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THE MAXIMUM NORM ANALYSIS OF SCHWARZ METHOD FOR ELLIPTIC QUASI-VARIATIONAL INEQUALITIES

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Abstract. In this paper, we present a maximum norm analysis of an overlapping Schwartz method on non matching grids for a quasi-variational inequality, where the obstacle and the second member depend on the solution. Our result improves and generalizes some previous results.

1. Introduction

Historically, Schwarz method has been introduced by Herman Amondus Schawarz, in order to resolve a purely theoretical matters. The Schawarz alternating method has been used to solve the stationary or evolutionary boundary valued problems, on domain which consists of two or more overlapping sub-domains, see for example [6, 7]. The solution is approximated by an infinite sequence of function, the result which is the resolution of a sequence of stationary or evolutionary boundary valued problems, in each of sub-domain.

In this work, we are interested in the analysis of error estimates in uniform norm for the quasi-variational inequality. Our goal is to generalize and improve some previous results given in [2–4, 10, 11] which concerning analysis of error estimates in uniform norm for the elliptic quasi-variational inequality. As in [2] they got the following approximation:

$$\|u_i - u_{ih}^{n+1}\|_\infty \le Ch^2|\log h|^3, \quad i = 1, 2,$$

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for the problem

$$\begin{cases} a(u, v-u) \geq (f, v-u) \text{ in } \Omega, & \text{for all } v \in K, \\ u \leq \psi, \quad v \leq \psi, \end{cases}$$

where K is a convex, closed and not empty set. In [4], they have obtained the same approximation for the following problem:

$$\begin{cases} a(u, v-u) \geq (f(u), v-u) \text{ in } \Omega, & \text{for all } v \in K(u), \\ u \leq \psi, \quad v \leq \psi, \end{cases}$$

also, for the non-coercive variational inequality, it has been reached in [11], the same approximation mentioned above. In [10], the authors studied a quasi-variational inequality related to control ergodic problem

$$\begin{cases} b(u_\alpha, v-u_\alpha) \geq (f + ru_\alpha, v-u_\alpha), & \alpha \in (0,1), \\ u_\alpha \leq Mu_\alpha, \quad v \leq Mu_\alpha, \end{cases}$$

and they got the following result:

$$\|u_{\alpha_i} - u_{\alpha_i h}^{n+1}\|_\infty \leq C\alpha^{-2} h^2 |\log h|^4, \quad i = 1, 2.$$

Finally in [3], the authors studied the following problem:

$$\begin{cases} a(u, v-u) \geq (f, v-u), & \text{for all } v \in K, \\ u \leq Mu, \quad Mu \geq 0, \\ Mu = k + \inf\limits_{\varepsilon \geq 0, x+\varepsilon \in \overline{\Omega}} u(x+\varepsilon), \\ \dfrac{\partial u}{\partial \eta} = \varphi \text{ in } \Gamma_0 \text{ and } u = 0 \text{ in } \Gamma/\Gamma_0, \end{cases}$$

and they obtained the following result:

$$\|u_i - u_{ih}^{n+1}\|_\infty \leq Ch^2 |\log h|^3, \quad i = 1, 2.$$

For our work, we claim about the general problem where the second member and the obstacle are related to the solution

$$\begin{cases} a(u, v-u) \geq (f(u), v-u) \text{ in } \Omega, & \text{for all } v \in K_g(u), \\ u \leq Mu, \quad v \leq Mu, \\ u = g \text{ on } \partial\Omega. \end{cases}$$

The outline of the paper, is as follows: in the second section, we will mention the same notations and assumptions, in the third section we will give our continuous problem, analogously in section four, we will define the discrete problem. Section five, is devoted to the L^∞-error analysis of the method.

2. Notation and Assumptions

Let Ω be an open in $\mathbb{R}^n$, with sufficiently smooth boundary $\partial\Omega$. For $u, v \in H^1(\Omega)$, consider the bilinear form as follows:

$$a(u,v) = \int_\Omega \left(\sum_{1\le i,j\le n} a_{ij}(x) \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + \sum_{1\le i\le n} a_i(x)\frac{\partial u}{\partial x_i} v + a_0(x) u.v \right) dx, \tag{2.1}$$

where $a_{ij}(x), a_i(x), a_0(x), x \in \overline{\Omega}$, $1 \le i, j \le n$, are sufficiently smooth coefficients and satisfying the following conditions:

$$\sum_{1\le i,j\le n} a_{ij}\xi_i\xi_j \ge \nu|\xi|^2, \quad \xi \in \mathbb{R}^n, \nu > 0,$$
$$a_0(x) \ge \beta > 0,$$

where β is a constant. The operator M is given by $Mu = k + \inf_{\varepsilon\ge 0, x+\varepsilon\in\overline{\Omega}} u(x+\varepsilon)$, where $k > 0$ and M satisfies

$$Mu \in W^{2,\infty}(\Omega), \quad Mu \ge 0 \text{ on } \partial\Omega : 0 \le g \le Mu, \tag{2.2}$$

where g is a regular function defined on $\partial\Omega$. Let f be a Lipschitzian non decreasing nonlinear function with rate α satisfying $\frac{\alpha}{\beta} < 1$ and $f \in L^\infty(\Omega)$, and $K_g(u)$ is an implicit convex and non empty set which defined as follows:

$$K_g(u) = \{v \in H^1(\Omega),\ v = g \text{ on } \partial\Omega,\ v \le Mu \text{ in } \Omega\}.$$

3. The Continuous Problem

We consider the following problem: Find $u \in k_g(u)$ the solution of

$$\begin{cases} a(u, v-u) \ge (f(u), v-u) \text{ in } \Omega, \quad \text{for all } v \in K_g(u), \\ u \le Mu, \quad v \le Mu, \\ u = g \text{ on } \partial\Omega. \end{cases} \tag{3.1}$$

We will present some results for our problem as the existence, uniqueness and other optimal properties which given in previous papers where we need them in the sequel.

Theorem 3.1 ([5]). *Under the previous conditions the problem* (3.1) *has an unique solution* $u \in K_g(u)$. *Moreover, we have*

$$u \in W^{2,p}(\Omega), \quad 2 \le p \le \infty.$$

Lemma 3.1 ([6]). *For all* u *and* $\tilde{u} \in K_g(u)$, *we have*

(a) *if* $u \le \tilde{u}$, *then* $Mu \le M\tilde{u}$ *and* $M(u+\lambda) = M(u) + \lambda$ *for all* $\lambda \in \mathbb{R}$;
(b) $\|Mu - M\tilde{u}\|_{L^\infty(\Omega)} \le \|u - \tilde{u}\|_{L^\infty(\Omega)}$.

3.1. **The continuous Schwarz sequences.** We decompose Ω in two sub-domains Ω_1, Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$ and u satisfies the local regularity condition:

$$u/_{\Omega_i} \in W^{2,p}(\Omega_i), \quad i = 1, 2, \text{ and } 2 \leq p < \infty,$$

denote by $\partial\Omega_i$ the boundary of Ω_i and $\Gamma_1 = \partial\Omega_1 \cap \Omega_2$, $\Gamma_2 = \partial\Omega_2 \cap \Omega_1$, $\Gamma_1 \cap \Gamma_2 = \emptyset$.

We define the following process. Choose $u_0 = k$ to be given, and define the alternating Schwarz sequences (u_1^{n+1}) on Ω_1 such that $u_1^{n+1} \in K(u_1^n)$ is solution of the following problem:

$$\begin{cases} a_1(u_1^{n+1}, v - u_1^{n+1}) \geq (f_1(u_1^n), v - u_1^{n+1}), \\ u_1^{n+1} \leq M u_1^n, \\ u_1^{n+1} = u_2^n \text{ on } \Gamma_1, \quad v = u_2^n \text{ on } \Gamma_1, \end{cases} \tag{3.2}$$

and (u_2^{n+1}) on Ω_2 such that $u_2^{n+1} \in K(u_2^n)$ solution of the following problem:

$$\begin{cases} a_2(u_2^{n+1}, v - u_2^{n+1}) \geq (f_2(u_2^n), v - u_2^{n+1}), \\ u_2^{n+1} \leq M u_2^n, \\ u_2^{n+1} = u_1^n \text{ on } \Gamma_2, \quad v = u_1^n \text{ on } \Gamma_2, \end{cases} \tag{3.3}$$

where $f_i = f/_{\Omega_i}$, $i = 1, 2$, and $(a_i(u, v)$ the form bilinear which defined in (2).

3.2. **Geometrical convergence.**

Theorem 3.2 ([3]). *The sequences (u_1^{n+1}), (u_2^{n+1}), $n \geq 0$, produced by the Schawarz alternating method converge geometrically to the solution u of the problem* (3.1)*, more precisely, there exist two constants $K_1, K_2 \in (0, 1)$ such that for all $n \geq 0$, we have*

$$\begin{aligned} \|u_1 - u_1^{n+1}\|_{L^\infty(\Omega_1)} &\leq K_1^n K_2^n \|u^0 - u\|_{L^\infty(\Gamma_1)}, \\ \|u_2 - u_2^{n+1}\|_{L^\infty(\Omega_2)} &\leq K_1^{n+1} K_2^n \|u^0 - u\|_{L^\infty(\Gamma_2)}. \end{aligned}$$

We will show an important proposition, which give the continuous dependence to the second member, the data g and the obstacle. We note that $u = \sigma(f(u), Mu, g)$, $\tilde{u} = \sigma(f(\tilde{u}), M\tilde{u}, \tilde{g})$, where $u, \tilde{u} \in K_g(u)$.

Proposition 3.1. *Under the previous hypotheses and notations, we have*

$$\|u - \tilde{u}\|_{L^\infty(\Omega_i)} \leq \|f(u) - f(\tilde{u})\|_{L^\infty(\Omega_i)} + \|Mu - M\tilde{u}\|_{L^\infty(\Omega_i)} + \|g - \tilde{g}\|_{L^\infty(\Gamma_i)},$$

where $\Gamma_i = \partial\Omega_i \cap \Omega_j$, $i, j = 1, 2$, and $i \neq j$.

Proof. Setting

$$\Phi = \|f(u) - f(\tilde{u})\|_{L^\infty(\Omega_i)} + \|Mu - M\tilde{u}\|_{L^\infty(\Omega_i)} + \|g - \tilde{g}\|_{L^\infty(\Gamma_i)},$$

we have

$$f(u) \leq f(\tilde{u}) + f(u) - f(\tilde{u}) \leq f(\tilde{u}) + \|f(u) - f(\tilde{u})\| \leq f(\tilde{u}) + \Phi.$$

Similarly, we have $g \leq \tilde{g} + \Phi$ and $Mu \leq M\tilde{u} + \phi$.

Now, making use of Lemma 3.2, we obtain

$$\sigma(f(u), Mu, g) \leq \sigma(f(\tilde{u}) + \Phi, M\tilde{u} + \Phi, \tilde{g} + \Phi) \\ \leq (f(\tilde{u}), M\tilde{u}, \tilde{g}) + \Phi,$$

so, $\sigma(f(u), Mu, g) - \sigma(f(\tilde{u}), M\tilde{u}, \tilde{g}) \leq \Phi$. Since $(f(u), Mu, g)$ and $(f(\tilde{u}), M\tilde{u}, \tilde{g})$ are symmetrical, we have $\sigma(f(\tilde{u}), M\tilde{u}, \tilde{g}) - \sigma(f(u), Mu, g) \leq \Phi$, and then

$$\|u - \tilde{u}\|_{L^\infty(\Omega_i)} \leq \|f(u) - f(\tilde{u})\|_{L^\infty(\Omega_i)} + \|Mu - M\tilde{u}\|_{L^\infty(\Omega_i)} + \|g - \tilde{g}\|_{L^\infty(\Gamma_i)}. \qquad \square$$

Remark 3.1. If $Mu = M\tilde{u}$, we have

$$\|u - \tilde{u}\|_{L^\infty(\Omega_i)} \leq \|f(u) - f(\tilde{u})\|_{L^\infty(\Omega_i)} + \|g - \tilde{g}\|_{L^\infty(\Gamma_i)}.$$

4. The Discrete Problem

We denote by V_h the standard piecewise linear finite element space, we consider the discrete quasi-variational inequality. Find $u_h \in K_{gh}(u_h)$ such that:

$$\text{(4.1)} \qquad \begin{cases} a(u_h, v - u_h) \geq (f(u_h), v - u_h), & \text{for all } u_h, v \in K_{gh}(u_h), \\ u_h \leq r_h M u_h, \\ u_h = \pi_h g \text{ on } \partial\Omega, \end{cases}$$

where $f \in L^\infty(\Omega)$; $Mu_h = k + \inf_{\varepsilon \geq 0, x+\varepsilon \in \overline{\Omega}} u_h(x + \varepsilon)$ and

$$K_{gh}(u_h) = \{v \in V_h : v = \pi_h g \text{ on } \partial\Omega, v \leq r_h M u_h \text{ in } \Omega\}.$$

We denote π_h the interpolation operator on $\partial\Omega$ and r_h is the usual finite element restriction operator in Ω.

4.1. **The discrete maximum principle.**

We assume that the respective matrices resulting from the discretization of problems (3.2), (3.1) are M-matrice [9].

Theorem 4.1 ([1])**.** *Let u and u_h be the solutions of problem* (3.1) *and* (4.1) *respectively, there exists a constant C_1 independent of h such that*

$$\|u - u_h\|_{L^\infty(\Omega)} \leq C_1 h^2 \log |h|^2.$$

Similarly, for the continuous case we will establish the discrete version of the lemma.

Lemma 4.1. *For all u_h and $\tilde{u_h} \in K_g(u_h)$ we have*

(a) *if $u_h \leq \tilde{u_h}$, then $Mu_h \leq M\tilde{u_h}$ and $M(u_h + \lambda) = M(u_h) + \lambda$ for all $\lambda \in \mathbb{R}$;*
(b) $\|Mu_h - M\tilde{u_h}\|_{L^\infty(\Omega)} \leq \|u_h - \tilde{u_h}\|_{L^\infty(\Omega)}$.

4.2. **The discrete Schwarz sequences.** For $i = 1, 2$, let $V_{h_i} = V_h(\Omega_i)$ be the space of continuous picewise linear function on τ_{h_i}, which vanish on $\partial\Omega \cap \partial\Omega_i$. For $w \in C(\Gamma_i)$, we define

$$V_{h_i}^{(w)} = \{v \in V_{h_i}, v = 0 \text{ on } \partial\Omega_i \cap \partial\Omega, v = \pi_{h_i}(w) \text{ on } \Gamma_2\},$$

where τ_{h_i} be a standard regular finite element triangulation in Ω_i, h_i being the mesh size. We suppose that the two triangulation are mutually independent on Ω_1, Ω_2, a triangle belonging to one triangulation does not necessarily belong to the other.

We now define the discrete countreparts of the continuous Schwarz sequences defined in (3.2) and (3.1) respectively, by $(u_{1h}^{n+1}) \in V_{h_1}^{(u_{2h}^n)}$, where (u_{1h}^{n+1}) is the solution of

$$\text{(4.2)} \qquad \begin{cases} a_1(u_{1h}^{n+1}, v - u_{1h}^{n+1}) \geq (f_1(u_{1h}^n), v - u_{1h}^{n+1}), \quad \text{for all } v \in V_{h_1}^{(u_{2h}^n)}, \\ u_{1h}^{n+1} \leq r_h M u_{1h}^n, \quad v \leq r_h m u_{1h}^n, \\ u_{1h}^{n+1} = u_{2h}^n \text{ on } \Gamma_1, \quad v = u_{2h}^n \text{ on } \Gamma_1, \end{cases}$$

and $(u_{2h}^{n+1}) \in V_{h2}^{(u_{1h}^{n+1})}$ such that (u_{2h}^{n+1}) is the solution of

$$\text{(4.3)} \qquad \begin{cases} a_2(u_{2h}^{n+1}, v - u_{2h}^{n+1}) \geq (f_2(u_{2h}^n), v - u_{2h}^{n+1}), \quad \text{for all } v \in V_{h_2}^{(u_{1h}^n)}, \\ u_{2h}^{n+1} \leq r_h M u_{2h}^n, \quad v \leq r_h M u_{2h}^n, \\ u_{2h}^{n+1} = u_{1h}^n \text{ on } \Gamma_2, \quad v = u_{2h}^n \text{ on } \Gamma_2. \end{cases}$$

We will finish this section by the discrete version of Proposition 3.1, this version plays an important role in the sequel.

Proposition 4.1. *Using the notations*

$$\begin{aligned} u_h =& \sigma(f(u_h), M u_h, \pi_h g), \\ \tilde{u_h} =& \sigma_h(f(\tilde{u_h}, M\tilde{u_h}, \pi_h \tilde{g}), \end{aligned}$$

where $u_h, \tilde{u_h} \in K_g(u_h)$, *we have*

$$\|u_h - \tilde{u_h}\|_{L^\infty(\Omega_i)} \leq \|f(u_h) - f(\tilde{u_h})\|_{L^\infty(\Omega_i)} + \|M u_h - M\tilde{u_h}\|_{L^\infty(\Omega_i)} + \|\pi_h g - \pi_h \tilde{g}\|_{L^\infty(\Gamma_i)},$$

$\Gamma_i = \partial\Omega_i \cap \Omega_j$, $i, j = 1, 2$, *and* $i \neq j$.

Proof. Similar for the continuous case. □

Remark 4.1. If $M u_h = M\tilde{u_h}$, we obtain

$$\|u_h - \tilde{u_h}\|_{L^\infty(\Omega_i)} \leq \|f(u_h) - f(\tilde{u_h})\|_{L^\infty(\Omega_i)} + \|\pi_h g - \pi_h \tilde{g}\|_{L^\infty(\Gamma_i)}.$$

5. L^∞-Error Estimate

We will use the algorithmic approach, which was used in [2, 4], but our problem is more complicated because the second member and the obstacle are related to the solution.

5.1. **Auxiliary sequences.** We introduce two discrete auxiliary sequences. Starting from $w_{ih}^0 = u_{ih}^0 = r_h M u_h^0 = k$, $i = 1,2$, define the sequences (w_{1h}^{n+1}) such that $w_{1h}^{n+1} \in V_{h_1}^{u_2^n}$

$$(5.1) \qquad \begin{cases} a_1(w_{1h}^{n+1}, v - w_{1h}^{n+1}) \geq (f_1(u_{1h}^n), v - w_{1h}^{n+1}), & \text{for all } v \in V_{h_1}^{(u_2^n)}, \\ w_{1h}^{n+1} \leq r_h M u_{1h}^n, \quad v \leq r_h M u_{1h}^n, \end{cases}$$

and (w_{2h}^{n+1}) such that $w_{2h}^{n+1} \in V_{h_2}^{(u_1^{n+1})}$ is a solution of

$$(5.2) \qquad \begin{cases} a_2(w_{2h}^{n+1}, v - w_{2h}^{n+1}) \geq (f_2({}_{2h}^n), v - w_{2h}^{n+1}), & \text{for all } v \in V_{h_2}^{(u_1^{n+1})}, \\ w_{2h}^{n+1} \leq r_h M u_{2h}^n, \quad v \leq r_h M u_{2h}^n. \end{cases}$$

Note that w_{ih}^{n+1} is the finite element approximation of u_i^{n+1} which defined in (3.2) and (3.1). The following lemma will play a crucial role in proving the main result of this paper. The demonstration of the lemma is an adaptation of the one in [2], given for the problem of variational inequality.

Lemma 5.1. *We have the following inequalities:*

$$\|u_1^{n+1} - u_{1h}^{n+1}\|_1 \leq \sum_{p=1}^{n+1} \|u_1^p - w_{1h}^p\|_1 + \sum_{p=0}^{n+1} \|u_2^p - w_{2h}^p\|_2,$$
$$\|u_2^{n+1} - u_{2h}^{n+1}\|_2 \leq \sum_{p=0}^{n+1} \|u_2^p - w_{2h}^p\|_2 + \sum_{p=1}^{n+1} \|u_1^p - w_{1h}^p\|_1.$$

Proof. In order to simplify the notation, we will adopt the following notations:

$$\begin{aligned} |\cdot|_1 =& \|\cdot\|_{L^\infty(\Gamma_1)}, \quad |\cdot|_2 = \|\cdot\|_{L^\infty(\Gamma_2)}, \\ \|\cdot\|_1 =& \|\cdot\|_{L^\infty(\Omega_1)}, \quad \|\cdot\|_2 = \|\cdot\|_{L^\infty(\Omega_2)}, \\ \pi_{h_1} =& \pi_{h_2} = \pi_h, \quad h_1 = h_2 = h. \end{aligned}$$

Started for $n = 0$, using the Remark 4.1, we get

$$\begin{aligned} \|u_1^1 - u_{1h}^1\|_1 \leq& \|u_1^1 - w_{1h}^1\|_1 + \|w_{1h}^1 - u_{1h}^1\|_1 \\ \leq& \|u_1^1 - w_{1h}^1\|_1 + \|f_1(u_1^0) - f_1(u_{1h}^0)\|_1 + |\pi_h M u_2^0 - \pi_h M u_{2h}^0|_1 \\ \leq& \|u_1^1 - w_{1h}^1\|_1 + |M u_2^0 - M u_{2h}^0|_1, \\ \|u_1^1 - u_{1h}^1\|_1 \leq& \|u_1^1 - w_{1h}^1\|_1 + \|M u_2^0 - M u_{2h}^0\|_2, \end{aligned}$$

and, from Lemma 4.1, we obtain

$$(5.3) \qquad \|u_1^1 - u_{1h}^1\|_1 \leq \|u_1^1 - w_{1h}^1\|_1 + \|u_2^0 - u_{2h}^0\|_2.$$

Similarly, we obtain

$$\begin{aligned}\|u_2^1-u_{2h}^1\|_2 \leq&\|u_2^1-w_{2h}^1\|_2+\|w_{2h}^1-u_{2h}^1\|_2\\ \leq&\|u_2^1-w_{2h}^1\|_2+\|f_2(u_{2h}^0)-f_2(u_{2h}^0)\|_2+|\pi_h Mu_1^1-\pi_h Mu_{1h}^1|_2\\ \leq&\|u_2^1-w_{2h}^1\|_2+|Mu_1^1-Mu_{1h}^1|_2\\ \leq&\|u_2^1-w_{2h}^1\|_2+\|Mu_1^1-Mu_{1h}^1\|_1\end{aligned}$$

and

$$\|u_2^1-u_{2h}^1\|_2 \leq \|u_2^1-w_{2h}^1\|_2+\|u_1^1-u_{1h}^1\|_1.$$

From (5.3), we get

$$\|u_2^1-u_{2h}^1\|_2 \leq \|u_1^1-w_{1h}^1\|_1+\|u_2^0-u_{2h}^0\|_2+\|u_2^1-w_{2h}^1\|_2, \tag{5.4}$$

so

$$\|u_1^1-u_{1h}^1\|_1 \leq \sum_{p=1}^{1}\|u_1^p-w_{1h}^p\|_1+\sum_{p=0}^{0}\|u_2^0-u_{2h}^0\|_2,$$

$$\|u_2^1-u_{2h}^1\|_2 \leq \sum_{p=0}^{1}\|u_2^p-w_{2h}^p\|_2+\sum_{p=1}^{1}\|u_1^p-w_{1h}^p\|_1.$$

For $n=1$, we have

$$\begin{aligned}\|u_1^2-u_{1h}^2\|_1 \leq&\|u_1^2-w_{1h}^2\|_1+\|w_{1h}^2-u_{1h}^2\|_1\\ \leq&\|u_1^2-w_{1h}^2\|_1+\|f(u_{1h}^1)-f(u_{1h}^1)\|_1+|\pi_h Mu_2^1-\pi_h Mu_{2h}^1|_1,\\ \leq&\|u_1^2-w_{1h}^2\|_1+|Mu_2^1-Mu_{2h}^1|_1\\ \leq&\|u_1^2-w_{1h}^2\|_1+\|u_2^1-u_{2h}^1\|_2.\end{aligned}$$

From (5.4), we get

$$\|u_1^2-u_{1h}^2\|_1 \leq \|u_2^1-w_{1h}^2\|_1+\|u_2^1-w_{2h}^1\|_2+\|u_1^1-w_{1h}^1\|_1+\|u_2^0-u_{2h}^0\|_2. \tag{5.5}$$

Similarly, we obtain

$$\begin{aligned}\|u_2^2-u_{2h}^2\|_2 \leq&\|u_2^2-w_{2h}^2\|_2+\|w_{2h}^2-u_{2h}^2\|_2\\ \leq&\|u_2^2-w_{2h}^2\|_2+\|f(u_{2h}^2)-f(u_{2h}^2)\|_2+|\pi_h Mu_1^2-\pi_h Mu_{1h}^2|_2\\ \leq&\|u_2^2-w_{2h}^2\|_2+\|u_1^2-u_{1h}^2\|_1.\end{aligned}$$

From (5.5), we get

$$\|u_2^2-u_{2h}^2\|_2 \leq \|u_2^2-w_{2h}^2\|_2+\|u_2^1-w_{1h}^2\|_1+\|u_2^1-w_{2h}^1\|_2+\|u_1^1-w_{1h}^1\|_1+\|u_2^0-u_{2h}^0\|_2,$$

where

$$\|u_1^2-u_{1h}^2\|_1 \leq \sum_{p=1}^{2}\|u_1^p-w_{1h}^p\|_1+\sum_{p=0}^{1}\|u_2^p-w_{2h}^p\|_2$$

and

$$\|u_2^2-u_{2h}^2\|_2 \leq \sum_{p=0}^{2}\|u_1^p-w_{2h}^p\|_2+\sum_{p=1}^{2}\|u_1^p-w_{1h}^p\|_1.$$

We go to the second step. Suppose that

$$\|u_2^n - u_{2h}^n\|_2 \le \sum_{p=0}^{n} \|u_2^p - w_{2h}^p\|_2 + \sum_{p=1}^{n} \|u_1^p - w_{1h}^p\|_1. \tag{5.6}$$

We claim the first inequality, for $i = 1$,

$$\begin{aligned}\|u_1^{n+1} - u_{1h}^{n+1}\|_1 \le& \|u_1^{n+1} - w_{1h}^{n+1}\|_1 + \|w_{1h}^{n+1} - u_{1h}^{n+1}\|_1 \\ \le& \|u_1^{n+1} - w_{1h}^{n+1}\|_1 + \|f_1(u_{1h}^n) - f_1(u_{1h}^n)\|_1 + |\pi_h M u_2^n - \pi_h M u_{2h}^n|_1 \\ \le& \|u_1^{n+1} - w_{1h}^{n+1}\|_1 + \|M u_2^n - M u_{2h}^n\|_2 \\ \le& \|u_1^{n+1} - w_{1h}^{n+1}\|_1 + \|u_2^n - u_{2h}^n\|_2.\end{aligned}$$

From (5.6), we get

$$\|u_1^{n+1} - u_{1h}^{n+1}\|_1 \le \|u_1^{n+1} - w_{1h}^{n+1}\|_1 + \sum_{p=0}^{n} \|u_1^p - w_{1h}^p\|_1 + \sum_{p=1}^{n} \|u_1^p - w_{1h}^p\|_1.$$

Consequently,

$$\|u_1^{n+1} - u_{1h}^{n+1}\|_1 \le \sum_{p=1}^{n+1} \|u_1^p - w_{1h}^p\|_1 + \sum_{p=0}^{n} \|u_2^p - w_{2h}^p\|_2. \tag{5.7}$$

For the second inequality, $i = 2$, we have

$$\begin{aligned}\|u_2^{n+1} - u_{2h}^{n+1}\|_2 \le& \|u_2^{n+1} - w_{2h}^{n+1}\|_2 + \|w_{2h}^{n+1} - u_{2h}^{n+1}\|_2 \\ \le& \|u_2^{n+1} - w_{2h}^{n+1}\|_2 + |\ f_2(u_{2h}^n) - f_2(u_{2h}^n)\|_2 + |\pi_h M u_1^{n+1} - \pi_h M u_{1h}^{n+1}|_2 \\ \le& \|u_2^{n+1} - w_{2h}^{n+1}\|_2 + \|M u_1^{n+1} - M u_{1h}^{n+1}\|_1 \\ \le& \|u_2^{n+1} - w_{2h}^{n+1}\|_2 + \|u_1^{n+1} - u_{1h}^{n+1}\|_1.\end{aligned}$$

From (5.7), we get

$$\|u_2^{n+1} - u_{2h}^{n+1}\|_2 \le \|u_2^{n+1} - w_{2h}^{n+1}\|_2 + \sum_{p=1}^{n+1} \|u_1^p - w_{1h}^p\|_1 + \sum_{p=0}^{n} \|u_2^p - w_{2h}^p\|_2.$$

Consequently,

$$\|u_2^{n+1} - u_{2h}^{n+1}\|_2 \le \sum_{p=0}^{n+1} \|u_2^p - w_{2h}^p\|_2 + \sum_{p=1}^{n+1} \|u_1^p - w_{1h}^p\|_1. \qquad \square$$

5.2. L^∞ **error estimate.** The main result is given as follows.

Theorem 5.1. *Setting $h = \max\{h_1, h_2\}$, so there exists a constant C independent of h and n such that*

$$\|u_i - u_{ih}^{n+1}\|_{L^\infty(\Omega_i)} \le C h^2 |\log h|^3, \quad i = 1, 2.$$

Proof. Indeed, let $K = \max\{k_1, k_2\}$, for $i = 1$ we have

$$\begin{aligned}\|u_1 - u_{1h}^{n+1}\|_{L^\infty(\Omega_1)} \leq&\|u_1 - u_1^{n+1}\|_{L^\infty(\Omega_1)} + \|u_1^{n+1} - u_{1h}^{n+1}\|_{L^\infty(\Omega_1)}\\ \leq&\|u_1 - u_1^{n+1}\|_{L^\infty(\Omega_1)} + \sum_{p=1}^{n+1}\|u_1^p - w_{1h}^p\|_1 + \sum_{p=0}^{n+1}\|u_2^p - w_{2h}^p\|_2\\ \leq&K^{2n}\|u^0 - u\|_{L^\infty(\Gamma_1)} + 2(n+1)C_1 h^2|\log h|^2,\end{aligned}$$

where we used Lemma 4.1 and Theorem 3.1, respectively. Now, setting $K^{2n} \leq h^2$ we get $\|u_1 - u_{1h}^{n+1}\|_{L^\infty(\Omega_1)} \leq Ch^2|\log h|^3$. Similarly, we obtain the same result for $i = 2$. □

Remark 5.1. Confirmation for what we mentioned previously that this result is a generalization to the previous works, we note that:

(a) if the second member and the obstacle are not related to the solution, we get [2];
(b) if only the obstacle is related to the solution, we get [3];
(c) if only the second member is related to the solution, we get [4, 10, 11].

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