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WARPED PRODUCT POINTWISE SEMI-SLANT SUBMANIFOLDS OF SASAKIAN MANIFOLDS

ION MIHAI[1], SIRAJ UDDIN[2], AND ADELA MIHAI[3]

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Abstract. Recently, B.-Y. Chen and O. J. Garay studied pointwise slant submanifolds of almost Hermitian manifolds. By using the notion of pointwise slant submanifolds, we investigate the geometry of pointwise semi-slant submanifolds and their warped products in Sasakian manifolds. We give non-trivial examples of such submanifolds and obtain several fundamental results, including a characterization for warped product pointwise semi-slant submanifolds of Sasakian manifolds.

1. Introduction

In [7], B.-Y. Chen introduced the notion of slant submanifolds of almost Hermitian manifolds as a natural generalization of holomorphic (invariant) and totally real (anti-invariant) submanifolds. Afterwards, the geometry of slant submanifolds became an active topic of research in differential geometry. Later, A. Lotta [20] has extended this study for almost contact metric manifolds. J. L. Cabrerizo et al. investigated slant submanifolds of a Sasakian manifold [6]. N. Papaghiuc introduced in [22] a class of submanifolds, called semi-slant submanifolds of almost Hermitian manifolds, which are the generalizations of slant and CR-submanifolds. Later on, Cabrerizo et al. [5] extended this idea for semi-slant submanifolds of contact metric manifolds and provided many examples of such submanifolds.

Next, as an extension of slant submanifolds of an almost Hermitian manifold, F. Etayo [16] introduced the notion of pointwise slant submanifolds of almost Hermitian manifolds. B.-Y. Chen and O. J. Garay [14] studied pointwise slant submanifolds of almost Hermitian manifolds. They have obtained several fundamental results, in

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particular, a characterization of these submanifolds. K. S. Park [23] has extended this study. B. Sahin studied pointwise semi-slant submanifolds and warped product pointwise semi-slant submanifolds by using the notion of pointwise slant submanifolds [26]. In [31], the authors considered pointwise slant submanifolds of an almost contact metric manifold such that the structure vector field ξ is tangent to the submanifold. They have obtained a simple characterization for such submanifolds and studied warped product pointwise pseudo-slant submanifolds of Sasakian manifolds.

In 1969, R. L. Bishop and B. O'Neill [3] introduced and studied warped product manifolds. 30 years later, around the beginning of this century, B.-Y. Chen initiated in [9, 10] the study of warped product CR-submanifolds of Kaehler manifolds. Chen's work in this line of research motivated many geometers to study the geometry of warped product submanifolds by using his idea for different structures on manifolds (see, for instance, [2, 17, 21] and [27]). For a detailed survey on warped product submanifolds we refer to Chen's books [11, 13] and his survey article [12] as well.

In [24], B. Sahin showed that there exists no proper warped product semi-slant submanifold of Kaehler manifolds. Then, he introduced the notion of warped product hemi-slant submanifolds of Kaehler manifolds [25]. He defined and studied warped product pointwise semi-slant submanifolds and showed that there exists a non-trivial warped product pointwise semi-slant submanifold of the form $M_T \times_f M_\theta$ in a Kaehler manifold $\tilde{M}$, where M_T and M_θ are invariant and proper pointwise slant submanifolds of $\tilde{M}$, respectively [26]. For almost contact metric manifolds, we have seen in [19] and [1] that there are no proper warped product semi-slant submanifolds in cosymplectic and Sasakian manifolds. Then, we have considered warped product pseudo-slant submanifolds (warped product hemi-slant submanifolds [25], in the same sense of almost Hermitian manifolds) of cosymplectic [28] and Sasakian manifolds [29].

K. S. Park [23] studied warped product pointwise semi-slant submanifolds. He proved that there do not exist warped product pointwise semi-slant submanifolds of the form $M_\theta \times_f M_T$ such that M_θ and M_T are proper pointwise slant and invariant submanifolds, respectively. Then he provided many examples and obtained several results for warped products by reversing these two factors, including sharp estimations for the squared norm of the second fundamental form in terms of the warping functions. Later, we also extended this idea in [31] to warped product pointwise pseudo-slant submanifolds of Sasakian manifolds.

In this paper, we study warped product pointwise semi-slant submanifolds of the form $M_T \times M_\theta$ of Sasakian manifolds.

The present paper is organized as follows: in Section 2, we give basic definitions and formulas needed for this paper. Section 3 is devoted to the study of pointwise semi-slant submanifolds of Sasakian manifolds; we define pointwise semi-slant submanifolds and in the definition of pointwise semi-slant submanifolds we assume that the structure vector field ξ is always tangent to the submanifold. We give two non-trivial examples of such submanifolds for the justification of our definition and a result which is useful to the next section. In Section 4, we study warped product pointwise semi-slant submanifolds of Sasakian manifolds. In [1], we have seen that there are no warped product semi-slant submanifolds of the form $M_T \times_f M_\theta$ in a Sasakian manifold other than contact CR-warped products, but if we assume that M_θ is a proper pointwise slant submanifold then there exists a non-trivial class of such warped products. In this section, we obtain several new results which are generalizations of warped product semi-slant submanifolds and contact CR-warped product submanifolds. In Section 5, we provide nontrivial examples of Riemannian product and warped product pointwise semi-slant submanifolds in Euclidean spaces.

2. Preliminaries

An *almost contact structure* (φ, ξ, η) on a $(2n+1)$-dimensional manifold $\tilde{M}$ is defined by a $(1,1)$ tensor field φ, a vector field ξ, called *characteristic* or *Reeb vector field*, and a 1-form η satisfying the following conditions

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \xi = 0, \quad \eta(\xi) = 1, \tag{2.1}$$

where $I : T\tilde{M} \to T\tilde{M}$ is the identity map [4]. There always exists a Riemannian metric g on an almost contact manifold $\tilde{M}$ satisfying the following compatibility condition

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

for any $X, Y \in \Gamma(T\tilde{M})$, the Lie algebra of vector fields on $\tilde{M}$. This metric g is called a *compatible metric* and the manifold $\tilde{M}$ together with the structure (φ, ξ, η, g) is called an *almost contact metric manifold*. As an immediate consequence of (2.2), one has $\eta(X) = g(X, \xi)$ and $g(\varphi X, Y) = -g(X, \varphi Y)$. If ξ is a Killing vector field with respect to g, then the contact metric structure is called a *K-contact structure*. A normal contact metric manifold is said to be a *Sasakian manifold*. In terms of the covariant derivative of φ, the Sasakian condition can be expressed by

$$(\tilde{\nabla}_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.3}$$

for all $X, Y \in \Gamma(T\tilde{M})$, where $\tilde{\nabla}$ is the Levi-Civita connection of g. From the formula (2.3), it follows that

$$\tilde{\nabla}_X \xi = -\varphi X, \tag{2.4}$$

for any $X \in \Gamma(T\tilde{M})$.

Let M be a Riemannian manifold isometrically immersed in $\tilde{M}$ and denote by the same symbol g the Riemannian metric induced on M. Let $\Gamma(TM)$ be the Lie algebra of vector fields in M and $\Gamma(T^\perp M)$ the set of all vector fields normal to M. The Gauss and Weingarten formulas are respectively given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.5}$$

and

$$\tilde{\nabla}_X N = -A_N X + \nabla^\perp_X N, \tag{2.6}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, where ∇ is the Levi-Civita connection on M, $\nabla^{\perp}$ is the normal connection in the normal bundle $T^{\perp}M$ and A_N is the shape operator of M with respect to the normal vector N. Moreover, $h : TM \times TM \to T^{\perp}M$ is the second fundamental form of M in $\tilde{M}$. Furthermore, A_N and h are related by [32]

$$g(h(X,Y),N) = g(A_N X, Y), \tag{2.7}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$.

For any X tangent to M, we write

$$\varphi X = PX + FX, \tag{2.8}$$

where PX and FX are the tangential and normal components of φX, respectively. Then P is an endomorphism of the tangent bundle TM and F is a normal bundle valued 1-form on TM. Similarly, for any vector field N normal to M, we put

$$\varphi N = tN + fN, \tag{2.9}$$

where tN and fN are the tangential and normal components of φN, respectively. Moreover, from (2.2) and (2.8), we have

$$g(PX,Y) = -g(X,PY), \tag{2.10}$$

for any $X, Y \in \Gamma(TM)$.

Throughout this paper, we assume the structure field ξ is tangent to M; otherwise M is a C-totally real submanifold [20]. Let M be a Riemannian manifold isometrically immersed in an almost contact metric manifold $(\tilde{M}, \varphi, \xi, \eta, g)$. A submanifold M of an almost contact metric manifold $\tilde{M}$ is said to be *slant* [6], if for each non-zero vector X tangent to M at $p \in M$ such that X is not proportional to ξ_p, the angle $\theta(X)$ between φX and T_pM is constant, i.e., it does not depend on the choice of $p \in M$ and $X \in T_pM - \langle \xi_p \rangle$.

A slant submanifold is said to be *proper slant* if neither $\theta = 0$ nor $\theta = \frac{\pi}{2}$. We note that on a slant submanifold if $\theta = 0$, then it is an invariant submanifold and if $\theta = \frac{\pi}{2}$, then it is an anti-invariant submanifold. A slant submanifold is said to be *proper slant* if it is neither invariant nor anti-invariant.

As a natural extension of slant submanifolds, F. Etayo [16] introduced pointwise slant submanifolds of an almost Hermitian manifold under the name of quasi-slant submanifolds. Later on, B.-Y. Chen and O. J. Garay studied pointwise slant submanifolds of almost Hermitian manifolds and obtained many interesting results [14]. In [31], the authors studied pointwise slant submanifolds of almost contact metric manifolds tangent to the structure vector field ξ.

A submanifold M of an almost contact metric manifold $\tilde{M}$ is said to be *pointwise slant* if for any nonzero vector X tangent to M at $p \in M$, such that X is not propotional to ξ_p, the angle $\theta(X)$ between φX and $T_p^*M = T_pM - \{0\}$ is independent of the choice of nonzero vector $X \in T_p^*M$. In this case, θ can be regarded as a function on M, which is called the *slant function* of the pointwise slant submanifold.

We note that every slant submanifold is a pointwise slant submanifold, but the converse is not true. We also note that a pointwise slant submanifold is *invariant* (respectively, *anti-invariant*) if for each point $p \in M$, the slant function $\theta = 0$ (respectively, $\theta = \frac{\pi}{2}$). A pointwise slant submanifold is slant if and only if the slant function θ is constant on M. Moreover, a pointwise slant submanifold is proper if neither $\theta = 0$, $\frac{\pi}{2}$ nor θ is constant.

In [31], we have obtained the following characterization theorem.

Theorem 2.1 ([31])**.** *Let M be a submanifold of an almost contact metric manifold $\tilde{M}$ such that $\xi \in \Gamma(TM)$. Then, M is pointwise slant if and only if*

$$P^2 = \cos^2\theta\,(-I + \eta \otimes \xi)\,, \tag{2.11}$$

for some real valued function θ defined on the tangent bundle TM of M.

The following relations are immediate consequences of Theorem 2.1.

Let M be a pointwise slant submanifold of an almost contact metric manifold $\tilde{M}$. Then, we have

$$g(PX, PY) = \cos^2\theta\,[g(X,Y) - \eta(X)\eta(Y)], \tag{2.12}$$

$$g(FX, FY) = \sin^2\theta\,[g(X,Y) - \eta(X)\eta(Y)], \tag{2.13}$$

for any $X, Y \in \Gamma(TM)$.

The next useful relations for a pointwise slant submanifold of an almost contact metric manifold was obtained in [31]

$$tFX = \sin^2\theta\,(-X + \eta(X)\xi), \quad fFX = -FPX, \tag{2.14}$$

for any $X \in \Gamma(TM)$.

3. Pointwise Semi-Slant Submanifolds

B. Sahin [26] defined and studied pointwise semi-slant submanifolds of Kaehler manifolds. In this section, we define and study pointwise semi-slant submanifolds of Sasakian manifolds.

Definition 3.1. A submanifold M of an almost contact metric manifold $\tilde{M}$ is said to be a *pointwise semi-slant* submanifold if there exists a pair of orthogonal distributions $\mathfrak{D}$ and $\mathfrak{D}^\theta$ on M such that

(i) the tangent bundle TM admits the orthogonal direct decomposition $TM = \mathfrak{D} \oplus \mathfrak{D}^\theta \oplus \langle\xi\rangle$;
(ii) the distribution $\mathfrak{D}$ is invariant under φ, i.e., $\varphi(\mathfrak{D}) = \mathfrak{D}$;
(iii) the distribution $\mathfrak{D}^\theta$ is pointwise slant with slant function θ.

Note that the normal bundle $T^\perp M$ of a pointwise semi-slant submanifold M is decomposed as

$$T^\perp M = F\mathfrak{D}^\theta \oplus \nu, \quad F\mathfrak{D}^\theta \perp \nu,$$

where ν is an invariant normal subbundle of $T^{\perp}M$ under φ.

If we denote the dimensions of $\mathfrak{D}$ and $\mathfrak{D}^{\theta}$ by m_1 and m_2, respectively, then we have the following.

(i) If $m_1 = 0$, then M is a pointwise slant submanifold.
(ii) If $m_2 = 0$, then M is an invariant submanifold.
(iii) If $m_1 = 0$ and $\theta = \frac{\pi}{2}$, then M is an anti-invariant submanifold.
(iv) If $m_1 \neq 0$ and $\theta = \frac{\pi}{2}$, then M is a contact CR-submanifold.
(v) If θ is constant on M, then M is a semi-slant submanifold with slant angle θ.

We also note that a pointwise semi-slant submanifold is *proper* if neither $m_1, m_2 = 0$ nor $\theta = 0, \frac{\pi}{2}$ and θ should not be a constant.

Now, we provide the following non-trivial examples of pointwise semi-slant submanifolds of an almost contact metric manifold.

Example 3.1. Let $(\mathbf{R}^7, \varphi, \xi, \eta, g)$ be an almost contact metric manifold with cartesian coordinates $(x_1,\, y_1,\, x_2,\, y_2,\, x_3,\, y_3,\, z)$ and the almost contact structure

$$\varphi\left(\frac{\partial}{\partial x_i}\right) = -\frac{\partial}{\partial y_i}, \quad \varphi\left(\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial x_j}, \quad \varphi\left(\frac{\partial}{\partial z}\right) = 0, \quad 1 \leq i, j \leq 3,$$

where $\xi = \frac{\partial}{\partial z}$, $\eta = dz$ and g is the standard Euclidean metric on $\mathbf{R}^7$. Then (φ, ξ, η, g) is an almost contact metric structure on $\mathbf{R}^7$. Consider a submanifold M of $\mathbf{R}^7$ defined by $\psi(u, v, w, t, z) = (u + v,\, -u + v,\, t\cos w,\, t\sin w,\, w\cos t,\, w\sin t,\, z)$, such that w, t $(w \neq t)$ are non-zero real numbers. Then the tangent space TM is spanned by the following vector fields

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial x_1} - \frac{\partial}{\partial y_1}, \quad X_2 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1}, \\
X_3 &= -t\sin w\frac{\partial}{\partial x_2} + t\cos w\frac{\partial}{\partial y_2} + \cos t\frac{\partial}{\partial x_3} + \sin t\frac{\partial}{\partial y_3}, \\
X_4 &= \cos w\frac{\partial}{\partial x_2} + \sin w\frac{\partial}{\partial y_2} - w\sin t\frac{\partial}{\partial x_3} + w\cos t\frac{\partial}{\partial y_3}, \quad X_5 = \frac{\partial}{\partial z}.
\end{aligned}$$

Thus, we observe that $\mathfrak{D} = \mathrm{Span}\{X_1,\, X_2\}$ is an invariant distribution and $\mathfrak{D}^{\theta} = \mathrm{Span}\{X_3,\, X_4\}$ is a pointwise slant distribution with pointwise slant function $\theta = \cos^{-1}((t - w)/\sqrt{(t^2 + 1)(w^2 + 1)})$. Hence, M is a pointwise semi-slant submanifold of $\mathbf{R}^7$ such that $\xi = \frac{\partial}{\partial z}$ is tangent to M.

Example 3.2. Consider a submanifold of $\mathbf{R}^7$ with almost contact structure φ given in Example 3.1. If the immersion $\psi : \mathbf{R}^5 \to \mathbf{R}^7$ is given by

$$\psi(u_1, u_2, u_3, u_4, t) = \left(u_1, \frac{u_3^2 + u_4^2}{2}, \cos u_4, -u_2, \frac{u_3^2 - u_4^2}{2}, \sin u_4, t\right), \quad u_4 \neq 0,$$

then the tangent space TM is spanned by X_1, X_2, X_3, X_4 and X_5, where

$$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = -\frac{\partial}{\partial y_1}, \quad X_3 = u_3 \frac{\partial}{\partial x_2} + u_3 \frac{\partial}{\partial y_2},$$

$$X_4 = u_4 \frac{\partial}{\partial x_2} - u_4 \frac{\partial}{\partial y_2} - \sin u_4 \frac{\partial}{\partial x_3} + \cos u_4 \frac{\partial}{\partial y_3}, \quad X_5 = \frac{\partial}{\partial t}.$$

Therefore, M is a pointwise semi-slant submanifold such that $\mathfrak{D} = \mathrm{Span}\{X_1,\, X_2\}$ is an invariant distribution and $\mathfrak{D}^\theta = \mathrm{Span}\{X_3,\, X_4\}$ is a pointwise slant distribution with pointwise slant function $\theta = \cos^{-1}\left(\sqrt{2}\, u_4/\sqrt{1+2u_4^2}\right)$.

Now, we obtain the following useful results for semi-slant submanifolds of a Sasakian manifold.

Lemma 3.1. *Let M be a pointwise semi-slant submanifold of a Sasakian manifold $\tilde{M}$. Then, we have*

(i) $\sin^2\theta\, g(\nabla_X Y, Z) = g(h(X, \varphi Y), FZ) - g(h(X,Y), FPZ)$,
(ii) $\sin^2\theta\, g(\nabla_Z W, X) = g(h(X,Z), FPW) - g(h(\varphi X, Z), FW)$,

for any $X, Y \in \Gamma(\mathfrak{D} \oplus \langle\xi\rangle)$ and $Z, W \in \Gamma(\mathfrak{D}^\theta)$.

Proof. The first and second parts of the lemma can be proved in a similar way. For any $X, Y \in \Gamma(\mathfrak{D} \oplus \langle\xi\rangle)$ and $Z \in \Gamma(\mathfrak{D}^\theta)$, we have

$$g(\nabla_X Y, Z) = g(\tilde{\nabla}_X Y, Z) = g(\varphi\tilde{\nabla}_X Y, \varphi Z).$$

From the covariant derivative formula of φ, we derive

$$g(\nabla_X Y, Z) = g(\tilde{\nabla}_X \varphi Y, \varphi Z) - g((\tilde{\nabla}_X \varphi)Y, \varphi Z).$$

Then, from (2.3), (2.8) and the orthogonality of the two distributions, we find

$$\begin{aligned} g(\nabla_X Y, Z) &= g(\tilde{\nabla}_X \varphi Y, PZ) + g(\tilde{\nabla}_X \varphi Y, FZ) \\ &= -g(\tilde{\nabla}_X PZ, \varphi Y) + g(h(X, \varphi Y), FZ) \\ &= g(\varphi\tilde{\nabla}_X PZ, Y) + g(h(X, \varphi Y), FZ). \end{aligned}$$

Again, from the covariant derivative formula of φ, we get

$$g(\nabla_X Y, Z) = g(\tilde{\nabla}_X \varphi PZ, Y) - g((\tilde{\nabla}_X \varphi)PZ, Y) + g(h(X, \varphi Y), FZ).$$

Using (2.3), (2.8) and the orthogonality of vector fields, we obtain

$$g(\nabla_X Y, Z) = g(\tilde{\nabla}_X P^2 Z, Y) + g(\tilde{\nabla}_X FPZ, Y) + g(h(X, \varphi Y), FZ).$$

Then, from (2.11) and (2.6), we have

$$\begin{aligned} g(\nabla_X Y, Z) = &-\cos^2\theta\, g(\tilde{\nabla}_X Z, Y) + \sin 2\theta\, X(\theta)\, g(Y,Z) - g(h(X,Y), FPZ) \\ &+ g(h(X, \varphi Y), FZ). \end{aligned}$$

From the orthogonality of the two distributions the above equation takes the form

$$g(\nabla_X Y, Z) = \cos^2\theta\, g(\tilde{\nabla}_X Y, Z) - g(h(X,Y), FPZ) + g(h(X, \varphi Y), FZ).$$

Hence, (i) follows from the above relation. In a similar way we can prove (ii). ☐

4. Warped Product Pointwise Semi-Slant Submanifolds

In this section, we study warped product submanifolds of Sasakian manifolds, by considering that one factor is a pointwise slant submanifold. In the following, first we give a brief introduction on warped product manifolds.

In [3], R. L. Bishop and B. O'Neill introduced the notion of warped product manifolds as follows: Let M_1 and M_2 be two Riemannian manifolds with Riemannian metrics g_1 and g_2, respectively, and a positive differentiable function f on M_1. Consider the product manifold $M_1 \times M_2$ with its projections $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$. Then their warped product manifold $M = M_1 \times_f M_2$ is the Riemannian manifold $M_1 \times M_2 = (M_1 \times M_2, g)$ equipped with the Riemannian metric

$$g(X,Y) = g_1(\pi_{1\star}X, \pi_{1\star}Y) + (f \circ \pi_1)^2 g_2(\pi_{2\star}X, \pi_{2\star}Y),$$

for any vector field X, Y tangent to M, where $\star$ is the symbol for the tangent maps. A warped product manifold $M = M_1 \times_f M_2$ is said to be *trivial* or simply a *Riemannian product manifold* if the warping function f is constant.

Let X be a vector field tangent to M_1 and Z be an another vector field on M_2; then from Lemma 7.3 of [3], we have

$$\nabla_X Z = \nabla_Z X = X(\ln f)Z, \tag{4.1}$$

where ∇ is the Levi-Civita connection on M. If $M = M_1 \times_f M_2$ is a warped product manifold then the base manifold M_1 is totally geodesic in M and the fiber M_2 is totally umbilical in M [3, 9].

By analogy to CR-warped products which are introduced by B.-Y. Chen in [9], one defines the warped product pointwise semi-slant submanifolds as follows.

Definition 4.1. A warped product of an invariant and a pointwise slant submanifolds, say M_T and M_θ of a Sasakian manifold $\tilde{M}$ is called a *warped product pointwise semi-slant submanifold*.

A warped product pointwise semi-slant submanifold is called *proper* if M_θ is a proper pointwise slant submanifold and M_T is an invariant submanifold of $\tilde{M}$.

The non-existence of warped product pointwise semi-slant submanifolds of the form $M_\theta \times_f M_T$ in Kaehler manifolds is proved in [26]. A similar result holds in Sasakian manifolds. On the other hand, there exist non-trivial warped product pointwise semi-slant submanifolds of the form $M_T \times M_\theta$ of Kaehler manifolds [26] and contact metric manifolds.

Note that a warped product pointwise semi-slant submanifold $M = M_T \times_f M_\theta$ is a warped product contact CR-submanifold if the slant function $\theta = \frac{\pi}{2}$. Similarly, the warped product pointwise semi-slant submanifold $M = M_T \times_f M_\theta$ is a warped product semi-slant submanifold if θ is constant on M, i.e., M_θ is a proper slant submanifold.

In this section, we study the warped product pointwise semi-slant submanifold of the form $M = M_T \times_f M_\theta$ of a Sasakian manifold $\tilde{M}$. To fill the gap in the earlier study, we obtain some results as a generalization.

On a warped product pointwise semi-slant submanifold $M = M_T \times_f M_\theta$, if we consider the structure vector field ξ tangent to M, then either $\xi \in \Gamma(TM_T)$ or $\xi \in \Gamma(TM_\theta)$. When ξ is tangent to M_θ, then it is easy to check that warped product is trivial (see [27]); therefore we always consider $\xi \in \Gamma(TM_T)$.

First, we prove the following useful results.

Lemma 4.1. *Let $M = M_T \times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a Sasakian manifold $\tilde{M}$ such that $\xi \in \Gamma(TM_T)$, where M_T is an invariant submanifold and M_θ is a proper pointwise slant submanifold of $\tilde{M}$. Then, we have*

$$g(h(X,W),FPZ) - g(h(X,PZ),FW) = \sin 2\theta\, X(\theta)\, g(Z,W), \tag{4.2}$$

for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$.

Proof. For any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$, we have

$$g(\tilde{\nabla}_X Z, W) = X(\ln f)\, g(Z,W). \tag{4.3}$$

On the other hand, we can obtain $g(\tilde{\nabla}_X Z, W) = g(\varphi\tilde{\nabla}_X Z, \varphi W)$. Using the covariant derivative formula of φ, we get

$$g(\tilde{\nabla}_X Z, W) = g(\tilde{\nabla}_X \varphi Z, \varphi W) - g((\tilde{\nabla}_X \varphi)Z, \varphi W).$$

The second term in the right hand side of above relation is identically zero by using (2.3) and the orthogonality of vector fields. Then, from (2.5), (2.8), (4.1) and the orthogonality of vector fields, we find

$$\begin{aligned} g(\tilde{\nabla}_X Z, W) =& g(\tilde{\nabla}_X PZ, PW) + g(\tilde{\nabla}_X PZ, FW) + g(\tilde{\nabla}_X FZ, \varphi W) \\ =& X(\ln f)\, g(PZ, PW) + g(h(X,PZ),FW) - g(\varphi\tilde{\nabla}_X FZ, W) \\ =& \cos^2\theta\, X(\ln f)\, g(Z,W) + g(h(X,PZ),FW) - g(\tilde{\nabla}_X \varphi FZ, W) \\ &+ g((\tilde{\nabla}_X \varphi)FZ, W). \end{aligned}$$

Again, the last term in the above equation is zero by using (2.3) and the orthogonality of vector fields. Then, from (2.9) and (2.14), we derive

$$\begin{aligned} g(\tilde{\nabla}_X Z, W) =& \cos^2\theta\, X(\ln f)\, g(Z,W) + g(h(X,PZ),FW) + \sin^2\theta\, g(\tilde{\nabla}_X Z, W) \\ &+ \sin 2\theta\, X(\theta)\, g(Z,W) + g(\tilde{\nabla}_Z FPX, Y). \end{aligned} \tag{4.4}$$

Hence, the result follows from (4.3) and (4.4) by using (2.6)–(2.7) and (4.1). □

Lemma 4.2. *Let $M = M_T \times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a Sasakian manifold $\tilde{M}$ such that $\xi \in \Gamma(TM_T)$, where M_T and M_θ are invariant and pointwise slant submanifolds of $\tilde{M}$, respectively. Then*

(i) *$g(PZ, W) = -\xi(\ln f)\, g(Z,W)$;*
(ii) *$g(h(X,Y),FZ) = 0$;*

(iii) $g(h(X,Z),FW) = X(\ln f)\, g(PZ,W) - \varphi X(\ln f)\, g(Z,W) - \eta(X)\, g(Z,W)$,

for any $X, Y \in \Gamma(TM_T)$ *and* $Z, W \in \Gamma(TM_\theta)$.

Proof. From (2.4), (2.5) and (2.8), we have $\nabla_Z \xi = -PZ$, for any $Z \in \Gamma(TM_\theta)$. Using (4.1) and taking the inner product with $W \in \Gamma(TM_\theta)$, we get (i). For the other parts of the lemma, considering any $X, Y \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_\theta)$, we have

$$g(h(X,Y),FZ) = g(\tilde{\nabla}_X Y, FZ) = g(\tilde{\nabla}_X Y, \varphi Z) - g(\tilde{\nabla}_X Y, PZ).$$

From (2.2) and (4.1), we get

$$g(h(X,Y),FZ) = -g(\varphi \tilde{\nabla}_X Y, Z) + X(\ln f)\, g(Y, PZ).$$

By covariant derivative formula of φ and the orthogonality of vector fields, we find

$$g(h(X,Y),FZ) = g((\tilde{\nabla}_X \varphi)Y, Z) - g(\tilde{\nabla}_X \varphi Y, Z).$$

Using (2.3) and the fact that $\xi \in \Gamma(TM_T)$, the first term in the right hand side of above equation vanishes identically and then by using (4.1) and the orthogonality of vector fields, we find (ii). Now, for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$, we have

$$g(h(X,Z),FW) = g(\tilde{\nabla}_Z X, FW) = g(\tilde{\nabla}_X Z, \varphi W) - g(\tilde{\nabla}_X Z, PW).$$

Again, using the covariant derivative formula of the Riemannain connection and (4.1), we get

$$g(h(X,Z),FW) = g((\tilde{\nabla}_Z \varphi)X, W) - g(\tilde{\nabla}_Z \varphi X, W) - X(\ln f)\, g(Z, PW).$$

Then from (2.3), (2.5) and (4.1), we derive

$$g(h(X,Z),FW) = -\eta(X)\, g(Z,W) - \varphi X(\ln f)\, g(Z,W) - X(\ln f)\, g(Z,PW),$$

which is the third part of the lemma. Hence, the proof is complete. □

Lemma 4.3. *Let* $M = M_T \times_f M_\theta$ *be a warped product pointwise semi-slant submanifold of a Sasakian manifold* $\tilde{M}$ *such that* $\xi \in \Gamma(TM_T)$, *where* M_T *is an invariant submanifold and* M_θ *is a pointwise slant submanifold of* $\tilde{M}$. *Then*

$$g(h(\varphi X,Z),FW) = X(\ln f)\, g(Z,W) - \eta(X)\, g(Z,PW) - \varphi X(\ln f)\, g(Z,PW), \tag{4.5}$$

for any $X \in \Gamma(TM_T)$ *and* $Z, W \in \Gamma(TM_\theta)$.

Proof. Interchanging X by φX, for any $X \in \Gamma(TM_T)$ in Lemma 4.2 (iii) and using the first part of Lemma 4.2, we get the required result. □

Lemma 4.4. *Let* $M = M_T \times_f M_\theta$ *be a warped product pointwise semi-slant submanifold of a Sasakian manifold* $\tilde{M}$ *such that* $\xi \in \Gamma(TM_T)$, *where* M_T *and* M_θ *are invariant and pointwise slant submanifolds of* $\tilde{M}$, *respectively. Then, we have*

(4.6)
$$g(h(X,PZ),FW) = \varphi X(\ln f)\, g(Z,PW) - \eta(X)\, g(PZ,W) - \cos^2\theta\, X(\ln f)\, g(Z,W),$$

for any $X \in \Gamma(TM_T)$ *and* $Z, W \in \Gamma(TM_\theta)$.

Proof. Interchange Z by PZ, for any $Z \in \Gamma(TM_\theta)$ in Lemma 4.2 (iii) and after using (2.12), we get (4.6). ☐

Similarly, if we interchange W by PW, for any $W \in \Gamma(TM_\theta)$ in Lemma 4.2 (iii), then we can obtain the following result.

Lemma 4.5. *Let $M = M_T \times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a Sasakian manifold $\tilde{M}$ such that $\xi \in \Gamma(TM_T)$, where M_T and M_θ are invariant and pointwise slant submanifolds of $\tilde{M}$, respectively. Then*

$$g(h(X,Z), FPW) = \cos^2\theta\, X(\ln f)\, g(Z,W) - \varphi X(\ln f)\, g(Z,PW) - \eta(X)\, g(Z,PW), \tag{4.7}$$

for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$.

Lemma 4.6. *Let $M = M_T \times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a Sasakian manifold $\tilde{M}$ such that $\xi \in \Gamma(TM_T)$, where M_T and M_θ are invariant and proper pointwise slant submanifolds of $\tilde{M}$, respectively. Then, we have*

$$g(A_{FW}\varphi X, Z) - g(A_{FPW}X, Z) = \sin^2\theta\, X(\ln f)\, g(Z,W), \tag{4.8}$$

for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$.

Proof. Subtracting (4.7) from (4.5), we get (4.8). ☐

A warped product submanifold $M = M_1 \times_f M_2$ of a Sasakian manifold $\tilde{M}$ is said to be *mixed totally geodesic* if $h(X,Z) = 0$, for any $X \in \Gamma(TM_1)$ and $Z \in \Gamma(TM_2)$, where M_1 and M_2 are any Riemannian submanifolds of $\tilde{M}$.

From Lemma 4.6, we obtain the following result.

Theorem 4.1. *Let $M = M_T \times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a Sasakian manifold $\tilde{M}$. If M is mixed totally geodesic, then either M is warped product of invariant submanifolds or the warping function f is constant on M.*

Proof. From (4.8) and the mixed totally geodesic condition, we have

$$\sin^2\theta\, X(\ln f)\, g(Z,W) = 0.$$

Since g is a Riemannian metric, then either $\sin^2\theta = 0$ or $X(\ln f) = 0$. Therefore, either M is warped product of invariant submanifolds or f is constant on M, thus, the proof is complete. ☐

Lemma 4.7. *Let $M = M_T \times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a Sasakian manifold $\tilde{M}$ such that $\xi \in \Gamma(TM_T)$, where M_T and M_θ are invariant and pointwise slant submanifolds of $\tilde{M}$, respectively. Then, we have*

$$g(A_{FPZ}W, X) - g(A_{FW}PZ, X) = 2\cos^2\theta\, X(\ln f)\, g(Z,W), \tag{4.9}$$

for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$.

Proof. Interchanging Z and W in (4.7) and using (2.10), we get

(4.10)
$$g(h(X,W),FPZ)=\cos^2\theta\,X(\ln f)\,g(Z,W)+\varphi X(\ln f)\,g(Z,PW)+\eta(X)\,g(Z,PW),$$

for any $X\in\Gamma(TM_T)$ and $Z,W\in\Gamma(TM_\theta)$. Subtracting (4.6) from (4.10), we find (4.9). □

Also, with the help of Lemma 4.7, we find the following result.

Theorem 4.2. *Let $M=M_T\times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a Sasakian manifold $\tilde{M}$. If M is mixed totally geodesic, then either M is a contact CR-warped product of the form $M_T\times_f M_\perp$ or the warping function f is constant on M.*

Proof. From (4.9) and the mixed totally geodesic condition, we have
$$\cos^2\theta\,X(\ln f)\,g(Z,W)=0.$$
Since g is a Riemannian metric, then either $\cos^2\theta=0$ or $X(\ln f)=0$. Therefore, either M is a contact CR-warped product or f is constant on M, which ends the proof. □

From Theorem 4.1 and Theorem 4.2, we conclude the following result.

Corollary 4.1. *There does not exist any mixed totally geodesic proper warped product pointwise semi-slant submanifold $M=M_T\times_f M_\theta$ of a Sasakian manifold.*

Also, from Lemma 4.1 and Lemma 4.7, we have the following result.

Theorem 4.3. *Let $M=M_T\times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a Sasakian manifold $\tilde{M}$ such that $\xi\in\Gamma(TM_T)$, where M_T is an invariant submanifold and M_θ is a pointwise slant submanifold of $\tilde{M}$. Then, either M is a contact CR-warped product of the form $M=M_T\times_f M_\perp$ or $\nabla(\ln f)=\tan\theta\,\nabla\theta$, for any $X\in\Gamma(TM_T)$, where ∇f is the gradient of f.*

Proof. From Lemma 4.1 and Lemma 4.7, we have
$$\cos^2\theta\{X(\ln f)-\tan\theta\,X(\theta)\}\,g(Z,W)=0.$$
Since g is a Riemannian metric, therefore, we conclude that either $\cos^2\theta=0$ or $X(\ln f)-\tan\theta\,X(\theta)=0$. Consequently, either $\theta=\frac{\pi}{2}$ or $X(\ln f)=\tan\theta\,X(\theta)$, which proves the theorem completely. □

As an application of Theorem 4.3, we have the following consequence.

Remark 4.1. If we consider that the slant function θ is constant, i.e., M_θ is a proper slant submanifold in Theorem 4.3, then $Z(\ln f)=0$, i.e., there are no warped product semi-slant submanifolds of the form $M_T\times_f M_\theta$ in Sasakian manifolds. Hence, Theorem 3.3 of [1] is a special case of Theorem 4.3.

In order to give a characterization result for pointwise semi-slant submanifolds of a Sasakian manifold, we need the following well-known result of Hiepko [18].

Theorem 4.4 (Hiepko's Theorem). *Let $\mathfrak{D}_1$ and $\mathfrak{D}_2$ be two orthogonal distribution on a Riemannian manifold M. Suppose that both $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are involutive such that $\mathfrak{D}_1$ is a totally geodesic foliation and $\mathfrak{D}_2$ is a spherical foliation. Then M is locally isometric to a non-trivial warped product $M_1 \times_f M_2$, where M_1 and M_2 are integral manifolds of $\mathfrak{D}_1$ and $\mathfrak{D}_2$, respectively.*

Theorem 4.5. *Let M be a pointwise semi-slant submanifold of a Sasakian manifold $\tilde{M}$. Then M is locally a non-trivial warped product submanifold of the form $M_T \times_f M_\theta$, where M_T is an invariant submanifold and M_θ is a proper pointwise slant submanifold of $\tilde{M}$ if and only if*

$$A_{FW}\varphi X - A_{FPW}X = \sin^2\theta\, X(\mu)W, \quad \textit{for all } X \in \Gamma(\mathfrak{D} \oplus \langle\xi\rangle), W \in \Gamma(\mathfrak{D}^\theta), \tag{4.11}$$

for some smooth function μ on M satisfying $Z(\mu) = 0$ for any $Z \in \Gamma(\mathfrak{D}^\theta)$.

Proof. Let $M = M_T \times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a Sasakian manifold $\tilde{M}$. Then for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$, from Lemma 4.2 (ii) we have

$$g(A_{FW}X, Y) = 0. \tag{4.12}$$

Interchanging X by φX in (4.12), we get $g(A_{FW}\varphi X, Y) = 0$, which means that $A_{FW}\varphi X$ has no component in TM_T. Similarly, if we interchange W by PW in (4.12) then, we get $g(A_{FPW}X, Y) = 0$, i.e., $A_{FPW}X$ also has no component in TM_T. Therefore, $A_{FW}\varphi X - A_{FPW}X$ lies in TM_θ, using this fact with Lemma 4.6, we find (4.11).

Conversely, if M is a pointwise semi-slant submanifold such that (4.11) holds, then from Lemma 3.1 (i), we have

$$g(\nabla_X Y, W) = \csc^2\theta\, g(A_{FW}\varphi Y - A_{FPW}Y, X),$$

for any $X, Y \in \Gamma(\mathfrak{D} \oplus \langle\xi\rangle)$ and $W \in \Gamma(\mathfrak{D}^\theta)$. From (4.11), we arrive at

$$g(\nabla_X Y, W) = Y(\mu)g(X, W) = 0,$$

which means that the leaves of the distribution $\mathfrak{D} \oplus \langle\xi\rangle$ are totally geodesic in M. Also, from Lemma 3.1 (ii), we have

$$g(\nabla_Z W, X) = \csc^2\theta\, g(A_{FPW}X - A_{FW}\varphi X, Z), \tag{4.13}$$

for any $Z, W \in \Gamma(\mathfrak{D}^\theta)$ and $X \in \Gamma(\mathfrak{D} \oplus \langle\xi\rangle)$. By polarization, we derive

$$g(\nabla_W Z, X) = \csc^2\theta\, g(A_{FPZ}X - A_{FZ}\varphi X, W). \tag{4.14}$$

Substracting (4.14) from (4.13), we get

$$\sin^2\theta\, g([Z, W], X) = g(A_{FZ}\varphi X - A_{FPZ}X, W) - g(A_{FW}\varphi X - A_{FPW}X, Z).$$

Using (4.11), we get

$$\sin^2\theta\, g([Z,W],X) = X(\mu)\, g(Z,W) - X(\mu)\, g(W,Z) = 0.$$

Since M is proper pointwise semi-slant, then $\sin^2\theta \neq 0$, thus we conclude that the pointwise slant distribution $\mathfrak{D}^\theta$ is integrable. Let us consider M_θ to be a leaf of $\mathfrak{D}^\theta$ and h^θ is the second fundamental form of M_θ in M. Then from (4.14), we have

$$g(h^\theta(Z,W),X) = g(\nabla_Z W, X) = -\csc^2\theta\, g(A_{FW}\varphi X - A_{FPW}X, Z).$$

Using (4.11), we find that

$$g(h^\theta(Z,W),X) = -X(\mu)\, g(Z,W).$$

Then from the definition of the gradient of a function, we arrive at

$$h^\theta(Z,W) = -(\vec{\nabla}\mu)\, g(Z,W).$$

Hence, M_θ is a totally umbilical submanifold of M with the mean curvature vector $H^\theta = -\vec{\nabla}\mu$, where $\vec{\nabla}\mu$ is the gradient of the function μ. Since $Z(\mu) = 0$, for any $Z \in \Gamma(\mathfrak{D}^\theta)$, then we can show that $H^\theta = -\vec{\nabla}\mu$ is parallel with respect to the normal connection, say D^n of M_θ in M (see [25, 26], [28]). Thus, M_θ is a totally umbilical submanifold of M with a non vanishing parallel mean curvature vector $H^\theta = -\vec{\nabla}\mu$, i.e., M_θ is an extrinsic sphere in M. Then from Heipko's Theorem [18], we conclude that M is a warped product manifold of M_T and M_θ, where M_T and M_θ are integral manifolds of $\mathfrak{D} \oplus \langle\xi\rangle$ and $\mathfrak{D}^\theta$, respectively. Thus, the proof is complete. □

As an application of Theorem 4.5, if we consider $\theta = \frac{\pi}{2}$ in Theorem 4.5, then by interchanging X by φX in (4.11), we get the condition (74) of Theorem 3.2 in [21]; thus the Theorem 4.5 is true for contact CR-warped product submanifolds of the form $M_T \times_f M_\perp$. Hence, Theorem 3.2 of [21] is a special case of Theorem 4.5 as follows.

Corollary 4.2 (Theorem 3.2 of [21]). *A strictly proper CR-submanifold M of a Sasakian manifold $\tilde{M}$ tangent to the structure vector field ξ is locally a contact CR-warped product if and only if*

$$A_{\varphi Z}X = (\eta(X) - \varphi X(\mu))\, Z, \quad X \in \Gamma(\mathfrak{D} \oplus \langle\xi\rangle), Z \in \Gamma(\mathfrak{D}^\perp), \tag{4.15}$$

for some function μ on M satisfying $W\mu = 0$, for all $W \in \Gamma(\mathfrak{D}^\perp)$.

5. Examples

In this section, we provide the following non-trivial examples of Riemannian products and warped product pointwise semi-slant submanifolds in Euclidean spaces.

Example 5.1. Let M be a submanifold of Euclidean 7-space $\mathbb{R}^7$ with its cartesian coordinates $(x_1, \ldots, x_3, y_1, \ldots, y_3, t)$ and the almost contact structure

$$\varphi\left(\frac{\partial}{\partial x_i}\right) = -\frac{\partial}{\partial y_i}, \quad \varphi\left(\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial x_j}, \quad \varphi\left(\frac{\partial}{\partial t}\right) = 0, \quad 1 \leq i,j \leq 3.$$

If M is given by the equations

$$\begin{aligned} x_1 &= u_1, \quad x_2 = u_3 \cos u_4, \quad x_3 = \frac{u_3^2}{2}, \quad y_1 = u_2, \quad y_2 = u_3 \sin u_4, \\ y_3 &= u_4, \quad t = t, \end{aligned}$$

for any non-zero function u_3 on M, then tangent space TM of M is spanned by X_1, X_2, X_3, X_4 and X_5, where

$$\begin{aligned} &X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial y_1}, \quad X_3 = \cos u_4 \frac{\partial}{\partial x_2} + u_3 \frac{\partial}{\partial x_3} + \sin u_4 \frac{\partial}{\partial y_2}, \\ &X_4 = -u_3 \sin u_4 \frac{\partial}{\partial x_2} + u_3 \sin u_4 \frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_3}, \quad X_5 = \frac{\partial}{\partial t}. \end{aligned}$$

Then, M is a pointwise semi-slant submanifold with invariant distribution $\mathfrak{D} = \mathrm{Span}\{\mathrm{X}_1, \mathrm{X}_2\}$ and the pointwise slant distribution $\mathfrak{D}^\theta = \mathrm{Span}\{X_3, X_4\}$. Clearly, the slant function is $\theta = \cos^{-1}(2u_3/\sqrt{1+u_3^2})$. Moreover, $\mathfrak{D}$ and $\mathfrak{D}^\theta$ are integrable. If M_T and M_θ are integral manifolds of $\mathfrak{D}$ and $\mathfrak{D}^\theta$, respectively, then, $M = M_T \times M_\theta$ is a Riemannian product of M_T and M_θ in $\mathbb{R}^9$.

Example 5.2. Consider the Euclidean 9-space $\mathbb{R}^9$ with its Cartesian coordinates $(x_1, \ldots, x_4, y_1, \ldots, y_4, t)$ and the almost contact structure

$$\varphi\left(\frac{\partial}{\partial x_i}\right) = -\frac{\partial}{\partial y_i}, \quad \varphi\left(\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial x_j}, \quad \varphi\left(\frac{\partial}{\partial t}\right) = 0, \quad 1 \leq i, j \leq 4.$$

Let M be a submanifold of $\mathbb{R}^9$ defined by the immersion ψ as follows:

$$\psi(u, v, w, s, t) = \left(u + v, \frac{1}{2}w^2, s\cos w, s\sin w, -u + v, \frac{1}{2}s^2, -w\sin s, w\cos s, t\right),$$

for any non-zero real numbers w and s. The tangent space of M is spanned by the following vectors

$$\begin{aligned} &X_1 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial y_1}, \quad X_2 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1}, \\ &X_3 = w\frac{\partial}{\partial x_2} - s\sin w \frac{\partial}{\partial x_3} + s\cos w \frac{\partial}{\partial x_4} - \sin s \frac{\partial}{\partial y_3} + \cos v \frac{\partial}{\partial y_4}, \\ &X_4 = \cos w \frac{\partial}{\partial x_3} + \sin w \frac{\partial}{\partial x_4} + s\frac{\partial}{\partial y_2} - w\cos s \frac{\partial}{\partial y_3} - w\sin s \frac{\partial}{\partial y_4}, \quad X_5 = \frac{\partial}{\partial t}. \end{aligned}$$

Then, M is a pointwise semi-slant submanifold such that the structure vector field $\xi = \frac{\partial}{\partial t}$ is tangent to M and $\mathfrak{D} = \mathrm{Span}\{X_1, X_2\}$ is an invariant distribution and $\mathfrak{D}^\theta = \mathrm{Span}\{X_3, X_4\}$ is a pointwise slant distribution with slant function $\theta = \cos^{-1}\left(\frac{(1-ws)\sin(w-s)-ws}{1+w^2+s^2}\right)$. It is easy to observe that both the distributions are integrable. If we denote the integral manifolds of $\mathfrak{D}$ and $\mathfrak{D}^\theta$ by M_T and M_θ, respectively, then M is a Riemannian product of invariant and pointwise slant submanifolds in $\mathbb{R}^9$.

Example 5.3. Let M be a submanifold of $\mathbb{R}^{13}$ given by the immersion $\psi : \mathbb{R}^5 \to \mathbb{R}^{13}$ as follows:

$$\begin{aligned}\psi(u_1, v_1, u_2, v_2, t) =&(u_1 - v_1,\ u_1\cos(u_2+v_2),\ u_1\sin(u_2+v_2),\ v_2,\ u_1\cos(u_2-v_2),\\ &u_1\sin(u_2-v_2),\ u_1+v_1,\ v_1\cos(u_2+v_2),\ v_1\sin(u_2+v_2),\ u_2,\\ &v_1\cos(u_2-v_2),\ v_1\sin(u_2-v_2),\ t),\end{aligned}$$

for non-zero functions u_1 and v_1. We use the almost contact structure from Example 5.2. Then, we have

$$\begin{aligned}X_1 =&\frac{\partial}{\partial x_1} + \cos(u_2+v_2)\frac{\partial}{\partial x_2} + \sin(u_2+v_2)\frac{\partial}{\partial x_3} + \cos(u_2-v_2)\frac{\partial}{\partial x_5}\\ &+ \sin(u_2-v_2)\frac{\partial}{\partial x_6} + \frac{\partial}{\partial y_1},\\ X_2 =&-\frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1} + \cos(u_2+v_2)\frac{\partial}{\partial y_2} + \sin(u_2+v_2)\frac{\partial}{\partial y_3} + \cos(u_2-v_2)\frac{\partial}{\partial y_5}\\ &+ \sin(u_2-v_2)\frac{\partial}{\partial y_6},\\ X_3 =&-u_1\sin(u_2+v_2)\frac{\partial}{\partial x_2} + u_1\cos(u_2+v_2)\frac{\partial}{\partial x_3} - u_1\sin(u_2-v_2)\frac{\partial}{\partial x_5}\\ &+ u_1\cos(u_2-v_2)\,\frac{\partial}{\partial x_6} - v_1\sin(u_2+v_2)\,\frac{\partial}{\partial y_2},\ +v_1\cos(u_2+v_2)\,\frac{\partial}{\partial y_3}\\ &+ \frac{\partial}{\partial y_4} - v_1\sin(u_2-v_2)\frac{\partial}{\partial y_5} + v_1\cos(u_2-v_2)\,\frac{\partial}{\partial y_6},\\ X_4 =&-u_1\sin(u_2+v_2)\frac{\partial}{\partial x_2} + u_1\cos(u_2+v_2)\,\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4} + u_1\sin(u_2-v_2)\frac{\partial}{\partial x_5}\\ &- u_1\cos(u_2-v_2)\,\frac{\partial}{\partial x_6} - v_1\sin(u_2+v_2)\,\frac{\partial}{\partial y_2},\ +v_1\cos(u_2+v_2)\,\frac{\partial}{\partial y_3}\\ &+ v_1\sin(u_2-v_2)\frac{\partial}{\partial y_5} - v_1\cos(u_2-v_2)\,\frac{\partial}{\partial y_6},\\ X_5 =&\frac{\partial}{\partial t}.\end{aligned}$$

By easy and direct computations we find that $\mathfrak{D} = \mathrm{Span}\{X_1, X_2\}$ is an invariant distribution and $\mathfrak{D}^\theta = \mathrm{Span}\{X_3, X_4\}$ is a pointwise slant distribution with slant function $\theta = \cos^{-1}\left(\frac{1}{1+2u_1^2+2v_1^2}\right)$. Hence, M is a pointwise semi-slant submanifold of $\mathbb{R}^{13}$. It is easy to observe that both the distributions are integrable. If we denote the integral manifolds of $\mathfrak{D}$ and $\mathfrak{D}^\theta$ by M_T and M_θ, respectively, then the product metric structure of M is given by

$$g = 4(du_1^2 + dv_1^2) + (1 + 2u_1^2 + 2v_1^2)(du_2^2 + dv_2^2) = g_{M_T} + f^2 g_{M_\theta}.$$

Hence, $M = M_T \times_f M_\theta$ is a warped product submanifold in $\mathbb{R}^{13}$ with warping function $f = \sqrt{1 + 2u_1^2 + 2v_1^2}$.

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[1]Faculty of Mathematics,
University of Bucharest,
Str. Academiei 14, 010014 Bucharest, Romania
Email address: imihai@fmi.unibuc.ro

[2]Department of Mathematics, Faculty of Science,
King Abdulaziz University,
21589 Jeddah, Saudi Arabia
Email address: siraj.ch@gmail.com

[3]Department of Mathematics and Computer Science,
Technical University of Civil Engineering Bucharest,
Lacul Tei Bvd. 122-124, 020396 Bucharest, Romania
Email address: adela.mihai@utcb.ro