Kragujevac Journal of Mathematics
Volume 45(4) (2021), Pages 597–605.

ON n-ABSORBING IDEALS IN A LATTICE

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Abstract. Let L be a lattice, and let n be a positive integer. In this article, we introduce n-absorbing ideals in L. We give some properties of such ideals. We show that every n-absorbing ideal I of L has at most n minimal prime ideals. Also, we give some properties of 2-absorbing and weakly 2-absorbing ideals in L. In particular we show that in every non-zero distributive lattice L, 2-absorbing and weakly 2-absorbing ideals are equivalent.

1. Introduction

The concept of a 2-absorbing ideal in a commutative ring with identity, which is a generalization of prime ideals, was defined in [2] by Badawi. Anderson and Badawi [1] generalized the concept of a 2-absorbing ideal to an n-absorbing ideal. According to their definition, a proper ideal I of commutative ring R is called an n-absorbing ideal whenever $a_1a_2\cdots a_{n+1} \in I$, then there are n of the a_i's whose product is in I for every $a_1, \ldots, a_{n+1} \in R$. Badawi and Darani [3] studied weakly 2-absorbing ideals which are generalizations of weakly prime ideals. The concepts of 2-absorbing, weakly 2-absorbing, 2-absorbing primary and weakly 2-absorbing primary elements in multiplicative lattices are studied in [10] and [5] as generalizations of prime and weakly prime elements. The concepts of φ-prime, φ-primary ideals are recently introduced in [4, 7], and generalizations of these are studied in [12]. Celikel et al. in [6] extended the concepts of 2-absorbing elements to φ-2-absorbing elements and investigated some characterizations in some special lattices. In [16], Wasadikar and Gaikwad introduced 2-absorbing and weakly 2-absorbing ideals in lattices and studied their properties.

This article is organized as follows. In Section 2, we review some basic notions and properties from lattice theory. In Section 3, we study some basic properties of

Key words and phrases. Lattice, minimal ideal, 2-absorbing ideal, n-absorbing ideal.
2010 *Mathematics Subject Classification.* Primary: 03G10. Secondary: 03G99.
DOI 10.46793/KgJMat2104.597E
Received: June 12, 2018.
Accepted: March 25, 2019.

2-absorbing and weakly 2-absorbing ideal in a lattice. For example in Proposition 3.4, we show that 2-absorbing and weakly 2-absorbing ideals are equivalent in a distributive lattice. Also, we show that in a distributive lattice, an ideal I is a 2-absorbing ideal if and only if $I_i \wedge I_j \subseteq I$ for some ideals I_1, I_2, I_3 of I where $I_1 \wedge I_2 \wedge I_3 \subseteq I$. In Section 4, we introduce the concept of an n-absorbing ideal in a lattice and give some basic properties of these ideals. For example, we show that an n-absorbing ideal is m-absorbing for every $m \geq n$. In a major result of this section (Proposition 4.5) we show that a n-absorbing ideal has at most n minimal prime ideals.

2. Preliminaries

In this section, we recall some concepts from lattice theory, see [8]. A partially ordered set $(L; \leq)$ is a lattice if $\sup\{a, b\}$ and $\inf\{a, b\}$ exist for all $a, b \in L$. A nonempty subset I of a lattice L is called an ideal if it is a sublattice of L and $x \in I$ and $a \in L$ imply that $x \wedge a \in I$. An ideal I of L is proper if $I \neq L$. A proper ideal I of L is prime if $a \wedge b \in I$ implies that $a \in I$ or $b \in I$, and it is weakly prime if $0 \neq a \wedge b \in I$ implies that either $a \in I$ or $b \in I$. A prime ideal P of L is said to be a minimal prime ideal if there is no prime ideal which is properly contained in P. Also, a prime ideal P of L is said to be a minimal prime ideal belonging to an ideal I, if $I \subseteq P$ and there are no prime ideals strictly contained in P that contain I. If an ideal I of a lattice L is contained in a prime ideal P of a lattice L, then P contains a minimal prime ideal belonginig to I. Note that a minimal prime ideal belonging to the zero ideal of L is a minimal prime ideal of L. The set of minimal prime ideals belonging to the ideal I of L denoted by $\mathrm{Min}(I)$. Let I be an ideal of a distributive lattice L with 0, and let P be a prime ideal such that $P \supseteq I$. The prime ideal P is a element of $\mathrm{Min}(I)$ if and only if for each $x \in P$ there is a $y \notin P$ such that $x \wedge y \in I$. All these results can be found in [15].

For basic facts concerning the fractions of a lattice we refer to [9]. Let L be a non-empty distributive lattice with 0, and let S be a non-empty subset of L which is a complete sublattice. Define a binary relation $\sim_S$ on $L \times S$ by

$$(a, b) \sim_S (c, d) \Leftrightarrow (\exists t \in S)(a \wedge d) \wedge t = (b \wedge c) \wedge t.$$

The relation $\sim_S$ on $L \times S$ is an equivalence relation. The set of all equivalence classes of $\sim_S$ is denoted by $L/\sim_S$. In other words, $L/\sim_S = \{[(a, b)]_{\sim_S} : a \in L, b \in S\}$. Let $m = \bigwedge_{x \in S} x$, then $(a, m) \sim_S (b, m) \Leftrightarrow (a, m) \sim_{\{m\}} (b, m)$ and $L/\sim_S = L/\sim_{\{m\}}$

From now on, $L/\sim_S$ will be denoted by $S^{-1}L$ and it is called the fractions of L with respect to S. Any element $[(a, b)]_{\sim_S} \in S^{-1}L$ is shown by $\frac{a}{b}$. We can consider every S as a singleton $\{m\}$, where $m = \bigwedge_{x \in S} x$. Therefore, from now on we assume S to be the singleton $\{m\}$. So, we can write $\frac{a}{m}$ for $\frac{a}{b}$. For $\frac{a_1}{m}$ and $\frac{a_2}{m} \in S^{-1}L$, we have $\frac{a_1}{m} = \frac{a_2}{m}$ if and only if $a_1 \wedge m = a_2 \wedge m$. $(S^{-1}L, \leq)$ is a partially ordered set, where $\leq$ is defined as follows:

$$\frac{a}{m} \leq \frac{b}{m} \Leftrightarrow a \wedge m \leq b \wedge m.$$

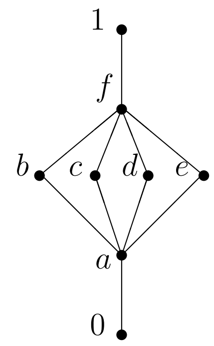

FIGURE 1.

The well-defined binary operations $\vee, \wedge : S^{-1}L \times S^{-1}L \to S^{-1}L$ are given by

$$\frac{a_1}{m} \wedge \frac{a_2}{m} = \frac{(a_1 \wedge a_2)}{m}$$

and

$$\frac{a_1}{m} \vee \frac{a_2}{m} = \frac{(a_1 \vee a_2)}{m}.$$

3. 2-Absorbing Ideals

In this section, we give some properties of 2-absorbing and weakly 2-absorbing ideals. We recall that from [16], a proper ideal I of lattice L is said to be a 2-absorbing ideal if for any $a_1, a_2, a_3 \in L$, $a_1 \wedge a_2 \wedge a_3 \in I$ implies $a_i \wedge a_j \in I$ for some $i, j \in \{1, 2, 3\}$ and weakly 2-absorbing ideal if for any $a_1, a_2, a_3 \in L$, $0 \neq a_1 \wedge a_2 \wedge a_3 \in I$ implies $a_i \wedge a_j \in I$ for some $i, j \in \{1, 2, 3\}$. Let I be a weakly 2-absorbing ideal of a lattice L and $a_1, a_2, a_3 \in L$. We say that (a_1, a_2, a_3) is a triple-zero of I if $a_1 \wedge a_2 \wedge a_3 = 0$ and for every $i, j \in \{1, 2, 3\}$, $a_i \wedge a_j \notin I$.

Example 3.1. Let $L = \{0, a, b, c, d, e, f, 1\}$ be a lattice, whose Hasse diagram is given in the Figure 1.

Consider the ideal $I = \downarrow a$. It is clear that I is a 2-absorbing ideal of L, but I is not a prime ideal of L.

Definition 3.1. Let I be an ideal of a lattice L. The radical of I, denoted by $\operatorname{Rad} I$, is the intersection all prime ideals P which contain I. If the set of prime ideals containing I is empty, then $\operatorname{Rad} I$ is defined to be L.

Proposition 3.1. *Every ideal I of a distributive lattice with 0 is the intersection of all prime ideals containing it, i.e.,* $\operatorname{Rad} I = I$.

Proof. See Page 64, Corollary 18 of [8]. □

Proposition 3.2. *Let I be a 2-absorbing ideal of distributive lattice L. Then there are at most 2 prime ideals of L minimal over I.*

Proof. Suppose that $\operatorname{Min}(I)$ has at least there elements. Let P_1, P_2 be two distinct prime ideals of L that are minimal over I. Hence, there is a $x_1 \in P_1 \setminus P_2$ and a $x_2 \in P_2 \setminus P_1$. First we show that $x_1 \wedge x_2 \in I$. By Lemma 3.1 of [11], there is $c_1 \in L \setminus P_2$ and $c_2 \in L \setminus P_1$ such that $x_1 \wedge c_2 \in I$ and $x_2 \wedge c_1 \in I$. Then $x_1 \wedge c_2 \wedge x_2 \in I$ and $x_2 \wedge c_1 \wedge x_1 \in I$, which implies that $(c_1 \vee c_2) \wedge x_1 \wedge x_2 \in I$. Since I is a 2-absorbing ideal of L, we conclude that $(c_1 \vee c_2) \wedge x_1 \in I$ or $(c_1 \vee c_2) \wedge x_2 \in I$ or $x_1 \wedge x_2 \in I$. If $(c_1 \vee c_2) \wedge x_1 \in I$, since $I \subseteq P_2$ and P_2 is a prime ideal, we have $x_1 \in P_2$ or $c_1 \vee c_2 \in P_2$, which is a contradiction. Therefore, $(c_1 \vee c_2) \wedge x_1 \notin I$. Similarity, $(c_1 \vee c_2) \wedge x_2 \notin I$ and so, $x_1 \wedge x_2 \in I$.

Now, suppose that there is a $P_3 \in \operatorname{Min}(I)$ such that P_3 is neither P_1 nor P_2. Then we can chose $y_1 \in P_1 \setminus (P_2 \cup P_3)$, $y_2 \in P_2 \setminus (P_1 \cup P_3)$, and $y_3 \in P_3 \setminus (P_1 \cup P_2)$. By the previous argument $y_1 \wedge y_2 \in I$. Since $I \subseteq P_1 \cap P_2 \cap P_3$ and $y_1 \wedge y_2 \in I$, we conclude that either $y_1 \in P_3$ or $y_2 \in P_3$, which is a contradiction. Hence, $\operatorname{Min}(I)$ contains at most two elements. □

Corollary 3.1. *Let I be a 2-absorbing ideal of a distributive lattice L. If I is not a prime ideal of L, then $|\operatorname{Min}(I)| = 2$.*

Proof. Let $|\operatorname{Min}(I)| \neq 2$. Then by Proposition 3.2, $|\operatorname{Min}(I)| = 1$. Let P be a minimal prime ideal of L such that $I \subseteq P$. Therefore by Proposition 3.1, $P = \operatorname{Rad} I = I$ and so I is a prime ideal which is a contradiction. Thus $|\operatorname{Min}(I)| = 2$. □

Proposition 3.3. *Suppose that I is a proper ideal of a distributive lattice L. Then the following statements are equivalent:*

(1) *I is a 2-absorbing ideal of L;*
(2) *If $I_1 \wedge I_2 \wedge I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of L, then $I_i \wedge I_j \subseteq I$ for some $i, j \in \{1, 2, 3\}$.*

Proof. (1) ⇒ (2). If I is a prime ideal, it is clear. Now, let I be not a prime ideal, by Corollary 3.1, we conclude that $\operatorname{Min}(I) = \{P_1, P_2\}$. Then by Proposition 3.1, $I = P_1 \cap P_2$. Now, let $I_1 \wedge I_2 \wedge I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of L. Then, $I_1 \wedge I_2 \wedge I_3 \subseteq P_i$ for $i = 1, 2$ and so, there exists $1 \leq i_1, i_2 \leq 3$ such that $I_{i_1} \subseteq P_1$ and $I_{i_2} \subseteq P_2$. Therefore, $I_{i_1} \cap I_{i_2} \subseteq P_1 \cap P_2 = I$.

(2) ⇒ (1). It is obvious. □

Proposition 3.4. *For every proper ideal $I \neq \{0\}$ in distributive lattice L, the following statements are equivalent:*

(1) *I is a 2-absorbing ideal;*
(2) *I is a weakly 2-absorbing ideal.*

Proof. (1) ⇒ (2). It is evident.

(2) ⇒ (1). Let I be a weakly 2-absorbing ideal of L that is not a 2-absorbing ideal. Then there exist $a_1, a_2, a_3 \in L$ such that $a_1 \wedge a_2 \wedge a_3 \in I$ and $a_i \wedge a_j \notin I$ for all $i \neq j \in \{1, 2, 3\}$. Consider $0 \neq a \in I$. Since $0 \neq (a_1 \vee a) \wedge (a_2 \vee a) \wedge (a_3 \vee a) \in I$, we

conclude that there exist $i, j \in \{1, 2, 3\}$ such that $(a_i \vee a) \wedge (a_j \vee a) \in I$. So $a_i \wedge a_j \in I$, for some $i, j \in \{1, 2, 3\}$, which is a contradiction. □

For an ideal I of a lattice L and $a, b \in L$, we define $a \wedge b \wedge I = \{a \wedge b \wedge i : i \in I\}$.

Proposition 3.5. *Let I be a weakly 2-absorbing ideal of distributive lattice L, and let (a_1, a_2, a_3) be a triple-zero of I for some $a_1, a_2, a_3 \in L$. Then the following statements hold:*

(1) $a_1 \wedge a_2 \wedge I = a_2 \wedge a_3 \wedge I = a_1 \wedge a_3 \wedge I = \{0\}$;
(2) $a_1 \wedge I = a_2 \wedge I = a_3 \wedge I = \{0\}$.

Proof. (1) See Theorem 3.1 of [16].

(2) Suppose that $a_1 \wedge a \neq 0$ for some $a \in I$. Then, by (1), we have

$$\begin{aligned} a_1 \wedge (a_2 \vee a) \wedge (a_3 \vee a) =& a_1 \wedge \big((a_2 \wedge a_3) \vee a)\big) \\ =&(a_1 \wedge a_2 \wedge a_3) \vee (a_1 \wedge a) \\ =&0 \vee (a_1 \wedge a) \\ =&a_1 \wedge a \\ \neq&0. \end{aligned}$$

Then, by Proposition 3.4, we have $a_1 \wedge a_2 \in I$ or $a_1 \wedge a_3 \in I$ or $a_2 \wedge a_3 \in I$, which is a contradiction. Thus $a_1 \wedge I = \{0\}$. Similarly, $a_2 \wedge I = a_3 \wedge I = \{0\}$. □

4. n-Absorbing Ideals

In this section, we introduce the concept of an n-absorbing ideal in a lattice and give some basic properties of them.

Definition 4.1. Let n be a positive integer. A proper ideal I of a lattice L is an n-absorbing ideal of L whenever $a_1 \wedge a_2 \wedge \cdots \wedge a_{n+1} \in I$, then there are n of the a_i's whose meet is in I for every $a_1, a_2, \ldots, a_{n+1} \in L$.

It is easy to see that if I is an n-absorbing ideal of L, then I is an m-absorbing ideal of L for all $m \geq n$. Also, a proper ideal I of L is n-absorbing if and only if whenever $a_1 \wedge a_2 \wedge \cdots \wedge a_m \in I$ for $a_1, \ldots, a_m \in I$ with $m \geq n$ then there are n of a_i's whose meet is in I.

Proposition 4.1. *If I_j is an n_j-absorbing ideal of L for each $1 \leq j \leq m$, then $\bigcap_{i=1}^m I_j$ is an n-absorbing ideal, where $n = \sum_{i=1}^m n_j$.*

Proof. Let $I_1, \ldots, I_m$ be proper ideals of L such that I_j is an n_j-absorbing and $k > n_1 + \cdots + n_m$. Suppose that $\bigwedge_{i=1}^k x_i \in \bigcap_{j=1}^m I_j$. Since for all j, I_j is n_j-absorbing ideal, a meet of n_j of these k elements belongs to I_j. Let the collection of those elements be denoted A_j and $A = \bigcup_{j=1}^m A_j$. Thus A has at most $n_1 + \cdots + n_m$ elements. Now since I_j is an ideal, the meet of all element of A must be in I_j for every $1 \leq j \leq m$. So $\bigcap_{j=1}^m I_j$ contains a meet of at most $n_1 + \cdots + n_m$ elements. Thus, the intersections of the I_j's is an $(n_1 + \cdots + n_m)$-absorbing ideal. □

Proposition 4.2. *If $\{I_\lambda\}_{\lambda\in\Lambda}$ is a non-empty chain of n-absorbing ideals of L, then $\bigcap_{\lambda\in\Lambda} I_\lambda$ is an n-absorbing ideal.*

Proof. Let $a_1,\ldots,a_{n+1} \in L$ such that $\bigwedge_{i=1}^{n+1} a_i \in J$ and $J = \bigcap_{\lambda\in\Lambda} I_\lambda$. Let $\widehat{a_i} = \bigwedge_{j\neq i} a_j$ and $\widehat{a_i} \notin J$ for all $1 \leq i \leq n$. Then for each $1 \leq i \leq n$, there exists an n-absorbing ideal I_{λ_i} such that $\widehat{a_i} \notin I_{\lambda_i}$. We may assume that $I_{\lambda_1} \subseteq \cdots \subseteq I_{\lambda_n}$. Consider $\mu \in \Lambda$. If $I_\mu \subseteq I_{\lambda_1} \subseteq \cdots \subseteq I_{\lambda_n}$, then $\widehat{a_i} \notin I_\mu$ for each $1 \leq i \leq n$. Now since $\bigwedge_{i=1}^{n+1} a_i \in J$ and I_μ is an n-absorbing ideal of L, we have $\widehat{a_{n+1}} \in I_\mu$. If there exists $1 \leq j \leq n$ such that $I_{\lambda_1} \subseteq \cdots \subseteq I_{\lambda_{j-1}} \subseteq I_\mu \subseteq I_{\lambda_j} \subseteq \cdots \subseteq I_{\lambda_n}$, then $\widehat{a_i} \in I_{\lambda_1}$ for each $1 \leq i \leq n$. Now since $\bigwedge_{i=1}^{n+1} a_i \in I_{\lambda_1}$ and I_{λ_1} is an n-absorbing ideal of L, we conclude that $\widehat{a_{n+1}} \in I_{\lambda_1}$ and so $\widehat{a_{n+1}} \in I_\mu$ for every $\mu \in \Lambda$. Therefore, $\widehat{a_{n+1}} \in J$. □

Proposition 4.3. *If I is an ideal of distributive lattice L such that $L \setminus I$ is closed under meet of $n+1$ elements, then I is an n-absorbing ideal.*

Proof. Let $a_1,\ldots,a_{n+1} \in L$ such that $\bigwedge_{i=1}^{n+1} a_i \in I$ and $\widehat{a_i} = \bigwedge_{j\neq i} a_j$ for each $1 \leq i \leq n+1$. Assume that $\widehat{a_i} \notin I$ for each $1 \leq i \leq n+1$. Since $L \setminus I$ is closed under the meet of $n+1$ elements, we have $\bigwedge_{i=1}^{n+1} a_i = \bigwedge_{i=1}^{n+1} \widehat{a_i} \in L \setminus I$ which is a contradiction. Which implies that I is an n-absorbing ideal. □

Let S be a non-empty subset of a lattice L. We say that S is a multiplicatively closed subset of L if $x \wedge y \in S$ for all x and y of S.

Proposition 4.4. *If S is a multiplicatively closed subset of L which does not meet the ideal I, then I is contained in an ideal M which is maximal with respect to the property of not meeting S and M is an n-absorbing ideal.*

Proof. Let $\mathcal{F} = \{J \mid \text{J is an ideal of L which does not meet } S \text{ and } I \subseteq J\}$. Since $I \in \mathcal{F}$, $\mathcal{F} \neq \emptyset$. Hence, by Zorn's Lemma, $(\mathcal{F}, \subseteq)$ has a maximal element say M. We show that M is an n-absorbing ideal. Let $a_1,\ldots,a_{n+1} \in L$ and for every $1 \leq i \leq n+1$, $\widehat{a_i} = \bigwedge_{j\neq i} a_j \notin M$. Then $(M\vee \downarrow \widehat{a_i}) \cap S \neq \emptyset$. Let $x_i \in (M\vee \downarrow \widehat{a_i}) \cap S$ for each $1 \leq i \leq n+1$. Since S is a multiplicatively closed subset of L, $\bigwedge_{i=1}^{n+1} x_i \in S$ and $\bigwedge_{i=1}^{n+1} x_i \in \bigcup_{i=1}^{n+1} (M\vee \downarrow \widehat{a_i})$. If $\bigwedge_{i=1}^{n+1} a_i \in M$, then $\bigwedge_{i=1}^{n+1} x_i \in M \cap S$ which is not true as $M \in \mathcal{F}$. Therefore, $\bigwedge_{i=1}^{n+1} a_i \notin M$ and so M is an n-absorbing ideal. □

Proposition 4.5. *Let I be an n-absorbing ideal of L. Then there are at most n prime ideals of L minimal over I.*

Proof. We may assume that $n \geq 2$, since an 1-absorbing ideal is a prime ideal. Suppose that $P_1, P_2, \ldots, P_n, P_{n+1}$ are distinct prime ideals of L minimal over I. Thus for each $1 \leq i \leq n$, there is an element x_i of $P_i \setminus \bigcup_{\substack{1\leq k\leq n+1\\ k\neq i}} P_k$. For each $1 \leq i \leq n$, there is an element $c_i \in L \setminus P_i$ such that $x_i \wedge c_i \in I$ and hence $x_1 \wedge \cdots \wedge x_n \wedge c_i \in I$. Therefore, $x_1 \wedge x_2 \wedge \cdots \wedge x_n \wedge (c_1 \vee c_2 \vee \cdots \vee c_n) \in I$. Since $x_i \in P_i \setminus \bigcup_{\substack{1\leq k\leq n+1\\ k\neq i}} P_k$ and $x_i \wedge c_i \in I \subseteq P_1 \cap P_2 \cap \cdots \cap P_n$ for each $1 \leq i \leq n$, we conclude that $c_i \in (\bigcap_{\substack{1\leq k\leq n\\ k\neq i}} P_k) \setminus P_i$

for each $1 \leq i \leq n$, and thus $c_1 \vee c_2 \vee \cdots \vee c_n \notin P_i$ for each $1 \leq i \leq n$. Hence,

$$(c_1 \vee c_2 \vee \cdots \vee c_n) \wedge \bigwedge_{\substack{1 \leq k \leq n \\ k \neq i}} x_k \notin P_i,$$

and so, $(c_1 \vee c_2 \vee \cdots \vee c_n) \wedge \bigwedge_{\substack{1 \leq k \leq n \\ k \neq i}} x_k \notin I$ for each $1 \leq i \leq n$. Since I is an n-absorbing ideal of L, we conclude that $x_1 \wedge \cdots \wedge x_n \in I \subseteq P_{n+1}$. Then $x_i \in P_{n+1}$ for some $1 \leq i \leq n$, which is a contradiction. Hence there are at most n prime ideals of L minimal over I. □

Let L be a distributive lattice and $S := \{m\} \subseteq L$. We recall from [9] that if I is an ideal of L, then $S^{-1}I$ is an ideal of $S^{-1}L$. Moreover, every ideal of $S^{-1}L$ can be represented as $S^{-1}I$, where I is an ideal of L.

Proposition 4.6. *Let I be an ideal of distributive lattice L and $S := \{m\} \subseteq L$. Then I is an n-absorbing ideal of L if and only if $S^{-1}I$ is an n-absorbing ideal of $S^{-1}L$.*

Proof. Let $\frac{a_1}{m}, \ldots, \frac{a_{n+1}}{m} \in S^{-1}L$ such that $\bigwedge_{i=1}^{n+1} \frac{a_i}{m} \in S^{-1}I$. Then $\frac{\bigwedge_{i=1}^{n+1} a_i}{m} \in S^{-1}I$ and so $\bigwedge_{i=1}^{n+1} a_i \in I$. Since I is a 2-absorbing ideal, we conclude that there exists an element i in $\{1, 2, \ldots, n+1\}$ such that $\widehat{a_i} \in I$, which implies that $\frac{\bigwedge a_j}{m} = \frac{\widehat{a_i}}{m} \in S^{-1}I$, where $\widehat{a_i} = \bigwedge_{j \neq i} a_j$. Hence $S^{-1}I$ is an 2-absorbing ideal of $S^{-1}L$.

Conversely, let $a_1, \ldots, a_{n+1} \in L$ such that $\bigwedge_{i=1}^{n+1} a_i \in I$. Then, $\bigwedge_{i=1}^{n+1} \frac{a_i}{m} = \frac{\bigwedge_{i=1}^{n+1} a_i}{m} \in S^{-1}I$. Since $S^{-1}I$ is an n-absorbing ideal of $S^{-1}L$, we infer that $\frac{\bigwedge_{i=1}^{n} a_i}{m} \in S^{-1}I$, and so $\bigwedge_{i=1}^{n} a_i \in I$. □

Let I be an n-absorbing ideal of a lattice L. Then I is a m-absorbing ideal for all integers $m \geq n$. Now, we put $\omega_L(L) = 0$ and if I is an n-absorbing ideal for some $n \in \mathbb{N}$, then we define $\omega_L(I) = \min\{n \in \mathbb{N} \mid I \text{ is an } n\text{-absorbing ideal of } L\}$, otherwise, set $\omega_L(I) = \infty$. Thus for any ideal I of L, we have $\omega(I) \in \mathbb{N} \cup \{0, \infty\}$ with $\omega(I) = 1$ if and only if I is a prime ideal of L, and $\omega(I) = 0$ if and only if $I = L$.

Proposition 4.7. *Let $f : L \to M$ be a homomorphism of lattices. Then the following statements hold.*

1. *If $f : L \to M$ is an epimorphism, and J is an n-absorbing ideal of M, then $f^{-1}(J)$ is an n-absorbing ideal of L. Moreover, $\omega_L(f^{-1}(J)) < \omega_M(J)$.*
2. *If f is an isomorphism, and I is an n-absorbing ideal of L, then $f(I)$ is an n-absorbing ideal of M.*

Proof. (1). Let $x_1, x_2, \ldots, x_{n+1} \in L$ such that $x_1 \wedge \cdots \wedge x_{n+1} \in f^{-1}(J)$, then

$$f(x_1) \wedge \cdots \wedge f(x_{n+1}) = f(x_1 \wedge \cdots \wedge x_{n+1}) \in J.$$

Then there is a meet of n of the $f(x_i)$'s that is in J, which implies that there is a meet of n of the x_i's that is in $f^{-1}(J)$. Then $f^{-1}(J)$ is an n-absorbing ideal of L.
(2). It is straightforward. □

Proposition 4.8. *Let I_1 be an m-absorbing ideal of a distributive bounded lattice L_1, and let I_2 be an n-absorbing ideal of a distributive bounded lattice L_2. Then $I_1 \times I_2$ is an $(m+n)$-absorbing ideal of the lattice $L_1 \times L_2$. Moreover $\omega_{L_1 \times L_2}(I_1 \times I_2) = \omega_{L_1}(I_1) + \omega_{L_2}(I_2)$.*

Proof. Let $L = L_1 \times L_2$. First we show that $I_1 \times I_2$ is an $(m+n)$-absorbing ideal. Let $\bigwedge_{i=1}^{n+m+1}(x_i, y_i) \in I_1 \times I_2$ for some $(x_1, y_1), \ldots, (x_{n+m+1}, y_{n+m+1}) \in I_1 \times I_2$. Since $\bigwedge_{i=1}^{n+m+1} x_i \in I_1$ and $\bigwedge_{i=1}^{n+m+1} y_i \in I_2$, we conclude that there exist

$$\{i_1, \ldots, i_m\}, \{j_1, \ldots, j_n\} \subseteq \{1, \ldots, n+m+1\},$$

such that $\bigwedge_{k=1}^{m} x_{i_k} \in I_1$ and $\bigwedge_{l=1}^{n} y_{j_l} \in I_2$, which implies that

$$(x_{i_1}, 1) \wedge \cdots \wedge (x_{i_m}, 1) \wedge (1, y_{j_1}) \wedge \cdots \wedge (1, y_{j_n}) = \left(\bigwedge_{k=1}^{m} x_{i_k}, \bigwedge_{l=1}^{n} y_{j_l} \right) \in I_1 \times I_2.$$

Now, we show that $\omega_L(I_1 \times I_2) = \omega_{L_1}(I_1) +_{L_2} (I_2)$. Let $\omega_{L_1}(I_1) = m < \infty$ and $\omega_{L_2}(I_2) = n < \infty$. Then, there are $x_1, \ldots, x_m \in L_1$ and $y_1, \ldots, y_n \in L_2$ such that satisfies the following statements:

- $x_1 \wedge \cdots \wedge x_m \in I_1$ and $y_1 \wedge \cdots \wedge y_n \in I_2$;
- for every $X \subsetneq \{x_1, \ldots, x_m\}$, $\bigwedge X \notin I_1$;
- for every $Y \subsetneq \{y_1, \ldots, y_n\}$, $\bigwedge Y \notin I_2$.

Thus,

$$(x_1, 1) \wedge \cdots \wedge (x_m, 1) \wedge (1, y_1) \wedge \cdots \wedge (1, y_n) = (x_1 \wedge \cdots \wedge x_m, y_1 \wedge \cdots \wedge y_n)$$

is an element of $I_1 \times I_2$, and also for proper subset S of

$$\{(x_1, 1), \ldots, (x_m, 1), (1, y_1), \ldots, (1, y_n)\},$$

$\bigwedge S \notin I_1 \times I_2$, which implies that $\omega_L(I_1 \times I_2) \geq m + n = \omega_{L_1}(I_1) + \omega_{L_2}(I_2)$.

Consider $N = m + n + 1$ and suppose that $(x_1, y_1), \ldots, (x_N, y_N) \in L$ such that $(x_1, y_1) \wedge \cdots \wedge (x_N, y_N) \in I_1 \times I_2$. Then $x_1 \wedge \cdots \wedge x_N \in I_1$ and $y_1 \wedge \cdots \wedge y_N \in I_2$, which implies that there are $\{i_1, \ldots, i_m\}, \{j_1, \ldots, j_n\} \subseteq \{1, \ldots, N\}$ such that $x_{i_1} \wedge \cdots \wedge x_{i_m} \in I_1$ and $y_{j_i} \wedge \cdots \wedge y_{j_m} \in I_2$. Let $K = \{i_1, \ldots, i_m\} \cup \{j_1, \ldots, j_n\}$, then $|K| \leq m + n$ and $\bigwedge_{k \in K}(x_k, y_k) \in I_1 \times I_2$, where $x_k = 1$ for every $k \notin \{i_1, \ldots, i_m\}$ and $y_k = 1$ for every $k \notin \{j_1, \ldots, j_n\}$. Hence, $\omega_L(I_1 \times I_2) \leq m + n = \omega_{L_1}(I_1) + \omega_{L_2}(I_2)$. Therefore, $\omega_L(I_1 \times I_2) = \omega_{L_1}(I_1) +_{L_2} (I_2)$. □

Corollary 4.1. *Let I_k be an ideal of a lattice L_k for each integer $1 \leq k \leq n$, and let $L = L_1 \times \cdots \times L_n$. Then $\omega_L(I_1 \times \cdots \times L_n) = \omega_{L_1}(I_1) + \cdots + \omega_{L_n}(I_n)$.*

Proof. By induction on n and Proposition 4.8, it is clear. □

Acknowledgements. The authors thank the anonymous referees for their valuable comments and suggestions for improving the article.

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