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GEOMETRIC INVARIANTS UNDER THE MÖBIUS ACTION OF THE GROUP $SL(2;\mathbb{R})$

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Abstract. In this paper we have introduced new invariant geometric objects in the homogeneous spaces of complex, dual and double numbers for the principal group $SL(2;\mathbb{R})$, in the Klein's Erlangen Program. We have considered the action as the Möbius action and have taken the spaces as the spaces of complex, dual and double numbers. Some new decompositions of $SL(2;\mathbb{R})$ have been used.

1. Introduction

In this paper, we have described and extended the geometry of the group $SL(2;\mathbb{R})$ on the two-dimensional space in the line of the Erlangen program defined by Felix Klein. The Erlangen program states that, have a geometric space and a transformation group, a geometry is the study of the invariance of geometric objects under a group action of that transformation group [11, 17]. A geometry is often referred to as a pair (G, X), where G is the transformation group and X is the geometric space. This pair is called Klein's geometry [14]. Vladimir V. Kisil has shown three geometries under linear fractional transformation of the group $SL(2;\mathbb{R})$ [8, 10]. By the geometric spaces taken, they are classified as elliptic, parabolic and hyperbolic cases [8]. The elliptic case is isomorphic to the upper half plane of the space of complex numbers. Similarly, parabolic case and hyperbolic cases are isomorphic to the upper half plane of spaces of dual and double numbers respectively, [16]. Similar work can be done for lower half plane also. Kisil in his paper [8] worked on the geometric objects which are lying strictly in the upper half planes of complex, dual and double numbers. He did not

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mention about other geometric objects. In this paper, we have taken the geometric objects of paper [8] which intercept the U-axis at two real points and those which have U-axis as a tangent and showed that they are also invariants under some restrictions.

The aim of our work is to extend the work of Kisil in the same line to include more invariant geometric objects in the existing $SL(2;\mathbb{R})$ geometry. The reason behind it is to fill the gaps in the existing geometries of $SL(2;\mathbb{R})$. Erlangen program of $SL(2;\mathbb{R})$ applications in mathematics and theoretical physics, e.g optics, classical mechanics (see Section 5), functional calculus [7] etc.

In this paper, we restrict ourselves to the geometry of the Lie group $SL(2;R)$. However interested readers may find construction of some other pairs of Klein's geometry in [14, 15] of the type (G, X), where G is a principal group and X be a geometric space.

In next section we have discussed some of the terminologies and results of our predecessors in this subject and in section 3 and 4 we mentioned our results. After that some applications in physics have been described in Section 5.

2. Preliminaries

In our paper, we shall use the following terminologies.

Definition 2.1 (Transformation group)**.** A transformation group G is a non-void set of mappings of a set X into itself with the following properties:

(a) the identity map is included in G;
(b) if $g_1 \in G$ and $g_2 \in G$, then $g_1 g_2 \in G$;
(c) if $g \in G$, then g^{-1} exists and belongs to G.

Definition 2.2 (Homogeneous space)**.** A topological space X together with an abstract group $(G, *)$, which acts on X transitively is said to be a homogeneous space [4].

Definition 2.3 (Isotropy subgroup)**.** For an abstract group G and for a group action of it on a set X, the set of elements $G_x = \{g \in G : \ g \cdot x = x\}$ forms a subgroup of G which is called the isotropy (fix) subgroup of G by x.

In his paper [8], Kisil have shown that if H is a one-dimensional subgroup of $SL(2;\mathbb{R})$, namely $K = \left\{ \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} : \ t \in \mathbb{R} \right\}$, $N = \left\{ \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix} : \ \nu \in \mathbb{R} \right\}$ and $A = \left\{ \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} : \ \alpha \ (> 0) \in \mathbb{R} \right\}$, then $SL(2;\mathbb{R})/H$ $(H = K, N, A)$ is a homogeneous space.

Dual numbers and double numbers are defined by $\mathbb{O} = \{u + iv : i^2 = 0,\ (u, v) \in \mathbb{R}^2\}$ and $\mathbb{D} = \{u + iv : i^2 = 1,\ (u, v) \in \mathbb{R}^2\}$, respectively.

Complex numbers with dual and double numbers are denoted as

$$\mathbb{R}^\sigma = \{a + ib : \ i^2 = \sigma = -1, 0, 1,\ (a, b) \in \mathbb{R}^2\}.$$

The Möbius action is defined as $g : SL(2;\mathbb{R}) \times \mathbb{R}^\sigma \to \mathbb{R}^\sigma$ by

$$g \cdot z = \frac{az+b}{cz+d},$$

for $z \in \mathbb{R}^\sigma$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2;\mathbb{R})$. It is a left action of $SL(2;\mathbb{R})$ on $\mathbb{R}^\sigma$, i.e., $g_1(g_2 \cdot z) = (g_1 g_2) \cdot z$.

To study the action we shall decompose $g \in SL(2;\mathbb{R})$ by the Iwasawa decomposition as $g = g_a g_n g_k$, where $g_a \in A$, $g_n \in N$ and $g_k \in K$ [8,12].

From now on througout this paper we shall denote elliptic case ($SL(2;\mathbb{R})/K$) which is isomorphic to the upper half plane of the space of complex numbers [8], parabolic case ($SL(2;\mathbb{R})/N$) which is isomorphic to the upper half plane of dual numbers and hyperbolic case ($SL(2;\mathbb{R})/A$) which is isomorphic to the upper half plane of double numbers namely and in short we denote it as EPH-cases.

In this paper we shall refer to **cycles** [8], as straight lines and one of the following. Circles in the elliptic case, parabolas (with vertical axis of symmetry) in the parabolic case and rectangular hyperbolas (with vertical axis of symmetry) in the hyperbolic case. Also, the word parabola and hyperbola in this paper always assume only one of the above described types.

The **center** of a cycle is referred to as :

1. center of a circle in the elliptic case;
2. focus of a parabola in the parabolic case and
3. center of a rectangular hyperbola in the hyperbolic case.

The **vertex** of a cycle is referred to as:

1. the lowest point of a circle;
2. vertex of a parabola and
3. vertex of a rectangular hyperbola, in each of the three EPH cases.

We can define the **radius** of a **cycle** as:

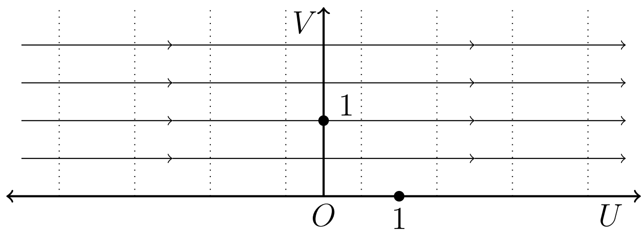
1. radius of a circle in the elliptic case;
2. distance between center and vertex of a parabola in the parabolic case;
3. distance between center and vertex of a hyperbola in the hyperbolic case.

In the next subsections 2.1 and 3.1, we shall discuss some of the results stated by Kisil [8]. We shall give their proof in details. Earlier most of the works (of subsections 2.1 and 3.1) have been proved using CAS (computer algebra system), by brute-force calculations [8,10] or, given a short proof.

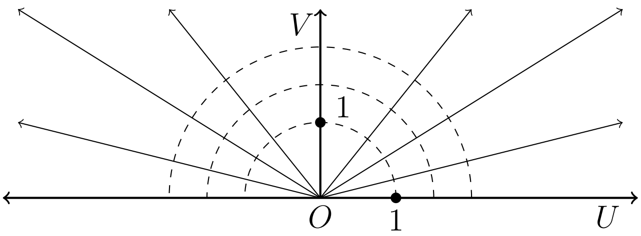
2.1. **Action of the subgroups.**

Lemma 2.1. *The action of the subgroup N under Möbius transformation on $\mathbb{R}^\sigma$ is $g_n \cdot (u,v) = (u+\nu,\ v)$, where $g_n = \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix}$ and $(u,v) \in \mathbb{R}^\sigma$, which defines shifts along the real axis U by ν (see [8]).*

Lemma 2.2. *The action of the subgroup A under Möbius transformation on $\mathbb{R}^\sigma$ is* $g_n \cdot (u, v) = \alpha^{-2}(u,\ v)$, *where* $g_a = \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix}$, $\alpha > 0$, *and* $(u, v) \in \mathbb{R}^\sigma$, *which defines dilations by the factor* α^{-2}, *which fixes the origin* $(0, 0)$ (*see* [8]).



(A) N-orbit



(B) A-orbit

Figure 1. Actions of different subgroups of $SL(2;\mathbb{R})$

Theorem 2.1. *A K-orbit in $\mathbb{R}^\sigma$ passing through the point $(0, t)$ has the following equation* (*see* [8])

$$u^2 - \sigma v^2 - v(t^{-1} - \sigma t) + 1 = 0. \tag{2.1}$$

Proof. If (u, v) be any point on the $K-$orbit passing through $(0, t)$ then it can be determined as follows.

$$\begin{aligned} u + iv &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot (it) = \frac{(it)\cos\theta - \sin\theta}{(it)\sin\theta + \cos\theta} \\ &= \frac{(-\sin\theta - it\cos\theta)(\cos\theta - it\sin\theta)}{\cos^2\theta - i^2\sin^2\theta}. \end{aligned}$$

There are three cases to follow.

(a) In the elliptic case ($i^2 = -1$),

$$u + iv = \frac{-\sin\theta\cos\theta + t^2\sin\theta\cos\theta}{\cos^2\theta + t^2\sin^2\theta} + i\frac{t}{\cos^2\theta + t^2\sin^2\theta},$$

then $u = \frac{(t^2-1)\sin\theta\cos\theta}{\cos^2\theta + t^2\sin^2\theta}$ and $v = \frac{t}{\cos^2\theta + t^2\sin^2\theta}$.

From the above expressions of u and v, we get $u^2 + v^2 - v(t^{-1} + t) = -1$.

(b) In the parabolic case ($i^2 = 0$),

$$u + iv = -\frac{\sin\theta\cos\theta}{\cos^2\theta} = i\frac{t}{\cos^2\theta},$$

then $u = -\tan\theta$ and $v = t\sec^2\theta$. Thus, $u^2 - vt^{-1} = -1$.

(c) In the hyperbolic case ($i^2 = 1$),

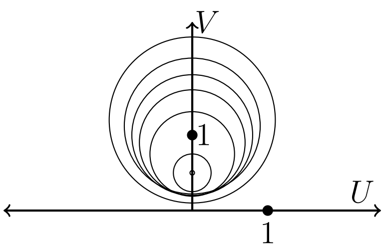
$$u + iv = -\frac{\sin\theta\cos\theta - t^2\sin\theta\cos\theta}{\cos^2\theta - t^2\sin^2\theta} + i\frac{t}{\cos^2\theta - t^2\sin^2\theta},$$

then $u = \frac{-(t^2+1)\sin\theta\cos\theta}{\cos^2\theta - t^2\sin^2\theta}$ and $v = \frac{t}{\cos^2\theta - t^2\sin^2\theta}$.

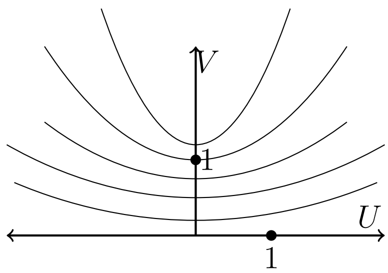
Thus, $u^2 - v^2 - v(t^{-1} - t) = -1$.

Combining the above three cases, we get the K-orbit as (Figure 2)

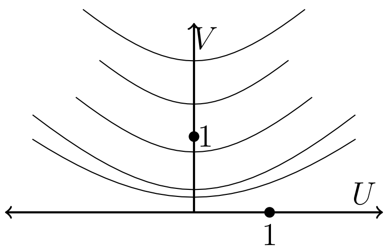
$$u^2 - \sigma v^2 - v(t^{-1} - \sigma t) + 1 = 0, \quad \text{for } \sigma = i^2 = -1, 0, 1.$$



(A) Elliptic case



(B) Parabolic case



(C) Hyperbolic case

FIGURE 2. Orbits of the subgroup K in *EPH*-cases

□

Remark 2.1. The shape of a geometric object in $\mathbb{R}^\sigma$ is solely dependent upon the action of the subgroup K.

Theorem 2.2. *The curvature of the K-orbit* (*at the vertex*) *in the elliptic, parabolic and hyperbolic cases are* (*see* [8])

$$\kappa = \frac{2t}{1 + \sigma t^2}, \quad \sigma = -1, 0, 1. \tag{2.2}$$

Proof. There are three cases to follow.

(a) We know that the curvature of a circle is inverse of its radius. As our K-orbit in the elliptic case is a circle with equation

$$u^2+\left(v-\frac{t^{-1}+t}{2}\right)^2=\left(\frac{t^{-1}-t}{2}\right)^2,$$

therefore its curvature would be $\kappa_e=\frac{2}{t^{-1}-t}=\frac{2t}{1-t^2}$.

(b) In the parabolic case, a K-orbit is a parabola with equation

$$u^2=\frac{1}{t}(v-t).$$

For determining the curvature, let us suppose that its parametric equation is $u=\frac{r}{t}$ and $v=t+\frac{r^2}{t}$, where r being an arbitrary parameter. The differential coefficients of u and v with respect to r at vertex (i.e when $r=0$) are $u'=\frac{1}{t},\ v'=0,\ u''=0$ and $v''=\frac{2}{t}$. Therefore, the curvature at the vertex is

$$\kappa_p\bigg|_{r=0}=\left|\frac{u'v''-u''v'}{(u'^2+v'^2)^{\frac{3}{2}}}\right|=2t.$$

(c) In the hyperbolic case, the K-orbit can be written as

$$\left(v+\frac{t^{-1}-t}{2}\right)^2-u^2=\left(\frac{t^{-1}+t}{2}\right)^2,$$

or, $(v-b)^2-u^2=r^2$ (say). We parametrize u and v as $u=r\tan\theta,\ v=b+r\sec\theta$, where θ is the parameter. The differential coefficients of u and v with respect to θ at the vertex (i.e., when $\theta=0$) are $u'=r,\ v'=0,\ u''=0$ and $v''=r$. Its curvature at the vertex is

$$\kappa_h\bigg|_{\theta=0}=\left|\frac{u'v''-u''v'}{(u'^2+v'^2)^{\frac{3}{2}}}\right|=\frac{1}{r}=\frac{2}{1+t^2}.$$

Combining the above three cases, we get the curvature of the K-orbit as

$$\kappa=\frac{2t}{1+\sigma t^2},\quad \sigma=-1,0,1.$$

□

Next theorem has been proved in short earlier in the paper [8] we shall give its detailed proof.

Theorem 2.3. *Möbius transformation preserves cycles in the upper half plane,* [8].

Proof. We know by the Lemmas 2.1 and 2.2 that the subgroups N and A produces shifts and dilations, respectively. Also, the subgroup K has orbits which are either circles, parabolas or hyperbolas. We have to prove that Möbius transformations preserve the cycles in the upper half plane.

Our first observation is that the subgroups A and N obviously preserve all circles, parabolas, hyperbolas and straight lines in all $\mathbb{R}^\sigma$. Thus we use subgroups A and N to fit a given cycle exactly on a particular orbit of subgroup K shown on Figure

2 of the corresponding type. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2;\mathbb{R})$ be an arbitrary Möbius transformation. We shall show that gC is of the same type of cycle as C.

To this end, for an arbitrary cycle C, we can find $g'_n \in N$ which puts the centre of C on the V-axis (see Figure 3).

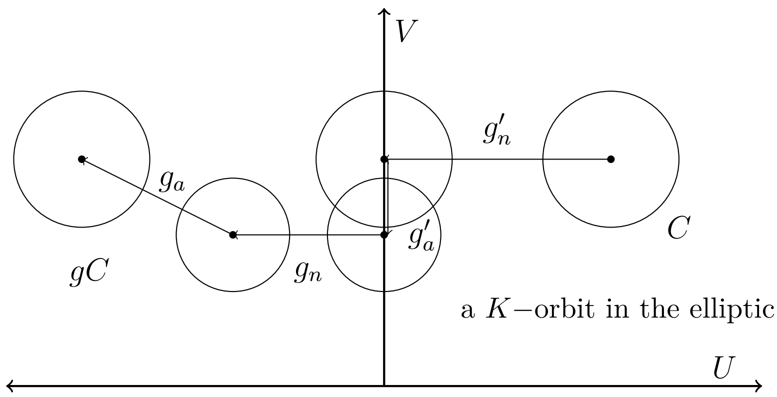


FIGURE 3. Pictorial representation of the proof of Theorem 2.3

Then there is a unique $g'_a \in A$ which scales it exactly to an orbit of K. To be more precise, we calculate the values of the shift factor and scaling factor in order to determine the factors of g'_n and g'_a for all the three EPH-cases.

Suppose we consider any cycle with centre (f, h) and vertex (u, v) and translate it to the V-axis by a shift factor f, then the co-ordinates of the center become $(0, h)$ and vertex become $(0, v)$. Therefore, the shift factor is f and of the form g'_n is $\begin{pmatrix} 1 & -f \\ 0 & 1 \end{pmatrix}$ in all three EPH-cases. Next we find the form of g'_a. Let us consider the cycle with the given vertex $(0, v_1)$ and curvature κ_1 at the point $(0, v_1)$. Now in order to fit this cycle into the orbit whose vertex and curvature are given by $(0, v)$ and κ, we calculate the scaling factor α, as follows. We have $v = \alpha v_1$ and $\kappa = \frac{\kappa_1}{\alpha}$, i.e., $\alpha = \frac{v}{v_1}$ and $v\kappa = v_1\kappa_1 = v_1\frac{2v_1}{1+\sigma v_1^2} = \frac{2}{\sigma + v_1^{-2}}$ (cf. Theorem 2.2). Therefore, $v_1 = \sqrt{\frac{v\kappa}{2-\sigma v\kappa}}$. This shows $\alpha = \frac{v}{v_1} = \sqrt{\frac{\kappa}{v(2-\sigma v\kappa)}}$, which is the scaling factor and $g'_a = \begin{pmatrix} \frac{1}{\sqrt{\alpha}} & 0 \\ 0 & \sqrt{\alpha} \end{pmatrix}$ for all the EPH cases.

Next, we show that this scaling factor α if exists then it is unique. Indeed, suppose that α and α' are both scaling factors used to bring the vertex $(0, v_1)$ to fit into the orbit with vertex $(0, v)$. Then we have $v = \alpha v_1$ and $v = \alpha' v_1$, i.e., $(\alpha - \alpha')v_1 = 0$. Now, we know that $v_1 \neq 0$, therefore, $\alpha - \alpha' = 0$, i.e., $\alpha = \alpha'$. This shows that the scaling factor is unique.

Let us take

$$g' = g(g'_a g'_n)^{-1} = \begin{pmatrix} \dfrac{a}{\sqrt{\alpha}} & \dfrac{ua+b}{\sqrt{\alpha}} \\ c\sqrt{\alpha} & \dfrac{uc+d}{\sqrt{\alpha}} \end{pmatrix}.$$

As $g' \in SL(2;\mathbb{R})$ using Iwasawa decomposition we get $g' = g_a g_n g_k$, for some $g_a \in A$, $g_n \in N$ and $g_k \in K$. Now,

$$\begin{aligned} gC &= g(g'_a g'_n)^{-1}(g'_a g'_n C) = (g_a g_n g_k)(g'_a g'_n C) \\ &= g_a g_n (g_k (g'_a g'_n C) = g_a g_n g'_a g'_n C \quad \text{(as K-orbits are K-invariant).} \end{aligned}$$

Also, g_a, g_n, g'_a, g'_n do not change the shape of the cycle therefore gC is the same type of cycle as C. □

3. Introduction of Two New Subgroups A' and N' of $SL(2;\mathbb{R})$

In the earlier section 2.1 where we have seen the importance of K-subgroups in $SL(2;\mathbb{R})$ geometry. In this section we introduce two subgroups of matrices A' and N' defined by

$$A' = \left\{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} : t \in \mathbb{R} \right\}, \quad N' = \left\{ \begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix} : \nu \in \mathbb{R} \right\}$$

in place of the subgroup K to obtain our results. We first describe their orbits.

Proposition 3.1. *The subgroups* $A' = \left\{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} : t \in \mathbb{R} \right\}$ *and* $N' = \left\{ \begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix} : \nu \in \mathbb{R} \right\}$ *are conjugate to* $A = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} : t \in \mathbb{R} \right\}$ *and* $N = \left\{ \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix} : \nu \in \mathbb{R} \right\}$, *respectively.*

Proof. This can be seen by the following two equations

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \cosh t & \sinh t \\ \cosh t & \sinh t \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -\nu \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix}.$$

Therefore, any matrix of A' is similar to a matrix of A and any matrix of N' is similar to a matrix of N. □

3.1. Orbits of the two new subgroups A' and N'.

Theorem 3.1. *The orbits of the subgroup A' are cycles*

$$u^2 - \sigma v^2 + (t^{-1} + \sigma t)v - 1 = 0, \qquad \text{where } \sigma = i^2 = -1, 0, 1,$$

which intercept the U-axis at two real points $(1,0)$ and $(-1,0)$ and have center on the V-axis in all the three elliptic, parabolic and hyperbolic cases.

Proof. We have the following three cases.

(a) **Elliptic case**. If (u,v) be an arbitrary point on the A'-orbit which passes through $(0,t)$, then

$$\begin{aligned} u + iv &= \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \cdot (it) = \frac{it\cosh\theta + \sinh\theta}{it\sinh\theta + \cosh\theta} \\ &= \frac{(1+t^2)\cosh\theta\sinh\theta}{\cosh^2\theta + t^2\sinh^2\theta} + i\frac{t}{\cosh^2\theta + t^2\sinh^2\theta}. \end{aligned}$$

Eliminating θ, we get $u^2 + v^2 = 1 + (t - t^{-1})v$.

(b) **Parabolic case**. Similar to the previous case if (u,v) is an arbitrary point on the orbit which passes through $(0,t)$, then

$$u + iv = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \cdot (it) = \frac{it\cosh\theta + \sinh\theta}{it\sinh\theta + \cosh\theta} = \frac{\sinh\theta}{\cosh\theta} + i\frac{t}{\cosh^2\theta}.$$

Therefore, the orbit would be $u^2 = -\frac{1}{t}(v - t)$.

(b) **Hyperbolic case** Again by taking an arbitrary point (u,v) in the orbit and assuming that it passes through $(0,t)$, we have

$$\begin{aligned} u + iv &= \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \cdot (it) = \frac{it\cosh\theta + \sinh\theta}{it\sinh\theta + \cosh\theta} \\ &= \frac{(1-t^2)\cosh\theta\sinh\theta}{\cosh^2\theta - t^2\sinh^2\theta} + i\frac{t}{\cosh^2\theta - t^2\sinh^2\theta}. \end{aligned}$$

Eliminating θ, we get $u^2 - v^2 = 1 - (t + t^{-1})v$, which is a hyperbola intercepting the U-axis at two real points.

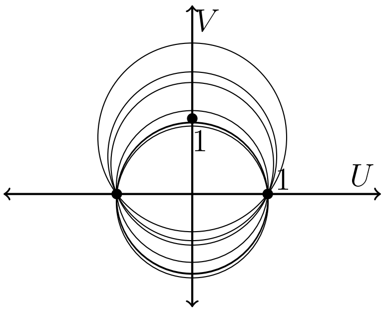
Combining the above three cases we get the A'-orbit as

$$u^2 - \sigma v^2 + (t^{-1} + \sigma t)v - 1 = 0, \quad \sigma = i^2 = -1, 0, 1. \tag{3.1}$$

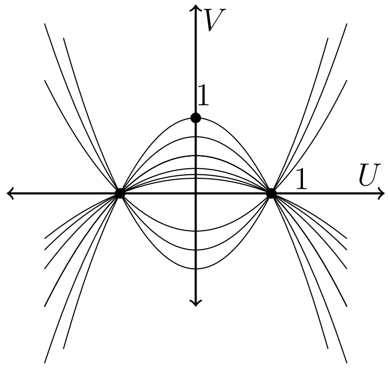
Putting $v = 0$ we get $u^2 = 1$, i.e., $u = \pm 1$. Thus, they are cycles intercepting the U-axis at $(1,0)$ and $(-1,0)$ unlike the K-orbit. □

Theorem 3.2. *In an A'-orbit*

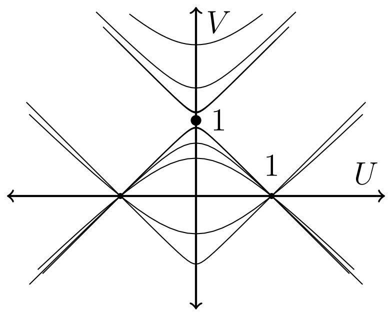
(a) *in the elliptic case, the relation between radius r_e and center $(0, v_e)$ is $r_e^2 - v_e^2 = 1$;*
(b) *in the parabolic case, the relation between vertex $(0, v_p)$ and center (focus) $(0, r_p/4)$ is $r_p v_p = 1$ and*



(A) Circles intercept the U-axis



(B) Parabolas intercepting the U-axis



(C) Rectangular hyperbolas intercepting the U-axis

FIGURE 4. The orbits of the subgroup A' in elliptic, parabolic and hyperbolic cases

(c) *in the hyperbolic case, the relation between center* $(0, v_h)$ *and* r_h *which is the distance between the center and vertex is* $v_h^2 - r_h^2 = 1$.

Proof. There are three cases to follow.

(a) **Elliptic case.** In this case, the A'-orbit is $u^2 + (v - v_e)^2 = r_e^2$, where r_e and v_e are of the form $r_e = \frac{t+t^{-1}}{2}$ and $v_e = \frac{t-t^{-1}}{2}$ (for some $t \in \mathbb{R}$). Therefore, by a simple calculation we can show $r_e^2 - v_e^2 = \left(\frac{t+t^{-1}}{2}\right)^2 - \left(\frac{t-t^{-1}}{2}\right)^2 = 1$.

(b) **Parabolic case.** In this case, the A'-orbit is $u^2 = r_p(v + v_p)^2$, we have center $(0, r_p/4)$ and vertex is $(0, v_p)$, then r_p and v_p are of the form $r_p = -\frac{1}{t}$ and $v_p = -t$ for some $t \in \mathbb{R}$. Therefore, $r_p v_p = 1$.

(c) **Hyperbolic case.** The A'-orbit in this case is $(v - v_h)^2 - u^2 = r_h^2$. We have $v_h = \frac{t+t^{-1}}{2}$ and $r_h = \frac{t-t^{-1}}{2}$ for some $t \in \mathbb{R}$, then $v_h^2 - r_h^2 = 1$. □

Remark 3.1. The subgroups A' and N' come naturally in Möbius action of $SL(2;\mathbb{R})$. In the next proposition we show that.

Proposition 3.2. *The isotropy (fix) subgroups of* $(0,1) \in \mathbb{R}^\sigma$ *under the Möbius action of* $SL(2;\mathbb{R})$ *are* K, A' *and* N'.

Proof. Let $i^2 = -1, 0, 1$, for the three cases elliptic, parabolic and hyperbolic. Then if $\frac{ai+b}{ci+d} = i$, then by calculation, we get

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix},$$

for the three cases, respectively. □

Theorem 3.3. *The orbits of the subgroup* N' *are*

$$u^2 - \sigma v^2 - v(t^{-1} - \sigma t) = 0, \quad \sigma = i^2 = -1, 0, 1.$$

They are circles, parabolas and rectangular hyperbolas which are tangent to the U*-axis and have center on the* V*-axis in the elliptic, parabolic and hyperbolic cases respectively.*

Proof. We have the following three cases.

(a) **Elliptic case.** If (u, v) be an arbitrary point on N'-orbit which passes through $(1, t)$ (If we take $(0, t)$ point, then it is only $(0, 0)$ in the parabolic case), then

$$u + iv = \begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix} \cdot (1 + it) = \frac{1 + it}{\nu(1 + it) + 1} = \frac{1 + \nu + \nu t^2}{(1 + \nu)^2 + \nu^2 t^2} + i \frac{t}{(1 + \nu)^2 + \nu^2 t^2}.$$

Eliminating ν we get the orbit as

$$u^2 + v^2 - v(t^{-1} + t) = 0. \tag{3.2}$$

(b) **Parabolic case.** If the cycle passes through $(1, t)$, then any arbitrary point (u, v) on the N'-orbit would be

$$u + iv = \begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix} \cdot (1 + it) = \frac{1 + it}{\nu + i\nu t + 1} = \frac{1}{1 + \nu} + i \frac{t}{(1 + \nu)^2}.$$

The orbit would be

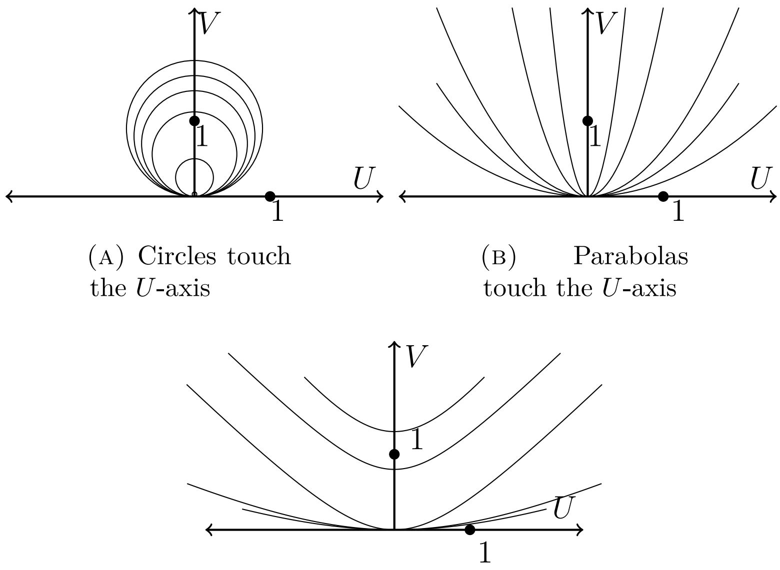
$$u^2 - vt^{-1} = 0. \tag{3.3}$$

(c) **Hyperbolic case.** If (u, v) be any arbitrary point on the orbit and it passes through $(1, t)$, then

$$u + iv = \begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix} \cdot (1 + it) = \frac{1 + it}{\nu + i\nu t + 1} = \frac{1 + \nu + \nu t^2}{(1 + \nu)^2 - \nu^2 t^2} + i \frac{t}{(1 + \nu)^2 - \nu^2 t^2}.$$

Eliminating ν we get the orbit as

$$u^2 - v^2 - v(t^{-1} - t) = 0.$$



(A) Circles touch the U-axis

(B) Parabolas touch the U-axis

(C) Rectangular hyperbolas touch the U-axis

FIGURE 5. Cycles which are tangent to the U-axis

Combining the above three cases we get

$$u^2 - \sigma v^2 - v(t^{-1} - \sigma t) = 0, \quad \sigma = -1, 0, 1. \tag{3.4}$$

As such the orbits of N' are cycles which are tangent to the U-axis. □

Remark 3.2. The center of a N'-orbit passing through $(1, t)$ is of the form $\left(0, \frac{t^{-1}-\sigma t}{2}\right)$ for some $t \in \mathbb{R}$ and $\sigma = -1, 0, 1$, in EPH cases.

Proposition 3.3. *Orbits of the isotropy subgroups of K, N' and A' of $(0, 1)$ in $\mathbb{R}^\sigma$ in elliptic, parabolic and hyperbolic cases are*

$$u^2 - \sigma v^2 - 2lv - \sigma = 0, \qquad \text{where } l \in \mathbb{R}.$$

4. Main Result

In this section we develop our work.

4.1. **Decomposition of $SL(2; \mathbb{R})$ using the subgroups A' and N'.** In this subsection we define two other decomposition of $SL(2; \mathbb{R})$ in order to include new invariances of types of cycles for the transformation group $SL(2; \mathbb{R})$.

Theorem 4.1. *Any matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2; \mathbb{R})$, where $d \neq 0$ can be represented uniquely as $g = g_a g_n g_{n'}$, where $g_a \in A, g_n \in N$ and $g_{n'} \in N'$.*

Proof. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, let $g_a = \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix}$, $g_n = \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix}$ and $g_{n'} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$. Then $g = g_a g_n g_{n'}$, i.e.,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} \cdot \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \begin{pmatrix} \alpha^{-1}(1+t\nu) & \alpha^{-1}\nu \\ \alpha t & \alpha \end{pmatrix}.$$

Therefore, $d = \alpha, t = \frac{c}{d}$ and $\nu = bd$.

The uniqueness can be obtained by using another decomposition with the same form and showing that the corresponding matrices are equal. □

Theorem 4.2. *Any matrix* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2;\mathbb{R})$, *where* $|d| > |c|$ *can be represented uniquely as* $g = g_a g_n g_{a'}$, *where* $g_a \in A, g_n \in N$ *and* $g_{a'} \in A'$.

Proof. Let $g_a = \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix}$, $g_n = \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix}$ and $g_{a'} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$ for $\alpha, t, \nu \in \mathbb{R}$ and $\alpha > 0$. Then $g = g_a g_n g_{a'}$, i.e.,

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} \cdot \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \\ &= \begin{pmatrix} \alpha^{-1}(\cosh t + \nu \sinh t) & \alpha^{-1}(\sinh t + \nu \cosh t) \\ \alpha \sinh t & \alpha \cosh t \end{pmatrix}. \end{aligned}$$

Thus, $\alpha = \sqrt{d^2 - c^2}$, $t = \tanh^{-1}\left(\frac{c}{d}\right)$ and $\nu = bd - ac$.

One can prove the uniqueness of the decomposition by taking another decomposition with the same form and showing that the corresponding matrices are equal. □

4.2. **Invariance of cycles in $\mathbb{R}^\sigma$.** We now introduce new types of cycles in the existing $SL(2;\mathbb{R})$ geometries [8,9] which have important applications (see Section 5).

Theorem 4.3. *If C is any arbitrary cycle which have U-axis as a tangent in the space of $\mathbb{R}^\sigma$ with center (u, v), then it is invariant under the Möbius action of* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2;\mathbb{R})$ *if* $u > -\frac{d}{c}$.

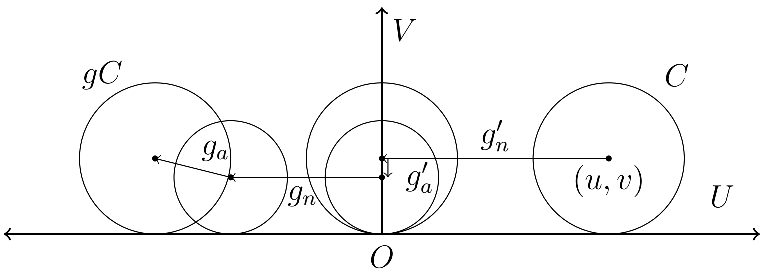


FIGURE 6. Pictorial representation of the proof of the Theorem 4.3

Proof. The Möbius action on the cycle C by the matrix $g'_n = \begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix}$ shifts its center (u, v) to $(0, v)$. Then applying Möbius action again on $g'_n C$ by $g'_a = \begin{pmatrix} \frac{1}{\sqrt{\alpha}} & 0 \\ 0 & \sqrt{\alpha} \end{pmatrix}$, $\alpha > 0$, we scale its center to $(0, v\alpha^{-1})$.

As N'-orbit have center of the form $\frac{t^{-1}-\sigma t}{2}$, by Remark 3.3, as such if we want $g'_a g'_n C$ to be a N'-orbit then we must have $v\alpha^{-1} = \frac{t^{-1}-\sigma t}{2}$, $\sigma = -1, 0, 1,\ t \in \mathbb{R}$. Therefore, the scaling factor $\alpha = \frac{2vt}{1-\sigma t^2}$.

The uniqueness of g'_a can be proved by taking another $g''_a = \begin{pmatrix} \frac{1}{\sqrt{\alpha'}} & 0 \\ 0 & \sqrt{\alpha'} \end{pmatrix}$. By the same calculations we get $\alpha' = \frac{2vt}{1-\sigma t^2}$. Therefore, $g'_a = g''_a$.

Now,

$$g' = g(g'_a g'_n)^{-1} = \begin{pmatrix} a\sqrt{\alpha} & (au+b)\sqrt{\alpha^{-1}} \\ c\sqrt{\alpha} & (cu+d)\sqrt{\alpha^{-1}} \end{pmatrix} \in SL(2; \mathbb{R}).$$

If $g(g'_a g'_n)^{-1}$ have a decomposition of the form of Theorem 4.1, then $cu + d > 0$, i.e., $u > -\frac{d}{c}$.

We can decompose $g(g'_a g'_n)^{-1} = g_a g_n g_{n'}$ for some $g_a \in A,\ g_n \in N$ and $g_{n'} \in N'$. Now,

$$\begin{aligned} gC &= g(g'_a g'_n)^{-1} g'_a g'_n C = g_a g_n g_{n'} (g'_a g'_n C) \\ &= g_a g_n g'_a g'_n C \quad \text{(as } g'_a g'_n C \text{ is a } N'\text{−orbit therefore it is } N'\text{-invariant).} \end{aligned}$$

As g_a, g'_a, g_n, g'_n do not change the shape of C, by Remark 2.1, therefore, gC would be the same type of cycle as C geometrically. □

Theorem 4.4. *If C is an arbitrary cycle with center (u_λ, v_λ) and radius r_λ which intercepts the U-axis at two real points then it is invariant under the Möbius action of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2; \mathbb{R})$ and if it satisfies $|c|\alpha < |cu_\lambda + d|$, where $\alpha = \sqrt{(v_\lambda^2 - r_\lambda^2)\sigma + r_\lambda v_\lambda (1 - |\sigma|)}$, $\sigma = -1, 0, 1$, and $\lambda = e, p, h$ in elliptic, parabolic and hyperbolic cases, respectively.*

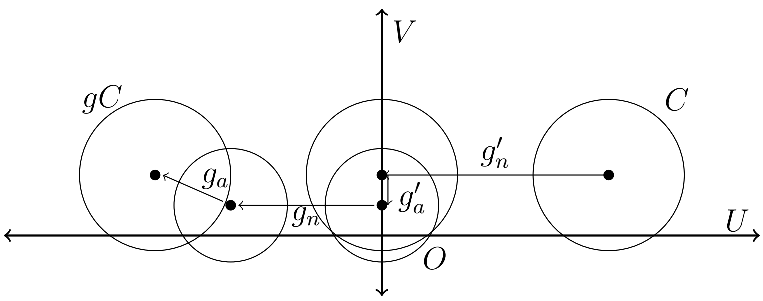


FIGURE 7. Pictorial representation of the proof of the Theorem 4.4

Proof. We consider the following three cases.

(a) **Elliptic case.** Let us take an arbitrary circle $(u-u_e)^2+(v-v_e^2)^2=r_e^2$ with center (u_e,v_e) and radius r_e which intercepts the U-axis at two real points therefore $|r_e|>|v_e|$.

First we translate the center of C to V-axis by a shift factor u_e, then the coordinates of the center become $(0,v_e)$ and the matrix which shifts is of the form $g'_n=\begin{pmatrix}1 & -u_e\\ 0 & 1\end{pmatrix}$. Next to fit the circle into an A'-orbit, we take Möbius action by the matrix $g'_a\in A$ defined as $g'_a=\begin{pmatrix}\frac{1}{\sqrt{\alpha_e}} & 0\\ 0 & \sqrt{\alpha_e}\end{pmatrix}$, $\alpha_e>0$. The radius and center (see Remark 2.1) become $r_e\alpha^{-1}$ and $(0,v_e\alpha^{-1})$. If $g'_ag'_nC$ is an A'-orbit, then using the Theorem 3.2 (of A'-orbit) we get $(r\alpha_e^{-1})^2-(v_e\alpha_e^{-1})^2=1$, i.e., $\alpha_e=\sqrt{r_e^2-v_e^2}$.

(b) **Parabolic case.** In this case we take an arbitrary parabola (with vertical axis of symmetry) $(u-u_p)^2=r_p(v+v_p)$ which intercepts the U-axis at two real points. Applying Möbius action by $g'_n=\begin{pmatrix}1 & -u_p\\ 0 & 1\end{pmatrix}$, we can shift the center to $(0,v_p)$, by Lemma 2.1, and applying $g'_a=\begin{pmatrix}\frac{1}{\sqrt{\alpha_p}} & 0\\ 0 & \sqrt{\alpha_p}\end{pmatrix}$, $\alpha_p>0$, we get the parabola as $u^2=r_p\alpha_p^{-1}(v+v_p\alpha_p^{-1})$. If this is an A'-orbit then by the Theorem 3.2 we get $(r_p\alpha_p^{-1})(v_p\alpha_p^{-1})=1$, i.e., $\alpha_p=\sqrt{r_pv_p}$.

(c) **Hyperbolic case.** We take an arbitrary hyperbola (with vertical axis of symmetry) $(v-v_h)^2-(u-u_h)^2=r_h^2$ with center (u_h,v_h) which intercepts U-axis at two real points then $|v_h|>|r_h|$. To fit this into a A'-orbit we apply Möbius actions by two matrices

$$g'_n=\begin{pmatrix}1 & -u_h\\ 0 & 1\end{pmatrix},\quad g'_a=\begin{pmatrix}\frac{1}{\sqrt{\alpha_h}} & 0\\ 0 & \sqrt{\alpha_h}\end{pmatrix},\quad \alpha_h>0.$$

Therefore, the equation transforms to the hyperbola $(v-v_h\alpha_h^{-1})^2-u^2=(r_h\alpha_h^{-1})^2$. To fit this into a A'-orbit using the Theorem 3.2, we get $(v_h\alpha_h^{-1})^2-(r_h\alpha_h^{-1})^2=1$, i.e., $\alpha_h=\sqrt{v_h^2-r_h^2}$.

Combining the above three cases we have $\alpha_\lambda=\sqrt{(v_\lambda^2-r_\lambda^2)\sigma+r_\lambda v_\lambda(1-|\sigma|)}$ for $\sigma=-1,0,1$ and $\lambda=e,p,h$.

Let us take α in place of α_e,α_p and α_h and u in place of u_e,u_p and u_h. Uniqueness of α can be proved by taking another α' and doing the same calculations we get $\alpha'=\sqrt{(v_\lambda^2-r_\lambda^2)\sigma+r_\lambda v_\lambda(1-|\sigma|)}$ for $\sigma=-1,0,1$ and $\lambda=e,p,h$. Therefore, $\alpha=\alpha'$ and α is unique.

Now,

$$g'=g(g'_ag'_n)^{-1}=\begin{pmatrix}a\sqrt{\alpha} & \frac{au+b}{\sqrt{\alpha}}\\ c\sqrt{\alpha} & \frac{cu+d}{\sqrt{\alpha}}\end{pmatrix}\in SL(2;\mathbb{R}).$$

If g' have decomposition of the form of Theorem 4.2, then $|c|\alpha < |cu + d|$. Thus, $g(g'_a g'_n)^{-1} = g_a g_n g_{a'}$ for some $g_a \in A, g_n \in N$ and $g_{a'} \in A'$. Therefore,

$$\begin{aligned} gC &= g(g'_a g'_n)^{-1} g'_a g'_n C = g_a g_n g_{a'}(g'_a g'_n C) \\ &= g_a g_n g'_a g'_n C \quad \text{(as } g'_a g'_n C \text{ is a } A'\text{-orbit therefore it is } A'\text{-invariant).} \end{aligned}$$

As g_a, g'_a, g_n, g'_n do not change shape of the cycle therefore gC is the same cycle as C. □

5. Discussion on Some Physical Applications of the Action of the Subgroups of the Group $SL(2;\mathbb{R})$

In optics, paraxial system is largely depend upon *paraxial groups* which are used to solve paraxial wave equation [1]. In fact, $SL(2;\mathbb{R})$ acts as a ray transfer matrix [3]. In chapter 2 of [3], the authors found the refractive matrix of the form

$$\mathcal{R} = \begin{pmatrix} 1 & 0 \\ \left(-\frac{n_1 - n_2}{r}\right) & 1 \end{pmatrix}.$$

That is two paraxial systems are dependent upon the relation $\begin{pmatrix} y_2 \\ V_2 \end{pmatrix} = \mathcal{R} \begin{pmatrix} y_1 \\ V_1 \end{pmatrix}$. This is the direct application of action of N' group.

In classical mechanics Galilean relativity principle [17] states that laws of mechanics will be invariant under the following linear transformation

$$\begin{pmatrix} t_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} \begin{pmatrix} t_1 \\ x_1 \end{pmatrix},$$

where t is time and x is spatial component. The matrix belongs to the group N'.

These two examples show that if these ordered pairs obey the conditions of our theorems, in the previous section 4 then they are invariant under $SL(2;\mathbb{R})$ action.

6. Conclusion and Future Work

In the elliptic and parabolic cases the upper half plane is preserved but in the hyperbolic case it is not true, which has some implications in geometry, physics and analysis [8].

We have obtained new invariant objects in the three homogeneous spaces. It applies to all the fields where $SL(2;\mathbb{R})$ is used.

In future we can study the projective space of cycles. A generic cycle can be represented as $C = \begin{pmatrix} g + if & c \\ a & -g + if \end{pmatrix}$, where i is a hypercomplex unit and C is $a(x^2 - y^2) - 2gx - 2fy + c = 0$. A cycle C is transformed to gCg^{-1} under the Möbius action of $g \in SL(2;\mathbb{R})$ [2]. After extending the cycle group by us in the previous theorems we can now include more values in $(a, g, f, c) \in \mathbb{R}^4$ and investigate further.

We can use our theory to find invariant metric [5] such as

$$ds^2 = \frac{du^2 - \sigma dv^2}{v^2}, \quad \sigma = -1, 0, 1.$$

In future, we can enrich Erlangen program of $SL(2;\mathbb{R})$ by including new invariants in all the three spaces. This theory can be used to make function theories on $\mathbb{R}^2$ [6, 13]. We can also extend our study of invariants to higher dimensional Lie groups, e.g., $SL(3;\mathbb{R})$, $SL(3;\mathbb{C})$, etc.

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