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DIFFERENTIAL SUBORDINATION RESULTS FOR HOLOMORPHIC FUNCTIONS RELATED TO GENERALIZED DIFFERENTIAL OPERATOR

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ABSTRACT. In the present investigation, we use the principle of subordination to introduce a new family for holomorphic functions defined by generalized differential operator. Also we establish some interesting geometric properties for functions belonging to this family.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}_m$ stands for the family of functions f of the form:

$$f(z) = z + \sum_{n=m+1}^{\infty} a_n z^n \quad (m \in \mathbb{N} = \{1, 2, \dots\}, z \in U), \tag{1.1}$$

which are holomorphic in the open unit disk $U = \{z \in C : |z| < 1\}$.

For two functions f and g holomorphic in U, we say that the function f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)(z \in U)$, if there exists a Schwarz function w holomorphic in U with $w(0) = 0$ and $|w(z)| < 1$, $z \in U$, such that $f(z) = g(w(z))$, $z \in U$. In particular, if the function g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$ (see [6]).

If $f \in \mathcal{A}_m$ is given by (1.1) and $g \in \mathcal{A}_m$ given by

$$g(z) = z + \sum_{n=m+1}^{\infty} b_n z^n \quad (m \in \mathbb{N} = \{1, 2, \dots\}, z \in U),$$

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then the Hadamard product (or convolution) $f * g$ of f and g is defined by

$$(f * g)(z) = z + \sum_{n=m+1}^{\infty} a_n b_n z^n = (g * f)(z).$$

A function $f \in \mathcal{A}_m$ is said to be starlike of order ρ in U if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \rho \quad (0 \le \rho < 1, z \in U).$$

Indicate the class of all starlike functions of order ρ in U by $S^*(\rho)$.

A function $f \in \mathcal{A}_m$ is said to be prestarlike of order ρ in U if

$$\frac{z}{(1-z)^{2(1-\rho)}} * f(z) \in S^*(\rho) \quad (\rho < 1).$$

Indicate the class of all prestarlike functions of order ρ in U by $\operatorname{Re}(\rho)$.

Clearly a function $f \in \mathcal{A}_m$ is in the class $\operatorname{Re}(0)$ if and only if f is convex univalent in U and $\operatorname{Re}(\frac{1}{2}) = S^*(\frac{1}{2})$.

For $\sigma \in N_0 = N \cup \{0\}$, $\alpha, \delta \ge 0$, $\tau, \lambda, \beta > 0$ and $\alpha \ne \lambda$, we consider the generalized differential operator $A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta) : \mathcal{A}_m \to \mathcal{A}_m$, introduced by Amourah and Darus [2], where

$$A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f(z) = z + \sum_{n=m+1}^{\infty} \left[1 + \frac{(n-1)\left((\lambda-\alpha)\beta + n\delta\right)}{\tau+\lambda}\right]^{\sigma} a_n z^n. \tag{1.2}$$

It is readily verified from (1.2) that

$$\begin{aligned} z\left(A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f(z)\right)' =& \frac{\tau+\lambda}{(\lambda-\alpha)\beta+n\delta} A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)f(z) \\ &+ \left(1 - \frac{\tau+\lambda}{(\lambda-\alpha)\beta+n\delta}\right) A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f(z). \end{aligned} \tag{1.3}$$

Here, we would point out some of the special cases of the operator defined by (1.2) can be found in [1, 4, 5, 11].

Let H be the class of functions h with $h(0) = 1$, which are holomorphic and convex univalent in U.

Definition 1.1. A function $f \in \mathcal{A}_m$ is said to be in the class $\mathcal{M}(\eta,\sigma,\tau,\lambda,\delta,\alpha,\beta,m;h)$ if it satisfies the subordination condition:

(1.4)

$$\frac{1}{z}\left[\left(1 - \frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}\right) A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f(z) + \frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta} A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)f(z)\right] \prec h(z),$$

where $\eta \in C$, $\sigma \in N_0 = N \cup \{0\}$, $\alpha, \delta \ge 0$, $\tau, \lambda, \beta > 0$, $\alpha \ne \lambda$ and $h \in H$.

Now, we need the following lemmas that will be used to prove our main results.

Lemma 1.1 ([8]). *Let g be holomorphic in U and let h be holomorphic and convex univalent in U with $h(0) = g(0)$. If*

$$g(z) + \frac{1}{\mu} z g'(z) \prec h(z), \tag{1.5}$$

where $\operatorname{Re}(\mu) \geq 0$ *and* $\mu \neq 0$*, then*

$$g(z) \prec \breve{h}(z) = \mu z^{-\mu} \int_0^z t^{\mu-1} h(t) dt \prec h(z)$$

and $\breve{h}$ is the best dominant of (1.5).

Lemma 1.2 ([10]). *Let $\rho < 1$, $f \in S^*(\rho)$ and $g \in \operatorname{Re}(\rho)$. Then, for any holomorphic function F in U*

$$\frac{g * (fF)}{g * f}(U) \subset \bar{co}\,(F(U)),$$

where $\bar{co}\,(F(U))$ denotes the closed convex hull of $F(U)$.

Such type of study was carried out by various authors for another classes, like, Liu [7], Prajapat and Raina [9], Atshan and Wanas [3], Wanas [12] and Wanas and Majeed [13].

2. Main Results

Theorem 2.1. *Let $0 \leq \eta < \varepsilon$. Then*

$$\mathcal{M}(\varepsilon, \sigma, \tau, \lambda, \delta, \alpha, \beta, m; h) \subset \mathcal{M}(\eta, \sigma, \tau, \lambda, \delta, \alpha, \beta, m; h).$$

Proof. Let $0 \leq \eta < \varepsilon$ and $f \in \mathcal{M}(\varepsilon, \sigma, \tau, \lambda, \delta, \alpha, \beta, m; h)$. Assume that

$$g(z) = \frac{A^{\sigma}_{\tau,\lambda,\delta}(\alpha, \beta) f(z)}{z} = 1 + \sum_{n=m+1}^{\infty} \left[1 + \frac{(n-1)\left((\lambda - \alpha)\beta + n\delta\right)}{\tau + \lambda}\right]^{\sigma} a_n z^{n-1}. \tag{2.1}$$

It is obvious that the function g is holomorphic in U and $g(0) = 1$. Since $f \in \mathcal{M}(\varepsilon, \sigma, \tau, \lambda, \delta, \alpha, \beta, m; h)$, then we deduce that

(2.2)

$$\frac{1}{z}\left[\left(1 - \frac{\varepsilon(\tau + \lambda)}{(\lambda - \alpha)\beta + n\delta}\right) A^{\sigma}_{\tau,\lambda,\delta}(\alpha, \beta) f(z) + \frac{\varepsilon(\tau + \lambda)}{(\lambda - \alpha)\beta + n\delta} A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha, \beta) f(z)\right] \prec h(z).$$

Differentiating both sides of (2.1) with respect to z and using (1.3) and (2.2), we find that

$$\begin{aligned}
&\frac{1}{z}\left[\left(1 - \frac{\varepsilon(\tau + \lambda)}{(\lambda - \alpha)\beta + n\delta}\right) A^{\sigma}_{\tau,\lambda,\delta}(\alpha, \beta) f(z) + \frac{\varepsilon(\tau + \lambda)}{(\lambda - \alpha)\beta + n\delta} A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha, \beta) f(z)\right] \\
=&\frac{1}{z}\left[(1 - \varepsilon) A^{\sigma}_{\tau,\lambda,\delta}(\alpha, \beta) f(z) + \varepsilon\left(1 - \frac{\tau + \lambda}{(\lambda - \alpha)\beta + n\delta}\right) A^{\sigma}_{\tau,\lambda,\delta}(\alpha, \beta) f(z)\right. \\
&\left. + \frac{\varepsilon(\tau + \lambda)}{(\lambda - \alpha)\beta + n\delta} A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha, \beta) f(z)\right]
\end{aligned}$$

$$=(1-\varepsilon)\frac{A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f(z)}{z}+\varepsilon\left(A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f(z)\right)'$$
$$=\frac{A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f(z)}{z}+\varepsilon z\left(\frac{A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f(z)}{z}\right)'$$
$$=g(z)+\varepsilon z g'(z)\prec h(z).$$

An application of Lemma 1.1 with $\mu=\frac{1}{\varepsilon}$, yields

$$g(z)\prec h(z). \tag{2.3}$$

Evidently, $0\leq\frac{\eta}{\varepsilon}<1$ and that h is convex univalent in U, it follows from (2.1), (2.2) and (2.3) that

$$\frac{1}{z}\left[\left(1-\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}\right)A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f(z)+\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)f(z)\right]$$
$$=\frac{\eta}{\varepsilon z}\left[\left(1-\frac{\varepsilon(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}\right)A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f(z)+\frac{\varepsilon(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)f(z)\right]$$
$$+\left(1-\frac{\eta}{\varepsilon}\right)g(z)\prec h(z).$$

Hence, $f\in\mathcal{M}(\eta,\sigma,\tau,\lambda,\delta,\alpha,\beta,m;h)$ and the proof of Theorem 2.1 is completed. ☐

Theorem 2.2. *Let* $\operatorname{Re}\left\{\frac{\tau+\lambda}{(\lambda-\alpha)\beta+n\delta}\right\}\geq 0$ *and* $\frac{\tau+\lambda}{(\lambda-\alpha)\beta+n\delta}\neq 0$. *Then*

$$\mathcal{M}(\eta,\sigma+1,\tau,\lambda,\delta,\alpha,\beta,m;h)\subset\mathcal{M}(\eta,\sigma,\tau,\lambda,\delta,\alpha,\beta,m;h).$$

Proof. Let $f\in\mathcal{M}(\eta,\sigma+1,\tau,\lambda,\delta,\alpha,\beta,m;h)$ and suppose that

$$g(z)=\frac{1}{z}\left[\left(1-\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}\right)A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f(z)+\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)f(z)\right]. \tag{2.4}$$

By taking the derivatives in the both sides of (2.4) with respect to z and using (1.3), we conclude that

$$\begin{aligned} g(z)+zg'(z) &= \frac{1}{z}\left[\left(1-\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}\right)\left(1-\frac{\tau+\lambda}{(\lambda-\alpha)\beta+n\delta}\right)A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f(z)\right.\\ &+\left(1+\eta\left(1-\frac{2(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}\right)\right)\frac{\tau+\lambda}{(\lambda-\alpha)\beta+n\delta}A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)f(z)\\ &\left.+\eta\left(\frac{\tau+\lambda}{(\lambda-\alpha)\beta+n\delta}\right)^2A^{\sigma+2}_{\tau,\lambda,\delta}(\alpha,\beta)f(z)\right]. \end{aligned} \tag{2.5}$$

In the light of (2.4) and (2.5), we deduce that

$$\frac{\tau+\lambda}{(\lambda-\alpha)\beta+n\delta}g(z)+zg'(z)$$

$$=\frac{1}{z}\left[\left(1-\frac{\eta\left(\tau+\lambda\right)}{\left(\lambda-\alpha\right)\beta+n\delta}\right)\left(\frac{\tau+\lambda}{\left(\lambda-\alpha\right)\beta+n\delta}\right)A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)f\left(z\right)\right.$$
$$\left.+\eta\left(\frac{\tau+\lambda}{\left(\lambda-\alpha\right)\beta+n\delta}\right)^{2}A^{\sigma+2}_{\tau,\lambda,\delta}(\alpha,\beta)f\left(z\right)\right],$$

that is

$$g(z)+\frac{\left(\lambda-\alpha\right)\beta+n\delta}{\tau+\lambda}zg'(z)=\frac{1}{z}\left[\left(1-\frac{\eta\left(\tau+\lambda\right)}{\left(\lambda-\alpha\right)\beta+n\delta}\right)A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)f\left(z\right)\right.$$
$$\left.+\frac{\eta\left(\tau+\lambda\right)}{\left(\lambda-\alpha\right)\beta+n\delta}A^{\sigma+2}_{\tau,\lambda,\delta}(\alpha,\beta)f\left(z\right)\right]. \tag{2.6}$$

Since $f\in\mathcal{M}(\eta,\sigma+1,\tau,\lambda,\delta,\alpha,\beta,m;h)$, then it follows from (2.6) that

$$g(z)+\frac{\left(\lambda-\alpha\right)\beta+n\delta}{\tau+\lambda}zg'(z)\prec h(z),$$

where

$$\operatorname{Re}\left\{\frac{\tau+\lambda}{\left(\lambda-\alpha\right)\beta+n\delta}\right\}\geq 0,\qquad \frac{\tau+\lambda}{\left(\lambda-\alpha\right)\beta+n\delta}\neq 0.$$

An application of Lemma 1.1, with $\mu=\frac{\tau+\lambda}{(\lambda-\alpha)\beta+n\delta}$, yields $g(z)\prec h(z)$. In view of (2.4), we have

$$\frac{1}{z}\left[\left(1-\frac{\eta\left(\tau+\lambda\right)}{\left(\lambda-\alpha\right)\beta+n\delta}\right)A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f\left(z\right)+\frac{\eta\left(\tau+\lambda\right)}{\left(\lambda-\alpha\right)\beta+n\delta}A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)f\left(z\right)\right]\prec h(z).$$

This shows that $f\in\mathcal{M}(\eta,\sigma,\tau,\lambda,\delta,\alpha,\beta,m;h)$ and the proof of Theorem 2.2 is completed. □

Theorem 2.3. *Let $\eta>0$, $\gamma>0$ and $f\in\mathcal{M}(\eta,\sigma,\tau,\lambda,\delta,\alpha,\beta,m;\gamma h+1-\gamma)$. If $\gamma\leq\gamma_0$, where*

$$\gamma_0=\frac{1}{2}\left(1-\frac{1}{\eta}\int_0^1\frac{u^{\frac{1}{\eta}-1}}{1+u}du\right)^{-1}, \tag{2.7}$$

then $f\in\mathcal{M}(0,\sigma,\tau,\lambda,\delta,\alpha,\beta,m;h)$. The bound γ_0 is the sharp when $h(z)=\frac{1}{1-z}$.

Proof. Assume that

$$g(z)=\frac{A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f\left(z\right)}{z}. \tag{2.8}$$

Let $f\in\mathcal{M}(\eta,\sigma,\tau,\lambda,\delta,\alpha,\beta,m;\gamma h+1-\gamma)$ with $\eta>0$ and $\gamma>0$. Then we obtain

$$g(z)+\eta zg'(z)$$
$$=\frac{1}{z}\left[\left(1-\frac{\eta\left(\tau+\lambda\right)}{\left(\lambda-\alpha\right)\beta+n\delta}\right)A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f\left(z\right)+\frac{\eta\left(\tau+\lambda\right)}{\left(\lambda-\alpha\right)\beta+n\delta}A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)f\left(z\right)\right]$$
$$\prec\gamma h(z)+1-\gamma.$$

Making use of Lemma 1.1, we observe that

$$g(z) \prec \frac{\gamma}{\eta} z^{-\frac{1}{\eta}} \int_0^z t^{\frac{1}{\eta}-1} h(t) dt + 1 - \gamma = (h * \phi)(z), \tag{2.9}$$

where

$$\phi(z) = \frac{\gamma}{\eta} z^{-\frac{1}{\eta}} \int_0^z \frac{t^{\frac{1}{\eta}-1}}{1-t} dt + 1 - \gamma. \tag{2.10}$$

If $0 < \gamma \leq \gamma_0$, where $\gamma_0 > 1$ is given by (2.7), then we find from (2.10) that

$$\operatorname{Re}(\phi(z)) = \frac{\gamma}{\eta} \int_0^1 u^{\frac{1}{\eta}-1} \operatorname{Re}\left(\frac{1}{1-uz}\right) du + 1 - \gamma > \frac{\gamma}{\eta} \int_0^1 \frac{u^{\frac{1}{\eta}-1}}{1+u} du + 1 - \gamma \geq \frac{1}{2}. \tag{2.11}$$

By using (2.8) and (2.9), we have

$$\frac{A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta) f(z)}{z} \prec (h * \phi)(z). \tag{2.12}$$

In the light of (2.11), we note that the function $\phi(z)$ has the Herglotz representation

$$\phi(z) = \int_{|x|=1} \frac{d\mu(x)}{1-xz} \quad (z \in U), \tag{2.13}$$

where $\mu(x)$ is a probability measure defined on the unit circle $|x| = 1$ and

$$\int_{|x|=1} d\mu(x) = 1.$$

Since h is convex univalent in U, then we deduce from (2.12) and (2.13) that

$$\frac{A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta) f(z)}{z} \prec (h * \phi)(z) = \int_{|x|=1} \phi(xz)\, d\mu(x) \prec h(z).$$

This shows that $f \in \mathcal{M}(0, \sigma, \tau, \lambda, \delta, \alpha, \beta, m; h)$. For $h(z) = \frac{1}{1-z}$ and $f \in \mathcal{A}_m$ defined by

$$\frac{A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta) f(z)}{z} = \frac{\gamma}{\eta} z^{-\frac{1}{\eta}} \int_0^z \frac{t^{\frac{1}{\eta}-1}}{1-t} dt + 1 - \gamma,$$

we obtain

$$\frac{1}{z}\left[\left(1 - \frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta + n\delta}\right) A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta) f(z) + \frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta + n\delta} A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta) f(z)\right]$$
$$= \gamma h(z) + 1 - \gamma.$$

Thus, $f \in \mathcal{M}(\eta, \sigma, \tau, \lambda, \delta, \alpha, \beta, m; \gamma h + 1 - \gamma)$. Also, for $\gamma > \gamma_0$, we have

$$Re\left\{\frac{A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta) f(z)}{z}\right\} \longrightarrow \frac{\gamma}{\eta} \int_0^1 \frac{u^{\frac{1}{\eta}-1}}{1+u} du + 1 - \gamma < \frac{1}{2} \quad (z \to -1),$$

which implies that $f \notin \mathcal{M}(0, \sigma, \tau, \lambda, \delta, \alpha, \beta, m; h)$. Thus, the bound γ_0 cannot be increased when $h(z) = \frac{1}{1-z}$. This completes the proof of the theorem. □

Theorem 2.4. *Let $f \in \mathcal{M}(\eta,\sigma,\tau,\lambda,\delta,\alpha,\beta,m;h)$ be defined as in* (1.1). *Then the function I defined by*

$$I(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \quad (\mathrm{Re}(c) > -1),$$

is also in the class $\mathcal{M}(\eta,\sigma,\tau,\lambda,\delta,\alpha,\beta,m;h)$.

Proof. Let $f \in \mathcal{M}(\eta,\sigma,\tau,\lambda,\delta,\alpha,\beta,m;h)$ be defined as in (1.1). Then, we find that

$$\frac{1}{z}\left[\left(1-\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}\right)A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f(z)+\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)f(z)\right]\prec h(z). \tag{2.14}$$

We can easily see that

$$I(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt = z+\sum_{n=m+1}^{\infty}\frac{c+1}{c+n}a_n z^n. \tag{2.15}$$

We have from (2.15) that $I \in \mathcal{A}_m$ and

$$f(z) = \frac{cI(z)+zI'(z)}{c+1}. \tag{2.16}$$

Define the function J by

$$J(z) = \frac{1}{z}\left[\left(1-\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}\right)A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)I(z)+\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)I(z)\right]. \tag{2.17}$$

Differentiating both sides of (2.17) with respect to z and using (2.14) and (2.16), we obtain

$$\begin{aligned}
&\frac{1}{z}\left[\left(1-\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}\right)A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f(z)+\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)f(z)\right]\\
=&\frac{1}{z}\left[\left(1-\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}\right)A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)\left(\frac{cI(z)+zI'(z)}{c+1}\right)\right.\\
&\left.+\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)\left(\frac{cI(z)+zI'(z)}{c+1}\right)\right]\\
=&\frac{c}{z(c+1)}\left[\left(1-\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}\right)A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)I(z)\right.\\
&\left.+\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)I(z)\right]\\
&+\frac{1}{z(c+1)}\left[\left(1-\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}\right)A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)(zI'(z))\right.\\
&\left.+\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)(zI'(z))\right]\\
=&\frac{c}{c+1}J(z)+\frac{1}{c+1}(zJ'(z)+J(z)) = J(z)+\frac{1}{c+1}zJ'(z)\prec h(z).
\end{aligned}$$

An application of Lemma 1.1 with $\mu = c+1$, yields $J(z) \prec h(z)$. By using (2.17), we get

$$\frac{1}{z}\left[\left(1-\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}\right)A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)I(z)+\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)I(z)\right]\prec h(z),$$

which implies that $I \in \mathcal{M}(\eta,\sigma,\tau,\lambda,\delta,\alpha,\beta,m;h)$. □

Theorem 2.5. *Let $f \in \mathcal{M}(\eta,\sigma,\tau,\lambda,\delta,\alpha,\beta,m;h)$, $g \in \mathcal{A}_m$ and*

$$\operatorname{Re}\left\{\frac{g(z)}{z}\right\} > \frac{1}{2}. \tag{2.18}$$

Then $f * g \in \mathcal{M}(\eta,\sigma,\tau,\lambda,\delta,\alpha,\beta,m;h)$.

Proof. Let $f \in \mathcal{M}(\eta,\sigma,\tau,\lambda,\delta,\alpha,\beta,m;h)$ and $g \in \mathcal{A}_m$. Then, we have

$$\begin{aligned}(2.19)\qquad &\frac{1}{z}\left[\left(1-\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}\right)A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)(f*g)(z)\right.\\ &\left.+\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)(f*g)(z)\right]\\ =&\left(1-\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}\right)\left(\frac{g(z)}{z}\right)*\left(\frac{A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f(z)}{z}\right)\\ &+\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}\left(\frac{g(z)}{z}\right)*\left(\frac{A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)f(z)}{z}\right)=\left(\frac{g(z)}{z}\right)*\varphi(z),\end{aligned}$$

where

(2.20)

$$\begin{aligned}\varphi(z)=\frac{1}{z}&\left[\left(1-\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}\right)A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f(z)+\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)f(z)\right]\\ &\prec h(z).\end{aligned}$$

In view of (2.18), we note that the function $\frac{g(z)}{z}$ has the Herglotz representation

$$\frac{g(z)}{z}=\int_{|x|=1}\frac{d\mu(x)}{1-xz}\quad (z\in U), \tag{2.21}$$

where $\mu(x)$ is a probability measure defined on the unit circle $|x|=1$ and

$$\int_{|x|=1} d\mu(x)=1.$$

Since h is convex univalent in U, then we find from (2.19), (2.20) and (2.21) that

$$\begin{aligned}&\frac{1}{z}\left[\left(1-\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}\right)A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)(f*g)(z)+\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}\right.\\ &\left.\times A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)(f*g)(z)\right]\end{aligned}$$

$$= \int_{|x|=1} \varphi(xz)\, d\mu(x) \prec h(z).$$

This shows that $f * g \in \mathcal{M}(\eta, \sigma, \tau, \lambda, \delta, \alpha, \beta, m; h)$. □

Theorem 2.6. *Let $f \in \mathcal{M}(\eta, \sigma, \tau, \lambda, \delta, \alpha, \beta, m; h)$ and $g \in \mathcal{A}_m$ be prestarlike of order α, $(\alpha < 1)$. Then $f * g \in \mathcal{M}(\eta, \sigma, \tau, \lambda, \delta, \alpha, \beta, m; h)$.*

Proof. For $f \in \mathcal{M}(\eta, \sigma, \tau, \lambda, \delta, \alpha, \beta, m; h)$ and $g \in \mathcal{A}_m$, from (2.19) (used in the proof of Theorem 2.5), we can write

$$\frac{1}{z}\left[\left(1 - \frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta + n\delta}\right) A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)(f*g)(z) + \frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta + n\delta} \right. \tag{2.22}$$

$$\left. \times A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)(f*g)(z)\right] \tag{2.23}$$

$$= \frac{g(z) * (z\varphi(z))}{g(z) * z} \quad (z \in U),$$

where $\varphi(z)$ is defined as in (2.20). Since h is convex univalent in U, $\psi(z) \prec h(z)$, $g(z) \in \operatorname{Re}(\alpha)$ and $z \in S^*(\alpha)$, $\alpha < 1$, it follows from (2.22) and Lemma 1.2, we obtain the result. □

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