (t_{k-j+1} - t_{k-j})^{\alpha-1} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} v(s)\, ds \right] \\ &+ \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)} \left[I_k(y(t_k^-)) + \sum_{i=1}^{k-1} \prod_{j=1}^{k-i} (t_{k-j+1} - t_{k-j})^{\alpha-1} I_i(y(t_i^-)) \right] \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} v(s)\, ds. \end{aligned}$$

(ii) Since $t \in (\lambda_n T, T]$, then

$$h(y_n, \lambda_n)(t) = h(y, \lambda)(t) = \overline{x}.$$

Thus,

$$\|h(y_n, \lambda_n) - h(y, \lambda)\|_{PC_*} \to 0 \quad n \to \infty.$$

Consequently, h is continuous, and hence, $S_F(c)$ is contractible to the point $\overline{x}$. Therefore, $S_F(c)$ is an acyclic space. □

Example 3.1 (An application). Consider

$$(3.6) \qquad \begin{cases} {}^{RL}D^{\frac{1}{2}} y(t) \in F(t, y(t)) \quad \text{a.e. } t \in (0,1],\ t \neq \dfrac{1}{2}, \\ \lim\limits_{t \to 0^+} t^{\frac{1}{2}} y(t) = \dfrac{1}{4}, \\ \Delta^* y \,|_{t=\frac{1}{2}} = \dfrac{1}{9} \left| \sin\left(y\left(\dfrac{1}{2}^- \right) \right) \right| + 1, \end{cases}$$

where $T = 1$, $m = 1$, $t_1 = \frac{1}{2}$,

$$F(t,x) = \left[0,\ \frac{1}{9}\sin x + \frac{|x|}{t+9} + \frac{1}{9}\right]$$

and

$$I_1(u) = \frac{1}{9}|\sin(u)| + 1, \quad \text{for } u \in \mathbb{R}.$$

Clearly,

$$\sup_{v\in F(t,x)} |v| \leq \frac{1}{9} + \frac{1}{9}|\sin x| + \frac{|x|}{t+9},$$

$$H_d(F(t,x), F(t,y)) \leq \left[\frac{1}{t+9} + \frac{1}{9}\right] |x-y|$$

and

$$|I_1(u)| \leq \frac{1}{9}|\sin(u)| + 1, \quad |I_1(u) - I_1(v)| \leq \frac{1}{9}|u-v|.$$

Let $p(t) = \frac{1}{9} + \frac{1}{t+9}$. Then $\|p\|_\infty = \frac{2}{9}$ and

$$\frac{T^\alpha \|p\|_\infty \Gamma(\alpha)(1 + mT^{\alpha-1})}{\Gamma(2\alpha)} + \frac{mT_0^{\alpha-1}L}{\Gamma(\alpha)} \simeq 0,73965 < 1.$$

Theorem 3.2 confirms that (3.6) has at least one solution. In addition, it is clear that F is a mLL-selectionable multivalued map (i.e., the function $f(t,u) = \frac{1}{9}\sin u + \frac{|u|}{t+9} + \frac{1}{9}$ is measurable, locally-Lipchitzian) with compact and convex values. Consequently, the solution set is contractible and compact, and hence it is acyclic.

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[1]Department of Mathematics,
Djillali Liabes University of Sidi Bel-Abbès,
P.O. Box 89, Sidi Bel-Abbès 22000 , Algeria

Email address: beddani2004@yahoo.fr