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SOME RESULTS ON POST-WIDDER OPERATORS PRESERVING TEST FUNCTION x^r

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Abstract. In the present paper, we consider Post-Widder operators and its modified form which preserve the test function x^r, $r \in \mathbb{N}$. We estimate direct results in terms of modulus of continuity for the modified operators. Also, some estimates for polynomially bounded functions and linear combinations are considered. Our estimates improve in some sense the previous results for the original Post-Widder operators.

1. Introduction

The Post-Widder operators for $n \in \mathbb{N}$ and $x > 0$ considered by Widder [18] is defined by

$$P_n(f,x) = \frac{1}{n!}\left(\frac{n}{x}\right)^{n+1}\int_0^\infty t^n e^{-\frac{nt}{x}} f(t)\, dt. \tag{1.1}$$

These operators preserve constant functions only. The q analogue of these operators was recently studied by Aydin et al. [3]. Earlier Rathore and Singh [15] (also for related results see [9]) established an asymptotic formula, and deduced inverse and saturation theorems in simultaneous approximation. They considered a parameter p and defined the operators in following way

$$P_n^p(f,x) = \frac{1}{(n+p)!}\left(\frac{n}{x}\right)^{n+p+1}\int_0^\infty t^{n+p} e^{-\frac{nt}{x}} f(t)\, dt.$$

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The special case $p = 0$ provides the operator (1.1) and for $p = -1$, these operators reduce to the operators due to May [13], which preserve the linear functions and considered earlier in the book of Ditizian and Totik [5], by Li and Wang in [12], also in the papers of Draganov and Ivanov [6, 7]. Rempulska and Skorupka in [16] further modified the Post-Widder operators of the form considered by May [13] in order to preserve the test function e_2, where $e_r(x) = x^r$. It was observed in [16] that the modified form provide better approximation results over the form of [13], but in that case the modified form loses the preservation of test function e_1. It may be observed that at a time only two preservations can be made either constant and e_1 or constant and the function e_r, $r > 1$, $r \in \mathbb{N}$. Here we deal with the modification of Post-Widder operators which preserve constants and e_r, $r \in \mathbb{N}$.

Following [16], the r-th order moment $\mu_r^{P_n}(x) = P_n(e_r, x)$, where $e_r(t) = t^r$, $r \in \mathbb{N} \cup \{0\}$ are given by

$$P_n(e_r, x) = \frac{1}{n!}\left(\frac{n}{x}\right)^{n+1}\int_0^\infty t^n e^{-\frac{nt}{x}} t^r\,dt = \frac{1}{n!}\left(\frac{n}{x}\right)^{n+1}\int_0^\infty t^{n+r} e^{-\frac{nt}{x}}\,dt.$$

Put $nt/x = u$ implying $(n/x)dt = du$, thus

$$\begin{aligned}
P_n(e_r, x) =& \frac{1}{n!}\left(\frac{n}{x}\right)^{n+1}\int_0^\infty \left(\frac{x}{n}\right)^{n+r+1} u^{n+r} e^{-u}du \\
=& \frac{1}{n!}\left(\frac{x}{n}\right)^{r}\Gamma(n+r+1) \\
=& \frac{1}{n!}\left(\frac{x}{n}\right)^{r}(n+r)! \\
=& \frac{(n+r)(n+r-1)\cdots(n+1)n!}{n!}\cdot\frac{x^r}{n^r} = \frac{(n+1)_r x^r}{n^r},
\end{aligned}$$

where $(n)_r = n(n+1)(n+2)\cdots(n+r-1)$.

$$\mu_r^{P_n}(x) = \frac{(n+1)_r x^r}{n^r}, \tag{1.2}$$

with $(n)_0 = 1$. If the central moments are denoted by $T_m^{P_n}(x) = P_n((t-x)^m, x)$, then

$$\begin{aligned}
T_1^{P_n}(x) =& \frac{x}{n}, \\
T_2^{P_n}(x) =& \frac{(n+2)x^2}{n^2}.
\end{aligned}$$

From the results given in [5, Ch. 9], denoting the space of all real valued continuous and bounded functions on $(0, \infty)$ by $C_B(0, \infty)$ for every $f \in C_B(0, \infty)$ and $\delta > 0$ there holds

$$|f(t) - f(x)| \leq \omega(f, \delta)\left(\frac{|t-x|}{\delta} + 1\right),$$

where $\omega(f,\cdot)$ is the first order modulus of continuity of f defined by

$$\omega(f,\delta)=\sup_{0<h\leq\delta, x\geq 0}|f(x+h)-f(x)|.$$

Thus,

$$\begin{aligned}|P_n(f,x)-f(x)|\leq&\frac{1}{n!}\left(\frac{n}{x}\right)^{n+1}\int\limits_0^\infty t^n e^{-\frac{nt}{x}}|f(t)-f(x)|dt\\ \leq&\frac{1}{n!}\left(\frac{n}{x}\right)^{n+1}\int\limits_0^\infty t^n\, e^{-\frac{nt}{x}}\,\omega(f,\delta)\left(\frac{|t-x|}{\delta}+1\right)dt\\ \leq&\omega(f,\delta)\left(\frac{\sqrt{T_2^{P_n}(x)}}{\delta}+1\right).\end{aligned}$$

Choosing $\delta=\sqrt{T_2^{P_n}(x)}$, we immediately get

$$|P_n(f,x)-f(x)|\leq 2\omega\left(f,\frac{\sqrt{n+2}}{n}x\right). \tag{1.3}$$

2. Modified Post-Widder Operators Preserving e_r

Let us consider that the Post-Widder operators preserve the test function x^r, $r\in\mathbb{N}$, then we start with the following form

$$\widetilde{P}_{n,r}(f,x)=\frac{1}{n!}\left(\frac{n}{a_{n,r}(x)}\right)^{n+1}\int\limits_0^\infty t^n e^{-\frac{nt}{a_{n,r}(x)}}f(t)dt. \tag{2.1}$$

Here $\widetilde{P}_{n,r}$ preserves constants for any positive function $a_{n,r}(x)$. Then

$$\begin{aligned}x^r=\widetilde{P}_{n,r}(e_r,x)=&\frac{1}{n!}\left(\frac{n}{a_{n,r}(x)}\right)^{n+1}\int\limits_0^\infty t^{n+r}e^{-\frac{nt}{a_{n,r}(x)}}dt\\ =&\frac{(n+r)!}{n!}\left(\frac{a_{n,r}(x)}{n}\right)^r=(n+1)_r\left(\frac{a_{n,r}(x)}{n}\right)^r,\end{aligned}$$

implying

$$a_{n,r}(x)=\frac{nx}{((n+1)_r)^{1/r}}. \tag{2.2}$$

Thus, our modified operators $\widetilde{P}_{n,r}$, $r\in\mathbb{N}$, $x\in(0,\infty)$, take the following form

$$\widetilde{P}_{n,r}(f,x)=\int_0^\infty k_n(x,t)f(t)dt,$$

where

$$k_n(x,t)=\frac{1}{n!}\left[\frac{((n+1)_r)^{1/r}}{x}\right]^{n+1}t^n e^{-\frac{t}{x}((n+1)_r)^{1/r}},$$

with $\widetilde{P}_{n,r}(f,0)=f(0)$ which preserve the function x^r and the constant function.

Following (1.2), the m-th order moments are given by

$$\widetilde{P}_{n,r}(e_m,x)=\frac{(n+1)_m(a_{n,r}(x))^m}{n^m}=\frac{(n+1)_m}{((n+1)_r)^{m/r}}x^m.$$

Lemma 2.1. *The first few images of monomials are given by*

$$\begin{aligned}\widetilde{P}_{n,r}(e_0,x)=&1,\\ \widetilde{P}_{n,r}(e_1,x)=&\frac{(n+1)}{((n+1)_r)^{1/r}}x,\\ \widetilde{P}_{n,r}(e_2,x)=&\frac{(n+1)(n+2)}{((n+1)_r)^{2/r}}x^2.\end{aligned}$$

Remark 2.1. We may point out here that when $r=1$, then the operator $\widetilde{P}_{n,1}$ preserves constant as well as linear functions. When $r=2$ these preserve constant and test function x^2.

Lemma 2.2. *If we denote the central moment as $T_m^{\widetilde{P}_{n,r}}(x)=\widetilde{P}_{n,r}((t-x)^m,x)$, then we have the following recurrence relation:*

$$T_{m+1}^{\widetilde{P}_{n,r}}(x)=\frac{x^2}{((n+1)_r)^{1/r}}[T_m^{\widetilde{P}_{n,r}}(x)]'+\left[\frac{(n+1)}{((n+1)_r)^{1/r}}-1\right]xT_m^{\widetilde{P}_{n,r}}(x)+\frac{mx^2T_{m-1}^{\widetilde{P}_{n,r}}(x)}{((n+1)_r)^{1/r}}.$$

In particular

$$\begin{aligned}T_1^{\widetilde{P}_{n,r}}(x)&=\left[\frac{(n+1)}{((n+1)_r)^{1/r}}-1\right]x,\\ T_2^{\widetilde{P}_{n,r}}(x)&=\left[\frac{(n+1)_2}{((n+1)_r)^{2/r}}-2\frac{(n+1)}{((n+1)_r)^{1/r}}+1\right]x^2,\\ T_4^{\widetilde{P}_{n,r}}(x)=&\left[\frac{(n+1)_4}{((n+1)_r)^{4/r}}-4\frac{(n+1)_3}{((n+1)_r)^{3/r}}+6\frac{(n+1)_2}{((n+1)_r)^{2/r}}-4\frac{(n+1)}{((n+1)_r)^{1/r}}+1\right]x^4,\\ T_6^{\widetilde{P}_{n,r}}(x)=&\left[\frac{(n+1)_6}{((n+1)_r)^{6/r}}-6\frac{(n+1)_5}{((n+1)_r)^{5/r}}+15\frac{(n+1)_4}{((n+1)_r)^{4/r}}\right.\\ &\left.-20\frac{(n+1)_3}{((n+1)_r)^{3/r}}+15\frac{(n+1)_2}{((n+1)_r)^{2/r}}-6\frac{(n+1)}{((n+1)_r)^{1/r}}+1\right]x^6.\end{aligned}$$

For any $r\in\mathbb{N}$ we have $T_m^{\widetilde{P}_{n,r}}(x)=O(n^{-[(m+1)/2]})$.

Proof. The kernel $k_n(x,t)$ of our modified operators $\widetilde{P}_{n,r}$, satisfy the following identity

$$x^2\frac{\partial}{\partial x}k_n(x,t)=[(((n+1)_r)^{1/r})t-(n+1)x]k_n(x,t),$$

we have

$$\begin{aligned}
x^2[T_m^{\widetilde{P}_{n,r}}(x)]' =& \int_0^\infty x^2[k_n(x,t)]'_x(t-x)^m dt - mx^2 T_{m-1}^{\widetilde{P}_{n,r}}(x)\\
=& \int_0^\infty \left[((n+1)_r)^{1/r}t-(n+1)x\right] k_n(x,t)(t-x)^m dt - mx^2 T_{m-1}^{\widetilde{P}_{n,r}}(x)\\
=&((n+1)_r)^{1/r} T_{m+1}^{\widetilde{P}_{n,r}}(x) + \left[((n+1)_r)^{1/r}-(n+1)\right] x T_m^{\widetilde{P}_{n,r}}(x)\\
&- mx^2 T_{m-1}^{\widetilde{P}_{n,r}}(x).
\end{aligned}$$

This completes the proof of recurrence relation. From recurrence relation by induction on m, it is easy to verify that the magnitude of the central moments satisfy $T_m^{\widetilde{P}_{n,r}}(x) = O(n^{-[(m+1)/2]})$ for any $r \in \mathbb{N}$. The other consequences follow from the recurrence relation. $\square$

We have the following observations for our modified operator, corresponding to the estimate (1.3).

Let us suppose that the operators preserve the test functions e_1, e_2, e_3, e_4 respectively, then, by Lemma 2.2 for every continuous and bounded function f on $(0,\infty)$, we have the following estimates:

$$|\widetilde{P}_{n,1}(f,x)-f(x)| \le 2\omega\left(f, \frac{x}{\sqrt{n+1}}\right), \tag{2.3}$$

$$|\widetilde{P}_{n,2}(f,x)-f(x)| \le 2\omega\left(f, \sqrt{2}\sqrt{\left(1-\sqrt{\frac{n+1}{n+2}}\right)}x\right), \tag{2.4}$$

$$\begin{aligned}
&|\widetilde{P}_{n,3}(f,x)-f(x)|\\
\le& 2\omega\left(f, \sqrt{\left(\frac{[(n+1)(n+2)]^{1/3}}{(n+3)^{2/3}} - 2\frac{(n+1)^{2/3}}{[(n+2)(n+3)]^{1/3}}+1\right)}x\right),
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
&|\widetilde{P}_{n,4}(f,x)-f(x)|\\
\le& 2\omega\left(f, \sqrt{\left(\frac{[(n+1)(n+2)]^{1/2}}{[(n+3)(n+4)]^{1/2}} - 2\frac{(n+1)^{3/4}}{[(n+2)(n+3)(n+4)]^{1/4}}+1\right)}x\right).
\end{aligned} \tag{2.6}$$

If we compare the above results, with the estimate (1.3), we find that the error becomes smaller and monotonically decreasing for $n \in \mathbb{N}$, $x \in (0,\infty)$, till the preservation of the test function e_3 as the following is true for second order moments:

$$\begin{aligned}
\frac{\sqrt{n+2}}{n} \ge& \frac{1}{\sqrt{n+1}} \ge \sqrt{2}\sqrt{\left(1-\sqrt{\frac{n+1}{n+2}}\right)}\\
&> \sqrt{\left(\frac{[(n+1)(n+2)]^{1/3}}{(n+3)^{2/3}} - 2\frac{(n+1)^{2/3}}{[(n+2)(n+3)]^{1/3}}+1\right)},
\end{aligned}$$

TABLE 1. Table for approximation

n	$\sqrt{T_2^{\widetilde{P}_{n,1}}(x)}$	$\sqrt{T_2^{\widetilde{P}_{n,2}}(x)}$	$\sqrt{T_2^{\widetilde{P}_{n,3}}(x)}$	$\sqrt{T_2^{\widetilde{P}_{n,4}}(x)}$
5	0.408248x	0.385175x	0.378018x	0.381509x
15	0.250000x	0.244367x	0.242538x	0.243903x
25	0.196116x	0.193356x	0.192451x	0.193203x
35	0.166667x	0.164961x	0.164399x	0.164890x
40	0.156174x	0.154767x	0.154303x	0.154716x
75	0.114708x	0.114147x	0.113961x	0.114135x
10000	0.0099995x	0.00999913x	0.0099990x	0.00999913x

but for higher order preservation of test function for example preservation of test function e_4, one can not have better approximation, which is also shown in the above table. Although the convergence takes places in all the cases for n sufficiently large.

We prove below the direct estimate for the modified operators which preserve e_r. Let π_r denote the space of all algebraic polynomials of degree r and suppose $C_B(0,\infty)$ is the space of all continuous and bounded functions on $(0,\infty)$ endowed with the norm $\|f\| = \sup\{|f(x)| : x \in (0,\infty)\}$. Further let us consider the following K-functional:

$$K_2(f,\delta) = \inf_{g\in C_B^2(0,\infty)} \{\|f-g\| + \delta\|g''\|\},$$

where $\delta > 0$ and $C_B^2(0,\infty) = \{g \in C_B(0,\infty) : g', g'' \in C_B(0,\infty)\}$.

Theorem 2.1. *Let $f \in C_B(0,\infty)$, then for $r \in \mathbb{N}$ we have*

$$\left|\widetilde{P}_{n,r}(f,x) - f(x)\right| \le C\omega_2\left(f, \sqrt{\delta_{n,r}}\right) + \omega\left(f, \left|\frac{(n+1)}{((n+1)_r)^{1/r}} - 1\right| x\right),$$

where C is a positive constant and $\delta_{n,r}$ is given as

$$\delta_{n,r} = \left[\frac{(n+1)(2n+3)}{((n+1)_r)^{2/r}} - \frac{4(n+1)}{((n+1)_r)^{1/r}} + 2\right] x^2.$$

Proof. We introduce the auxiliary operators $P_{n,r} : C_B(0,\infty) \to C_B(0,\infty)$ as follows

$$P_{n,r}(f,x) = \widetilde{P}_{n,r}(f,x) - f\left(\frac{(n+1)x}{((n+1)_r)^{1/r}}\right) + f(x). \tag{2.7}$$

These are linear operators and preserve linear functions. As by Lemma 2.1 and the positivity of $\widetilde{P}_{n,r}(t,x)$, we have

$$P_{n,r}(t,x) = \widetilde{P}_{n,r}(t,x) - \frac{(n+1)x}{((n+1)_r)^{1/r}} + x = \frac{(n+1)x}{((n+1)_r)^{1/r}} - \frac{(n+1)x}{((n+1)_r)^{1/r}} + x = x.$$

Let $g \in C_B^2(0,\infty)$ and $x, t \in (0,\infty)$. By Taylor's formula, we have

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)du.$$

Then using Lemma 2.2 and by positivity of $\widetilde{P}_{n,r}$, we have

$$\begin{aligned}
|P_{n,r}(g,x)-g(x)| =& \left|P_{n,r}\left(\int_x^t (t-u)g''(u)du, x\right)\right| \\
=& \left|\widetilde{P}_{n,r}\left(\int_x^t (t-u)g''(u)du, x\right)\right. \\
& \left. - \int_x^{\frac{(n+1)x}{((n+1)_r)^{1/r}}} \left(\frac{(n+1)x}{((n+1)_r)^{1/r}} - u\right) g''(u)du\right| \\
\leq & \left[T_2^{\widetilde{P}_{n,r}}(x) + \left(\frac{(n+1)x}{((n+1)_r)^{1/r}} - x\right)^2\right] ||g''|| \\
:=& \delta_{n,r}||g''||. \qquad (2.8)
\end{aligned}$$

Next, by (2.7) and from Lemma 2.1, we have

$$|P_{n,r}(f,x)| \leq \widetilde{P}_{n,r}(1,x)||f|| + 2||f|| \leq 3||f||. \qquad (2.9)$$

Using (2.7), (2.8) and (2.9), we have

$$\begin{aligned}
\left|\widetilde{P}_{n,r}(f,x) - f(x)\right| \leq & |P_{n,r}(f-g,x) - (f-g)(x)| + |P_{n,r}(g,x) - g(x)| \\
& + \left|f\left(\frac{(n+1)x}{((n+1)_r)^{1/r}}\right) - f(x)\right| \\
\leq & 4||f-g|| + \delta_{n,r}||g''|| + \left|f\left(\frac{(n+1)x}{((n+1)_r)^{1/r}}\right) - f(x)\right| \\
\leq & C\left\{||f-g|| + \delta_{n,r}||g''||\right\} + \omega\left(f, \left|\frac{(n+1)x}{((n+1)_r)^{1/r}} - x\right|\right).
\end{aligned}$$

Finally, if we take the infimum over all $g \in C_B^2(0,\infty)$, and using the inequality due to Gonska [8], $K_2(f,\delta) \leq C\omega_2(f,\sqrt{\delta})$, $\delta > 0$, we get at once the desired result. ☐

Corollary 2.1. *Let $f \in \pi_1 + C_B(0,\infty)$. Then*

$$\left|\widetilde{P}_{n,1}(f,x) - f(x)\right| \leq C\omega_2\left(f, \frac{x}{\sqrt{(n+1)}}\right),$$

where C is certain positive constant.

Let us consider

$$B_2(0,\infty) := \left\{f : f \in C(0,\infty) \to \mathbb{R} \text{ and } |f(x)| \leq C(f)\left(1+x^2\right)\right\},$$

where $C(f)$ is a positive constant depending only on f and we denote $C_2(0,\infty) = C(0,\infty) \cap B_2(0,\infty)$, by $C_2^*(0,\infty)$, we denote the subspace of all $f \in C_2(0,\infty)$ for which $\lim\limits_{x\to\infty} \frac{f(x)}{1+x^2} < \infty$. The weighted modulus of continuity $\Omega(f,\delta)$ defined on infinite interval $\mathbb{R}^+$ (see [11]) is defined as

$$\mathbf{\Omega}\left(f,\delta\right)=\sup_{|h|<\delta,\ x\in\mathbb{R}^+}\frac{\left|f\left(x+h\right)-f\left(x\right)\right|}{\left(1+h^2+x^2+h^2x^2\right)},\qquad \text{for each } f\in C_2\left(0,\infty\right).$$

We now estimate the following quantitative Voronovskaja-type asymptotic formula.

Theorem 2.2. *Let $f''\in C_2^*\left(0,\infty\right)$ and $r\in\mathbb{N}$. Then for $x>0$, we have*

$$\begin{aligned}&\left|\widetilde{P}_{n,r}(f,x)-f\left(x\right)-\left[\frac{(n+1)}{((n+1)_r)^{1/r}}-1\right]xf'(x)\right.\\&\left.-\left[\frac{(n+1)_2}{((n+1)_r)^{2/r}}-2\frac{(n+1)}{((n+1)_r)^{1/r}}+1\right]\frac{x^2}{2}f''\left(x\right)\right|\\&\leq C\left(1+x^2\right)\Omega\left(f'',n^{-1/2}\right)\left[T_2^{\widetilde{P}_{n,r}}(x)+n^2\ T_6^{\widetilde{P}_{n,r}}(x)\right],\end{aligned}$$

where C is certain absolute constant.

Proof. By Taylor's expansion,

$$f(t)=\sum_{i=0}^{2}(t-x)^i\frac{f^{(i)}(x)}{i!}+h\left(t,x\right)\left(t-x\right)^2,$$

where

$$h\left(t,x\right):=\frac{f''\left(\eta\right)-f''\left(x\right)}{2},$$

we have

$$\begin{aligned}\widetilde{P}_{n,r}(f,x)-f(x)=&\widetilde{P}_{n,r}\left(\left(f(t)-f(x),x\right)\right.\\=&T_1^{\widetilde{P}_{n,r}}(x)f'(x)+\frac{T_2^{\widetilde{P}_{n,r}}(x)}{2}f''(x)+\widetilde{P}_{n,r}\left(h\left(t,x\right)\left(t-x\right)^2,x\right),\end{aligned}$$

and h is a continuous function which vanishes at 0 and η lies between x and t.

Proceeding along the lines of [1, Theorem 2], we have

$$\widetilde{P}_{n,r}\left(\left|h\left(t,x\right)\right|\left(t-x\right)^2,x\right)\leq C\left(1+x^2\right)\Omega\left(f'',\delta\right)\left(T_2^{\widetilde{P}_{n,r}}(x)+\frac{1}{\delta^4}\ T_6^{\widetilde{P}_{n,r}}(x)\right).$$

Finally, using Lemma 2.2 and choosing $\delta=\frac{1}{\sqrt{n}}$, we get the desired result. ☐

Corollary 2.2. *Let $f''\in C_2^*\left(0,\infty\right)$, then for $x>0$, we have*

$$\begin{aligned}&\left|\widetilde{P}_{n,1}(f,x)-f\left(x\right)-\frac{x^2}{2(n+1)}f''\left(x\right)\right|\\&\leq C\left(1+x^2\right)\left(\frac{x^2}{(n+1)}+\frac{n^2x^6}{(n+1)^5}(5n^2+160n+265)\right)\Omega\left(f'',n^{-1/2}\right),\end{aligned}$$

where C is certain absolute constant.

3. Estimates for Polynomially Bounded Functions

In this section, we are going to extend our point wise estimates for unbounded functions defined on $(0,\infty)$ and having polynomial growth of order greater than 2. In recent years several authors in [2,4,17] and [14] have studied such problems. We recall that in Section 4, we used $\Omega(f;\delta)$ appropriate for functions with polynomial growth of order at most 2. To overcome this R. Păltănea introduced in [14] the weighted modulus $\omega_\varphi(f;h)$ defined as

$$\omega_\varphi(f;h)=\sup\left\{|f(x)-f(y)|: x\geq 0, y\geq 0, |x-y|\leq h\varphi\left(\frac{x+y}{2}\right)\right\},\quad h\geq 0,$$

where $\varphi(x)=\frac{\sqrt{x}}{1+x^m}$, $x\in(0,\infty)$, $m\in\mathbb{N}$, $m\geq 2$. We consider here those functions, for which we have the property

$$\lim_{h\to 0}\omega_\varphi(f;h)=0.$$

It is easy to verify that this property is fulfilled for f an algebraic polynomial of degree less than or equal to m. This follows from Theorem 2 in [14], which states that $\lim_{h\to 0}\omega_\varphi(f;h)=0$ whenever the function $f\circ e_2$ is uniformly continuous on $(0,1]$ and the function $f\circ e_v$, $v=\frac{2}{2m+1}$ is uniformly continuous on $[1,\infty)$, where $e_v(x)=x^v$, $x\geq 0$. Let us denote by $W_\varphi(0,\infty)$ the subspace of all real functions defined on $(0,\infty)$, for which the two conditions mentioned above hold true. In [17] we studied positive linear operators $L_n: E\to C(0,\infty)$, where E is a subspace of $C(0,\infty)$, such that $C_k(0,\infty)\subset E$ with $k=\max\{m+3,6,2m\}$ and

$$C_k(0,\infty)=\{f\in C(0,\infty)\text{ exists } M>0: |f(x)|\leq M(1+x^k)\text{ for all } x>0\},\quad k\in\mathbb{N}.$$

One of main results of [17] which we are going to apply for $\widetilde{P}_{n,r}$ is the following quantitative estimate in terms of weighted modulus $\omega_\varphi(f;h)$ (see [17, Theorem 2.2]).

Theorem 3.1 (Theorem A)**.** *Let $L_n: E\to C(0,\infty)$, $C_k(0,\infty)\subset E$, $k=\max\{m+3,6,2m\}$ be sequence of linear positive operators, preserving the linear functions. If $f\in C^2(0,\infty)\cap E$ and $f''\in W_\varphi(0,\infty)$, then we have for $x\in(0,\infty)$ that*

$$\begin{aligned}&\left|L_n(f,x)-f(x)-\frac{1}{2}f''(x)\mu_{n,2}^L(x)\right|\\ \text{(3.1)}\quad &\leq\frac{1}{2}\left[\mu_{n,2}^L(x)+\sqrt{2}\sqrt{L_n\left(\left[1+\left(x+\frac{|t-x|}{2}\right)^m\right]^2;x\right)}\right]\omega_\varphi\left(f'';\left(\frac{\mu_{n,6}^L}{x}\right)^{1/2}\right).\end{aligned}$$

Here $\mu_{n,k}^L(x)=L_n((t-x)^k,x)$ is the k-th order central moment of L_n.

We point out here that the statement in Theorem A can be extended for positive linear operators which don't preserve linear functions what we need in this case is to add the term $\mu_{n,1}^L(x)f'(x)$ in the left side of (3.1). As an application of Theorem A, we have the following result for modified Post-Widder operators.

Theorem 3.2. *Let $\widetilde{P}_{n,r} : E \to C(0,\infty)$, $C_k(0,\infty) \subset E$, $k = \max\{m+3, 6, 2m\}$, be the sequence of linear positive operators, preserving the test function e_r. If $f \in C^2(0,\infty) \cap E$ and $f'' \in W_\varphi(0,\infty)$, then we have for $x \in (0,\infty)$ that*

$$\begin{aligned}
&\Bigg|\widetilde{P}_{n,r}(f,x) - f(x) - \left[\frac{(n+1)}{((n+1)_r)^{1/r}} - 1\right] x f'(x) \\
&\quad - \frac{x^2}{2} f''(x) \left[\frac{(n+1)_2}{((n+1)_r)^{2/r}} - 2\frac{(n+1)}{((n+1)_r)^{1/r}} + 1\right]\Bigg| \qquad (3.2)\\
&\le \frac{1}{2}\left[\left(\frac{(n+1)_2}{((n+1)_r)^{2/r}} - 2\frac{(n+1)}{((n+1)_r)^{1/r}} + 1\right) x^2 \right. \qquad (3.3)\\
&\quad \left. + \sqrt{2}\sqrt{\widetilde{P}_{n,r}\left(\left[1 + \left(x + \frac{|t-x|}{2}\right)^m\right]^2 ; x\right)}\right] \\
&\quad \times \omega_\varphi\Bigg(f''; \Bigg(\Bigg[\frac{(n+1)_6}{((n+1)_r)^{6/r}} - 6\frac{(n+1)_5}{((n+1)_r)^{5/r}} + 15\frac{(n+1)_4}{((n+1)_r)^{4/r}} \\
&\quad - 20\frac{(n+1)_3}{((n+1)_r)^{3/r}} + 15\frac{(n+1)_2}{((n+1)_r)^{2/r}} - 6\frac{(n+1)}{((n+1)_r)^{1/r}} + 1\Bigg] x^5\Bigg)^{1/2}\Bigg).
\end{aligned}$$

We observe that the argument of $\omega_\varphi(f''; \delta)$ in above theorem is of order $\delta = O(n^{-3/2})$, $n \to \infty$.

Corollary 3.1. *Under the assumption of above theorem, if the operators preserve test function e_1, then we have*

$$\begin{aligned}
&\left|\widetilde{P}_{n,1}(f,x) - f(x) - \frac{x^2}{2(n+1)} f''(x)\right| \\
&\le \frac{1}{2}\left[\frac{x^2}{n+1} + \sqrt{2}\sqrt{\widetilde{P}_{n,1}\left(\left[1 + \left(x + \frac{|t-x|}{2}\right)^m\right]^2 ; x\right)}\right] \\
&\quad \times \omega_\varphi\Bigg(f''; \Bigg(\Bigg[\frac{(n+2)_5}{(n+1)^5} - 6\frac{(n+2)_4}{(n+1)^4} + 15\frac{(n+2)_3}{(n+1)^3} \\
&\quad - 20\frac{(n+2)_2}{(n+1)^2} + 15\frac{(n+2)}{(n+1)} - 5\Bigg] x^5\Bigg)^{1/2}\Bigg).
\end{aligned}$$

We consider $M_{n,k}^L = L_n(|t-x|^k, x)$ as the k-th order absolute moments of operators L_n.

The next main result of [17], which we are going to apply for the operators $P_{n,r}$ is the following quantitative variant of Voronovskaja theorem (see [17, Theorem 2.3]).

Theorem 3.3 (Theorem B)**.** *Let $L_n : E \to C(0,\infty)$, $C_k(0,\infty) \subset E$, $k = \max\{m+3, 4\}$, be sequence of linear positive operators, preserving the linear functions. If*

$f \in C^2(0,\infty) \cap E$ and $f'' \in W_\varphi(0,\infty)$, then we have for $x \in (0,\infty)$ that

$$
\begin{aligned}
&\left| L_n(f,x) - f(x) - \frac{1}{2} f''(x) \mu_{n,2}^L(x) \right| \\
\leq &\frac{1}{2} \left[\mu_{n,2}^L(x) + \frac{\sqrt{2}}{\sqrt{x}} \mu_{n,2}^L(x) C_{n,2,m}(x) \right] \omega_\varphi \left(f''; \sqrt{\frac{\mu_{n,4}^L(x)}{\mu_{n,2}^L(x)}} \right), \tag{3.4}
\end{aligned}
$$

where

$$C_{n,2,m}(x) = 1 + \frac{1}{M_{n,3}^L(x)} \sum_{k=0}^{m} \binom{m}{k} x^{m-k} \frac{M_{n,k+3}^L(x)}{2^k}.$$

We suppose for the operators L_n that

$$\frac{M_{n,k}^L}{M_{n,3}^L}, \quad 4 \leq k \leq m,$$

is a bounded ratio for fixed x and m, when $n \to \infty$.

Using Cauchy-Schwarz inequality, we have

$$\widetilde{P}_{n,r}(|t-x|^k, x) \leq \sqrt{\widetilde{P}_{n,r}((t-x)^{2k}, x)}$$

Applying Lemma 2.2, we have

$$\widetilde{P}_{n,r}(|t-x|^k, x) = O(n^{-k/2}), \quad n \to \infty,$$

and so it is easy to observe that $C_{n,2,m}(x)$ is a bounded term for fixed x and m when $n \to \infty$. Also as in Theorem A, we point out that the statement in Theorem B can be extended in a similar way for positive linear operators, which don't preserve linear functions. As an application of Theorem B we obtain the following quantitative asymptotic Voronovskaja theorem for $\widetilde{P}_{n,r}$.

Theorem 3.4. *Let $\widetilde{P}_{n,r} : E \to C(0,\infty)$, $C_k(0,\infty) \subset E$, $k = \max\{m+3, 4\}$, be sequence of linear positive operators, preserving the test function e_r. If $f \in C^2(0,\infty) \cap E$ and $f'' \in W_\varphi(0,\infty)$, then we have for $x \in (0,\infty)$ that*

$$
\begin{aligned}
&\left| \widetilde{P}_{n,r}(f,x) - f(x) - T_1^{\widetilde{P}_{n,r}}(x) f'(x) - \frac{T_2^{\widetilde{P}_{n,r}}(x)}{2} f''(x) \right| \\
\leq &\frac{T_2^{\widetilde{P}_{n,r}}(x)}{2} \left[1 + \frac{\sqrt{2}}{\sqrt{x}} C_{n,2,m}(x) \right] \omega_\varphi \left(f''; x\sqrt{\frac{T_4^{\widetilde{P}_{n,r}}(x)}{T_2^{\widetilde{P}_{n,r}}(x)}} \right),
\end{aligned}
$$

where

$$C_{n,2,m}(x) = 1 + \frac{1}{M_{n,3}^{\widetilde{P}_{n,r}}(x)} \sum_{k=0}^{m} \binom{m}{k} x^{m-k} \frac{M_{n,k+3}^{\widetilde{P}_{n,r}}(x)}{2^k}.$$

Corollary 3.2. *If f, f'' satisfy the same conditions as in the assumption of Theorem* 3.4*, then we have for $x \in (0,\infty)$ that*

$$\lim_{n\to\infty} n[\widetilde{P}_{n,r}(f,x) - f(x)] = xf'(x) + \frac{x^2}{2} f''(x).$$

Corollary 3.3. *Under the assumption of above Theorem* 3.4*, if the operators preserve test function e_1, then we have*

$$\left|\widetilde{P}_{n,1}(f,x) - f(x) - \frac{x^2}{2(n+1)} f''(x)\right|$$
$$\leq \frac{x^2}{2(n+1)} \left[1 + \frac{\sqrt{2}}{\sqrt{x}} C_{n,2,m}(x)\right] \omega_\varphi \left(f''; x\frac{\sqrt{3(n+3)}}{n+1}\right).$$

Our next goal in this section is to obtain estimates in terms of K-functional for polynomially bounded functions. We recall some notations from [4].

If $m \in \mathbb{N}$ is fixed we consider the weight

$$\rho(x) = \rho_m(x) = (1+x)^{-m}, \quad x \in I \equiv [0,\infty).$$

The polynomials weighted space associated to ρ is defined by

$$C_\rho(I) = \{f \in C(I) : ||f||_\rho < \infty\},$$

where

$$||f||_\rho = \sup_{x\geq 0} \rho(x)|f(x)|.$$

In [4] it was used

$$\phi(x) = \sqrt{(1+ax)(bx+c)}, \quad a \in \mathbb{N}_0,\ b > 0,\ c \geq 0.$$

Here we set $a = b = c = 1$, i.e.,

$$\phi(x) = 1 + x. \tag{3.5}$$

For $\lambda \in [0,1]$, $s = 1,2$, and $f \in C_\rho(I)$, we consider the K-functional

$$K_{s,\phi^\lambda}(f,t)_\rho = \inf\{||f-g||_\rho + t^s||\phi^{\lambda s} \cdot g^{(s)}||_\rho,\ g \in W^\infty_{s,\lambda}(\phi)\}, \tag{3.6}$$

where $W^\infty_{s,\lambda}(\phi)\}$ consists of all functions $g \in C_\rho(0,\infty)$ such that $g^{(s-1)}$ is absolutely continuous on every finite closed subinterval of $(0,\infty)$ and $||\phi^{\lambda s} \cdot g^{(s)}||_\rho < \infty$.

One of the main results in [4] is Theorem 1 which we cite here as (see [4, page 1498]).

Theorem 3.5 (Theorem C)**.** *For a positive integer $m, \rho(x) = (1+x)^{-m}$ and $\phi(x) = \sqrt{(1+ax)(bx+c)}$, $a \in \mathbb{N}_0$, $b > 0$, $c \geq 0$, and for positive linear operator $L_n : C_\rho(I) \to C(I)$, we suppose the following conditions:*

(i) $L_n(e_0) = e_0$;

(ii) *there exist a constant C_1 and a sequence $\{\alpha_n\}$, $\alpha_n \to 0$, $n \to \infty$, such that*

$$L_n((t-x)^2, x) \leq C_1 \alpha_n \phi^2(x);$$

(iii) *there exists a constant* $C_2 = C_2(m)$ *such that for each* $m \in \mathbb{N}$

$$L_n((1+t)^m) \le C_2(1+x)^m, \quad x \ge 0;$$

(iv) *there exists a constant* $C_3 = C_3(m)$, *such that for each* $m \in \mathbb{N}$

$$\rho(x)L_n((t-x)^2/\rho(t);x) \le C_3\alpha_n\phi^2(x), \quad x \ge 0.$$

Then for $\lambda \in [0,1]$ *there exists a constant* $C_4 = C_4(m,\lambda)$ *such that for any* $f \in C_\rho(I)$, $x \in I$, $n \in \mathbb{N}$, *one has*

$$\rho(x)\left|f(x) - L_n(f;x)\right| \le C_4 K_{1,\phi^\lambda}\left(f;\sqrt{\alpha_n}\phi^{1-\lambda}(x)\right)_\rho, \quad x \ge 0, \tag{3.7}$$

where $K_{1,\phi^\lambda}(f;t)_\rho$ *is defined in* (3.6) *for* $s = 1$.

If in addition $L_n(e_1) = e_1$, *then there exists a constant* $C_5 = C_5(m,\lambda)$ *such that*

$$\rho(x)\left|f(x) - L_n(f;x)\right| \le C_5 K_{2,\phi^\lambda}\left(f;\sqrt{\alpha_n}\phi^{1-\lambda}(x)\right)_\rho, \tag{3.8}$$

where $K_{2,\phi^\lambda}(f;t)_\rho$ *is the* K*-functional defined in* (3.6) *for* $s = 2$.

We apply Theorem C for the modified Post-Widder operators $\widetilde{P}_{n,r}$. The condition (i) is trivial. The condition (ii) follows from the representation of second central moment $T_2^{\widetilde{P}_{n,r}}(x)$ in Lemma 2.2, with $\alpha_n = \frac{1}{n}$. The condition (iii) follows from representation of $\widetilde{P}_{n,r}(t^k;x) = \frac{(n+1)_k}{((n+1)_r)^{k/r}}x^k$ obtained in previous section. To verify condition (iv) we apply Cauchy-Schwarz inequality and, by Lemma 2.2,

$$\begin{aligned}(1+x)^{-m}\widetilde{P}_{n,r}\left((t-x)^2(1+t)^m;x\right) &\le (1+x)^{-m}\sqrt{\widetilde{P}_{n,r}\left((t-x)^4;x\right)}\sqrt{\widetilde{P}_{n,r}\left((1+t)^{2m};x\right)}\\ &\le (1+x)^{-m}C(r)T_2^{\widetilde{P}_{n,r}}(x)\sqrt{C_2(m)(1+x)^{2m}}\\ &\le C(r)\sqrt{C_2(m)}T_2^{\widetilde{P}_{n,r}}(x)\\ &\le C_3\alpha_n\phi^2(x), \quad x \ge 0,\end{aligned}$$

where we have used condition (iii) and representation of $T_2^{\widetilde{P}_{n,r}}$ in Lemma 2.2. Therefore, as a consequence from Theorem C, we obtain the following.

Theorem 3.6. *Let* $\rho(x) = (1+x)^{-m}$, $m \in \mathbb{N}$, $f \in C_\rho(I)$, $\phi(x) = 1+x$, $r \in \mathbb{N}$, $r \ge 2$, $\alpha = \frac{1}{n}$. *Then for* $\lambda \in [0,1]$ *we have*

$$\rho(x)\left|\widetilde{P}_{n,r}(f;x) - f(x)\right| \le C(m,\lambda)K_{1,\phi^\lambda}\left(f;\sqrt{\alpha_n}\phi^{1-\lambda}(x)\right)_\rho. \tag{3.9}$$

For $r = 1$ *we have*

$$\rho(x)\left|\widetilde{P}_{n,1}(f;x) - f(x)\right| \le C(m,\lambda)K_{2,\phi^\lambda}\left(f;\sqrt{\alpha_n}\phi^{1-\lambda}(x)\right)_\rho. \tag{3.10}$$

Remark 3.1. It is known (see [5]) that the K-functional $K_{s,\phi^\lambda}(f;t)_\rho$ is equivalent to Ditzian-Totik moduli $\omega^s_{\phi^\lambda}(f;t)_\rho$. In the most important cases $\lambda = 0$ (point-wise Becker-type estimate) and $\lambda = 1$ (estimate in norm) we can formulate (3.9) and (3.10) in terms of $\omega^s_{\phi^\lambda}(f;t)_\rho$. If we denote by $C_{\rho,\infty}$ the set of all continuous functions on $(0,\infty)$ such that $\rho(x)f(x)$ has finite limit as $x\to\infty$, then it can be proved that $\lim_{t\to 0^+}\omega^s_{\phi^\lambda}(f;t)_\rho=0$ whenever $f\in C_{\rho,\infty}(0,\infty)$. For $\lambda=0$ we get from here estimates (1.3) and (2.3)–(2.6) with some constant C independent of f and n in place of 2.

4. Linear Combinations of Modified Post-Widder Operators

From Corollary 2.1 and Corollary 3.2 we see that bounded continuous functions $f\in C_B(0,\infty)$ and also polynomially bounded functions can be approximated by $\widetilde{P}_{n,r}$ with order of approximation not greater than $O(n^{-1})$. To increase the order of approximation we consider the linear combinations $L_{n,r}$ of $\widetilde{P}_{n,r}$. For more general settings and recent results for approximation by positive linear operators we refer the readers to the monograph [9]. Following the idea from [9] we will consider the following linear combinations

$$L_{n,r}=\sum_{i=0}^{r}\alpha_i(n)\widetilde{P}_{n_i,r}, \tag{4.1}$$

where n_i, $i=0,1,\dots$, are different positive numbers, $r\in\mathbb{N}$. Determine $\alpha_i(n)$ such that $L_{n,r}p=p$ for all $p\in\pi_r$ the set of algebraic polynomials of degree less than or equal to r. This seems to be natural as the operators $\widetilde{P}_{n,r}$ don't preserve linear functions if $r\geq 2$. The requirement that each polynomial of degree at most r should be reproduced leads to the linear system

$$L_{n,r}(t^k,x)=x^k,\quad 0\leq k\leq r. \tag{4.2}$$

From the nice representation of the images of monomials obtained in Section 2, $\widetilde{P}_{n,r}(t^m,x)=\frac{(n+1)_m}{((n+1)_r)^{m/r}}x^m$ and (4.1), we obtain the system

$$\begin{cases}\alpha_0(n)+\alpha_1(n)+\alpha_2(n)+\cdots+\alpha_r(n)=1,\\ \displaystyle\sum_{i=0}^{r}\alpha_i(n)\frac{(n_i+1)_m}{((n_i+1)_r)^{m/r}}=1,\quad 1\leq m\leq r.\end{cases} \tag{4.3}$$

We observe that if $m=r$ in (4.3), the last equation coincides with the first one. So to have a unique solution for the coefficients $\alpha_i(n)$, we may impose additional condition

$$L_{n,r}(t^{r+1},x)=x^{r+1}. \tag{4.4}$$

Then the system (4.3) will have the form

$$\begin{cases}\alpha_0(n)+\alpha_1(n)+\alpha_2(n)+\cdots+\alpha_r(n)=1,\\ \displaystyle\sum_{i=0}^{r}\alpha_i(n)\frac{(n_i+1)_m}{((n_i+1)_r)^{m/r}}=1,\quad 1\leq m\leq r-1,\\ \displaystyle\sum_{i=0}^{r}\alpha_i(n)\frac{(n_i+1)_{r+1}}{((n_i+1)_r)^{(r+1)/r}}=1.\end{cases} \tag{4.5}$$

Then from (4.5), we observe that all the polynomials of degree up to $r+1$ will be preserved by the combinations $L_{n,r}$ from (4.1). To obtain the direct estimate for approximation by linear combinations $L_{n,r}$, one need two additional conditions

$$n = n_0 < n_1 < n_2 < \cdots < n_r \leq An, \quad A = A(r), \tag{4.6}$$

$$\sum_{i=0}^{r} |\alpha_i(n)| \leq C. \tag{4.7}$$

The condition (4.6) guarantees that

$$L_{n,r}(|t-x|^{r+2}, x) = O(n^{-(r+2)/2}), \quad n \to \infty \tag{4.8}$$

The fact that all polynomials of degree less or equal to $r+1$ are preserved allow us to consider approximating functions f from much larger class than $C_B(0,\infty)$, namely we consider $f \in \pi_{r+1} + C_B(0,\infty)$, where π_{r+1} is the set of all algebraic polynomials of degree $\leq r+1$. Then following the same method applied by the authors in [10, Theorem 4] we arrive at the proof of following theorem.

Theorem 4.1. *Let $f \in \pi_{r+1} + C_B(0,\infty)$. Then for every $x \in (0,\infty)$ and for $C > 0$, $n > r$ and if the coefficients $\alpha_i(n)$ and numbers n_i, $0 \leq i \leq r$, satisfy the conditions* (4.5)*,* (4.6) *and* (4.7)*, we have*

$$|L_{n,r}(f,x) - f(x)| \leq C\omega_{r+2}(f, n^{-1/2}). \tag{4.9}$$

Corollary 4.1. *If $f \in \pi_{r+1} + C_B(0,\infty)$ and $f^{(r+2)} \in C_B(0,\infty)$, then*

$$|L_{n,r}(f,x) - f(x)| \leq Cn^{-(r+2)/2}||f^{(r+2)}||_{C_B(0,\infty)}. \tag{4.10}$$

Let us consider the case $r = 1$. In this case, to determine the coefficients $\alpha_0(n)$ and $\alpha_1(n)$ from (4.5), we get

$$\begin{cases} \alpha_0(n) + \alpha_1(n) = 1, \\ \alpha_0(n)\dfrac{(n_0+1)_2}{(n_0+1)^2} + \alpha_0(n)\dfrac{(n_1+1)_2}{(n_1+1)^2} = 1. \end{cases}$$

The solution of this system is

$$\alpha_0(n) = -\frac{n_0+1}{n_1-n_0}, \quad \alpha_1(n) = \frac{n_1+1}{n_1-n_0}. \tag{4.11}$$

Obviously $\alpha_1(n) > 0$, $\alpha_0(n) < 0$. According to the conditions (4.6), (4.7), we must be careful with the choice of $\alpha_0(n)$, $\alpha_1(n)$. For example, if $n_0 = n$, $n_1 = 2n$, then (4.11) implies

$$\alpha_0(n) = -1 - \frac{1}{n}, \quad \alpha_1(n) = 2 + \frac{1}{n},$$

and (4.7) is satisfied. But if $n_0 = n$ and $n_1 = n+1$, then $\alpha_0(n) = -(n+1)$, $\alpha_1(n) = n+2$ and (4.7) is not fulfilled although (4.6) is true.

We observe that linear combinations $L_{n,1}$ preserve e_0, e_1, e_2. Therefore, if π_2 denotes the space of all algebraic polynomials of degree 2, we may consider the approximating functions f to be every $f \in \pi_2 + C_B(0,\infty)$, which means that we consider $f = g + h$, where $g \in \pi_2$ is an arbitrary algebraic polynomial of degree less than or equal to 2 and $g \in C_B(0,\infty)$ is an arbitrary bounded continuous function. With α_0, α_1 defined in (4.11) we have

Theorem 4.2. *Let $f \in \pi_2 + C_B(0,\infty)$. Then for every $C > 0$, $n > 1$, we have the following:*

$$|L_{n,1}(f,x) - f(x)| \leq C\omega_3(f, n^{-1/2}). \tag{4.12}$$

If $f''' \in C_B(0,\infty)$, then

$$|L_{n,1}(f,x) - f(x)| \leq Cn^{-3/2}||f'''||_{C_B(0,\infty)}. \tag{4.13}$$

Remark 4.1. We observe that the estimates with linear combinations $L_{n,r}$ from Theorem 4.1 and Corollary 4.1 are better than the previous estimates [16, (1.3)], also from [10]. The reason for this effect is that our modified Post-Widder operators $P_{n,r}$ preserve e_0 and e_r.

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