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FRACTIONAL ORDER OPERATIONAL MATRIX METHOD FOR SOLVING TWO-DIMENSIONAL NONLINEAR FRACTIONAL VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. This article presents a numerical method for solving nonlinear two-dimensional fractional Volterra integral equation. We derive the Hat basis functions operational matrix of the fractional order integration and use it to solve the two-dimensional fractional Volterra integro-differential equations. The method is described and illustrated with numerical examples. Also, we give the error analysis.

1. Introduction

Fractional differential and integral equations involving the Caputo fractional operator or the Riemann-Liouville fractional operator has been paid more and more attention. There are several numerical methods for solving fractional integro-differential equations. Such as Haar wavelet method [24], CAS wavelets [25], Bernstein polynomials [1], collocation method [23], fractional differential transform method [3], Block pulse operational matrix [20, 28].

Integro-differential equation of fractional order has been proved to be valuable tools to model the dynamics of many processes in various fields of science and engineering through strongly anomalous media. Indeed, we can find numerous applications in electro-chemistry, viscoelasticity, signal processing, economies, electromagnetic, etc. [9, 10, 18, 22].

Hat functions (HFs) are a powerful mathematical tool for solving various kinds of equations. The solution of stochastic Ito-Volterra integral equations based on stochastic operational matrix [11], E. Babolian et al. have applied this method for

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solving systems of nonlinear integral equations [5], M. H. Heydari et al. have applied Hat functions for solving nonlinear stochastic Ito integral equations [11,13]. F. Mirzaee and E. Hadadiyan have used two-dimensional Hat functions for solving space-time integral equations [17]. M. P. Tripathi et al. have applied HFs for solving fractional differential equations [27].

The operational matrix of integration has been determined for several types of orthogonal polynomials, such as Legendre polynomials [21], Laguerre series [12], and Block-pulse functions [4, 7], Triangular functions [15]. The operational matrix of fractional derivatives has been determined for some types of orthogonal polynomials, such as Legendre polynomials [26], Chebyshev polynomials [6], Triangular functions [8, 14].

In this paper, two dimensional Hat functions (2DHFs) will be used to solve the following nonlinear two-dimensional fractional integral equation

$$D_x^\alpha u(x,y) = f(x,y) + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^y \int_0^x (y-s)^{\alpha-1}(x-t)^{(\beta-1)} G(x,y,s,t,u(s,t))dsdt, \tag{1.1}$$

with the initial conditions

$$\frac{\partial^i}{\partial x^i} u(0,y) = \delta_i, \quad i = 0,1,\ldots,\rho-1, \rho-1 < \alpha \leq \rho,\ \rho \in \mathbb{N}, \tag{1.2}$$

where $(\alpha,\beta) \in (0,\infty)\times(0,\infty)$, $u \in L^1(\Omega)$, $\Omega := [0,a]\times[0,b]$, are known functions, (1.1) is the Caputo fractional differentiation operator and the unknown function $u(x,y)$ to be determined. In this work, we consider that, the nonlinear function has the following form $G(x,y,s,t,u) = k(x,y,s,t,)[u(s,t)]^P$, where p is positive integer. In this paper, we introduce a new operational method to solve nonlinear two dimensional fractional Volterra integro-differential equations. The method is based on reducing the equation to the system of algebraic equation by expanding the solution as Hat functions.

2. Riemann-Liouville and Caputo Fractional Derivatives

There are various types of definition for the fractional derivative. The most commonly used definitions are Riemann-Liouville and Caputo formulas. Riemann-Liouville fractional integration of order α is defined as

$$I_{x_0}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_{x_0}^x (x-t)^{\alpha-1} f(t)dt, \quad \alpha > 0, x > 0. \tag{2.1}$$

The following equations define Riemann-Liouville and Caputo fractional derivatives of order α, respectively,

$$\begin{aligned} D_{x_0}^\alpha f(x) =& \frac{d^m}{dx^m}[I_{x_0}^{m-\alpha} f(x)], \\ D_{*x_0}^\alpha f(x) =& I_{x_0}^{m-\alpha}\left[\frac{d^m}{dx^m} f(x)\right], \end{aligned} \tag{2.2}$$

where $m-1 \le \alpha < m$ and $n \in \mathbb{N}$. From (2.1) and (2.2), we have

$$D_{x_0}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_{x_0}^{x} (x-t)^{m-\alpha-1} f(t)dt, \quad x > x_0.$$

Lemma 2.1. *If $n-1 < \alpha \le n$, $n \in \mathbb{N}$, then $D_x^{\alpha} I^{\alpha} u(x,t) = u(x,t)$, and*

$$I^{\alpha} D_x^{\alpha} \mathrm{u}(x,t) = \mathrm{u}(x,t) - \sum_{k=0}^{n-1} \frac{\partial^k u(0^+,t)}{\partial x^k} \frac{x^k}{k!}, \quad x > 0.$$

Definition 2.1 ([2])**.** Let $(\alpha,\beta) \in (0,\infty)\times(0,\infty)$, $\theta = (0,0)$, $\Omega := [0,a]\times[0,b]$, and $u \in L^1(\Omega)$. The left-sided mixed Riemann-Liouille integral of order (α,β) of u is defined by

$$(I_{\theta}^{(\alpha,\beta)} u)(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^y \int_0^x (y-s)^{(\alpha-1)} (x-t)^{(\beta-1)} u(s,t) ds dt.$$

In particular

1. $(I_{\theta}^{(\alpha,\beta)} u)(x,y) = u(x,y)$;
2. $(I_{\theta}^{(\alpha,\beta)} u)(x,y) = \int_0^x \int_0^y u(s,t) dt ds$, $(x,y) \in \Omega$, $\sigma = (1,1)$;
3. $(I_{\theta}^{(\alpha,\beta)} u)(x,0) = (I_{\theta}^{(\alpha,\beta)})(0,y) = 0$, $x \in [0,a]$, $y \in [0,b]$;
4. $I_{\theta}^{\alpha,\beta} x^{\lambda} y^{\omega} = \frac{\Gamma(1+\lambda)\times\Gamma(1+\omega)}{\Gamma(1+\lambda+\alpha)\times\Gamma(1+\omega+\beta)} x^{\lambda+\alpha} y^{\omega+\beta}$, $(x,y) \in \Omega$, $\lambda, \omega \in (-1,\infty)$.

3. Review of Hat Functions and Their Properties

A set of HFs is usually defined on $[0,1]$ as:

$$\phi_0(t) = \begin{cases} \dfrac{h-t}{h}, & 0 \le t < h, \\ 0, & \text{otherwise}, \end{cases}$$

$$\phi_i(t) = \begin{cases} \dfrac{t-(i-1)h}{h}, & (i-1)h \le t < ih, \\ \dfrac{(i+1)h-t}{h}, & ih \le t < (i+1)h, \ i = 1,2,\dots,n-1, \\ 0, & \text{otherwise}, \end{cases}$$

$$\phi_n(t) = \begin{cases} \dfrac{t-(1-h)}{h}, & T-h \le t < T, \\ 0, & \text{otherwise}, \end{cases}$$

where $h = \frac{1}{n}$ and n is an arbitrary positive integer. Indeed, the unit interval [0,1] is divided into n equidistant subintervals. According to the definition of HFs, we have

$$\phi_i(jh) = \delta_{ij}, \tag{3.1}$$

where δ denotes the Kronecker delta function. By generalizing the definition of one-dimensional HFs, 2DHFs can be defined as follows

$$\Phi_{i,j}(x,y) = \Phi_i(x)\Phi_j(x), \quad i,j = 0,1,\dots,n. \tag{3.2}$$

By substituting (3.1) and (3.2), we have $\Phi_{i,j}(kh, lh) = \delta_{jl}\delta_{ik}$. Now, for the 2DHFs, we have

$$\phi_{i,j}(x, y)\phi_{k,l}(x, y) = 0, \quad |i - j| \geq 2 \text{ or } |j - l| \geq 2 \tag{3.3}$$

and

$$\sum_{i=0}^{n}\sum_{j=0}^{n} \phi_{i,j}(x, y) = 1.$$

An arbitrary function $U(x, y)$ can be expanded in vector form as:

$$U(x, y) \simeq U^T\Phi(x, y) = \Phi^T(x, y)U, \tag{3.4}$$

where $U = [u_0, u_1, \dots, u_n]^T$,

$$\Phi(x, y) = [\phi_{0,0}(x, y), \dots, \phi_{0,m}(x, y), \phi_{1,0}(x, y), \dots, \phi_{1,0}(x, y)]^T$$

and $u_{i,j} = u(ih, jh)$, $i, j = 0, 1, \dots, n$. The positive integer powers of $u(x, y)$ may be approximated by HFs as $[u(x, y)]^P \simeq C_P^T \cdot \Phi(x, y)$. Now, let $k(x, y, s, t)$ be an arbitrary function of two variables defined on $L^2([0, 1] \times [0, 1])$. It can be expanded by HFs as: $k(x, y, s, t) \simeq \Phi^T(x, y)K\Phi(s, t)$, where $\Phi(x, y)$ and $\Phi(s, t)$ are 2DHFs vectors of dimention $(n+1)^2$, and K is 2DHFs coefficients matrix of dimention $(n_1+1)^2 \times (n+1)^2$ with entries $a_{ij}, i = 0, 1, \dots, n_1,\ j = 0, 1, \dots, n_2$, as $a_{ij} = k(ih, jh)$. In this paper, for convenience, we put $n_1 = n_2 = n$. Moreover, from (3.3) follows:

$$\Phi(x, y)\Phi^T(x, y)$$
$$= \begin{pmatrix} \phi_0^2(x) & \phi_0(x)\phi_1(x) & & & \\ \phi_0(x)\phi_1(x) & \phi_1^2(x) & \phi_1(x)\phi_2(x) & & \\ & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \phi_{n-1}(x)\phi_n(x) \\ & & & \phi_{n-1}(x)\phi_n(x) & \phi_n^2(x) \end{pmatrix}$$
$$\otimes \begin{pmatrix} \phi_0^2(x) & \phi_0(x)\phi_1(x) & & & \\ \phi_0(x)\phi_1(x) & \phi_1^2(x) & \phi_1(x)\phi_2(x) & & \\ & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \phi_{n-1}(x)\phi_n(x) \\ & & & \phi_{n-1}(x)\phi_n(x) & \phi_n^2(x) \end{pmatrix}$$

and

$$P_1 = \int_0^1 \int_0^1 \Phi(x, y)\Phi^T(x, y)dxdy = \Upsilon_1 \otimes \Upsilon_1,$$

where P_1 is the following $(n+1)\times(n+1)$ matrix

$$P_1 = \frac{h}{6}\begin{pmatrix} 2 & 1 & & & & \\ 1 & 4 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 & \\ & & & 1 & 2 & \end{pmatrix}.$$

By considering (3.1), and expanding entries of $\Phi(x,y)\Phi^T(x,y)$ by 2DHFs, we have $\Phi(x,y)\Phi^T(x,y) \simeq \mathrm{diag}(\Phi(x,y))$. Now, suppose that Λ is a vector $(n+1)^2$. We obtain

$$\Phi(x,y)\Phi^T(x,y)\Lambda \simeq \tilde{\Lambda}\Phi(x,y), \tag{3.5}$$

where $\tilde{\Lambda} = \mathrm{diag}(\Lambda)$ is an $(n+1)^2\times(n+1)^2$-diagonal matrix. Furthermore, if A is an $(n+1)^2\times(n+1)^2$-matrix, we have

$$\Phi^T(x,y)A\Phi(x,y) \simeq \Phi^T(x,y)\hat{A}, \tag{3.6}$$

where $\hat{A}$ is an $(n+1)^2$-vector with elements equal to diagonal entries of matrix A. Now, we have

$$\begin{aligned}\int_0^y\int_0^x \Phi(s,t)dydt &= \int_0^y\int_0^x \Phi(s)\otimes\Phi(t)dsdt = \Big(\int_0^y \Phi(s)ds\Big)\otimes\Big(\int_0^x \Phi(t)dt\Big)\\ &\simeq (\Upsilon_1\Phi(x))\otimes(\Upsilon_2\Phi(y)) = (\Upsilon_1\otimes\Upsilon_2)\Phi(x,y) = P_2\Phi(x,y),\end{aligned}$$

where P_2 is the following $(n+1)\times(n+1)$ matrix

$$P_2 = \frac{h}{2}\begin{pmatrix} 0 & 1 & 1 & 1 & \cdots & 1\\ 0 & 1 & 2 & 2 & \cdots & 2\\ 0 & 0 & 1 & 2 & \cdots & 2\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 1\end{pmatrix}.$$

3.1. **Operational matrix of the fractional order integration (OMFI).** Our goal is to get, to derive the Hat OMFI. For this purpose, Block pulse fractional matrix for the one-dimensional case is presented as follows:

$$(I^\alpha f)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}b(\tau)d\tau = F^\alpha b(t),$$

where $\alpha\in\mathbb{R}$ is the order of the integration and $\Gamma(\alpha)$ is the Gamma function. Also, we define an m-set of Block Pulse Functions (BPFs) as

$$b_i(x) = \begin{cases} 1, & \dfrac{i}{m} \le x < \dfrac{(i+1)}{m},\\ 0, & \text{otherwise,}\end{cases}$$

where $i = 0, 1, 2, \dots, m-1$. The function $b_i(x)$ is disjoint and orthogonal, that is

$$b_j(x)b_i(x) = \begin{cases} b_j, & j = i, \\ 0, & j \neq i, \end{cases}$$

where F^α is the $m \times m$ fractional operational matrix of integration of order α for the BPFs (see [16]) where

$$(I^\alpha B_m)(x) \simeq F^\alpha B_m(x),$$

$$F^\alpha = \frac{1}{m^\alpha}\frac{1}{\Gamma(\alpha+2)}\begin{bmatrix} 1 & \xi_1 & \xi_2 & \xi_3 & \dots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \xi_2 & \dots & \xi_{m-1} \\ 0 & 0 & 1 & \xi_1 & \dots & \xi_{m-3} \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & \dots & 0 & 1 & \xi_1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix},$$

and $\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$. Our aim is to derive the Hat OMFI. For this purpose, we used the Riemann-Liouville fractional order integration, as following:

$$\begin{aligned}(I^\alpha u)(x,y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^y\int_0^x (y-s)^{\alpha-1}(x-t)^{\beta-1}u(s,t)dsdt \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}y^{\alpha-1}x^{\beta-1} * u(x,y),\end{aligned}$$

where $\alpha, \beta \in \mathbb{R}$ are the order of the integration, $\Gamma(\alpha)$ and $\Gamma(\beta)$ are the Gamma functions and $y^{\alpha-1} * u(x,y)$, $x^{\beta-1} * u(x,y)$ denote the convolution products of $y^{\alpha-1}$, $x^{\beta-1}$ and $u(x,y)$. Now if $u(x,y)$ is expanded in HFs, as shown in (3.4), the Riemann-Liouville fractional integration becomes

$$(I^\alpha u)(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}y^{\alpha-1}x^{\beta-1} * u(x,y) \approx C^T\frac{1}{\Gamma(\alpha)\Gamma(\beta)}y^{\alpha-1}x^{\beta-1} * \Phi(x,y).$$

Thus, if $y^{\alpha-1} * u(x,y)$ and $x^{\beta-1} * u(x,y)$ can be integrated, then by expanding the Hat functions, the Riemann-Liouville fractional order integration solve the HFs. Also, we define an m-set of BPF as

$$b_{i_1,i_2}(x,y) = \begin{cases} 1, & (i_1-1)h_1 \leqslant x < i_1h_1 \text{ and } (i_2-1)h_2 \leqslant y < i_2h_2, \\ 0, & \text{otherwise}, \end{cases}$$

where $i = 0, 1, 2, \dots, m-1$. The function $b_{i,j}(t)$ is disjoint and orthogonal, that is

$$b_{i_1,i_2}(x,y)b_{j_1,j_2}(x,y) = \begin{cases} b_{i_1,i_2}(x,y), & i_1 = j_1 \text{ and } i_2 = j_2, \\ 0, & \text{otherwise}. \end{cases}$$

The HFs can be expanded in to m-set of BPs functions as

$$\Phi(x,y) = \Psi_{m\times m}B_m(x,y), \tag{3.7}$$

where $B_m(x) = (b_0(x), b_1(x), \dots, b_i(x), \dots, b_{m-1}(x))^T$ (see [24,25]) and Ψ is an $MN \times MN$ product operational matrix. Next, we derive the Hat OMFI. We have the two dimensional BPFs operational matrix of fractional integration as:

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^y\int_0^x (y-s)^{\alpha-1}(x-t)^{\beta-1}U(s,t)dsdt = F^{\alpha,\beta}U(x,y),$$

where

$$F^{\alpha,\beta} = \frac{1}{m^\alpha m^\beta}\frac{1}{\Gamma(\alpha+2)\Gamma(\beta+2)} \times \begin{bmatrix} 1 & \xi_1 & \xi_2 & \xi_3 & \dots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \xi_2 & \dots & \xi_{m-1} \\ 0 & 0 & 1 & \xi_1 & \dots & \xi_{m-3} \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & \dots & 0 & 1 & \xi_1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & \eta_1 & \eta_2 & \eta_3 & \dots & \eta_{m-1} \\ 0 & 1 & \eta_1 & \eta_2 & \dots & \eta_{m-1} \\ 0 & 0 & 1 & \eta_1 & \dots & \eta_{m-3} \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & \dots & 0 & 1 & \eta_1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix},$$

$\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$ and $\eta_k = (k+1)^{\beta+1} - 2k^{\beta+1} + (k-1)^{\beta+1}$.

Fractional integration of the BPFs is given as the following

$$(I^{\alpha,\beta}B_m)(x,y) \approx F^{\alpha,\beta}B_m(x,y). \tag{3.8}$$

Now, we derive the HFs operational matrix of the fractional order integration. Let

$$(I^{\alpha,\beta}\Phi)(x,y) \approx P^{\alpha,\beta}_{m\times m}\Phi(x,y), \tag{3.9}$$

where matrix $P^{\alpha,\beta}_{m\times m}$ is called the Hat functions OMFI. Using (3.7) and (3.8), we have

$$(I^{\alpha,\beta}\Phi)(x,y) \approx (I^{\alpha,\beta}\Psi_{m\times m}B_m)(x,y) = \Psi_{m\times m}(I^{\alpha}B_m)(x,y) \approx \Psi_{m\times m}F^{\alpha,\beta}B_m(x,y). \tag{3.10}$$

By (3.9) and (3.10) we get

$$P^{\alpha,\beta}_{m\times m}\Phi(x,t) = \Psi_{m\times m}F^{\alpha,\beta}B_m(x,y) = \Psi_{m\times m}F^{\alpha,\beta}\Phi_{m\times m}\Psi^{-1}_{m\times m}.$$

Then, the Hat functions OMFI $P^{\alpha,\beta}_{m\times m}$ is given by

$$P^{\alpha,\beta}_{m\times m} = \Psi_{m\times m}F^{\alpha,\beta}\Psi^{-1}_{m\times m}. \tag{3.11}$$

4. Applying the Method

In this section, 2DHFs fractional operational matrix are applied to solving (1.1). Now, let

$$D^{\alpha}_{*}u(x,y) \simeq C^T\Phi(x,y). \tag{4.1}$$

By using (4.1) and (3.9) and Lemma 2.1, we have

$$u(x,y) = C^TP^{\alpha}_{m\times m}\Phi(x,y) + \sum_{k=0}^{m-1}\frac{\partial^k u(0^+,y)}{\partial x^k}\frac{x^k}{k!}, \quad x>0.$$

So, by replacing the supplementary initial conditions (1.2), in the above summation in the above equations and approximating it by Hat functions, we have

$$u(x,y) \cong (C^TP^{\alpha}_{m\times m} + C^T_p)\Phi(x,y),$$

where C_p is a column m-vector. Define $e = [e_0, e_1, \dots, e_{m-1}] = (C^T P^{\alpha}_{m\times m} + C_p^T)$, so, $u(x,y) \cong e\Phi(x,y)$. We could easily check out the correctness of the expression with induction $[u(x,y)]^q \cong [e_0^q, e_1^q, \dots, e_{m-1}^q]\Phi(x,y) = e_q \Phi_{m\times m}$, where $\tilde{e}_q = [e_0^q, e_1^q, \dots, e_{m-1}^q]$. The function $u(x,y), k(x,y,s,t)$ and $f(x,y)$ can be approximated by

$$\begin{aligned} u(x,y) =& U^T\Phi(x,y) = U\Phi^T(x,y),\\ F(x,y) =& F^T\Phi(x,y) = F\Phi^T(x,y),\\ [u(x,y)]^p =& \Phi^T(x,y)C_p,\\ k(x,y,s,t) =& \Phi^T(x,y)\cdot K\cdot \Phi(s,t). \end{aligned} \tag{4.2}$$

Now, with substituting (4.2) in (1.1), we have

$$D_x^{\alpha}u(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^y\int_0^x (y-s)^{\alpha-1}(x-t)^{(\beta-1)}G(x,y,s,t,u(s,t))dsdt + f(x,y).$$

Using (3.5), (3.6), (3.9), and (3.11), we have

$$\begin{aligned} & C\Phi^T(x,y)\\ =&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^y\int_0^x (y-s)^{\alpha-1}(x-t)^{(\beta-1)}k(x,y,s,t)[u(s,t)]^p dsdt + F\Phi^T(x,y)\\ =&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^y\int_0^x (y-s)^{\alpha-1}(x-t)^{(\beta-1)}\Phi^T(x,t)K\Phi(s,t)\Phi^T(x,y)C_p dsdt + F\Phi^T(x,y)\\ =&\Phi^T(x,y)K\tilde{C}_p\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^y\int_0^x (y-s)^{\alpha-1}(x-t)^{(\beta-1)}\Phi(s,t)dsdt + F\Phi^T(x,y)\\ =&\Phi^T(x,y)K\tilde{C}_pP^{\alpha,\beta}_{m\times m}\Phi(x,y) = \left(\widehat{K\tilde{c}pP^{\alpha,\beta}_{m\times m}}\right)^T\cdot\Phi(x,y) + F\Phi^T(x,y)\\ =&\left(\widehat{K\tilde{c}pP^{\alpha,\beta}_{m\times m}}\right)\cdot\Phi^T(x,y) + F\Phi^T(x,y). \end{aligned}$$

Set

$$B = \left(\widehat{K\tilde{c}pP^{\alpha,\beta}_{m\times m}}\right),$$

so,

$$C\Phi^T(x,y) = B\Phi^T(x,y) + F\Phi^T(x,y),$$

hence, we have

$$C = B + F, \tag{4.3}$$

which is a system of algebraic equations. By solving this system, we can obtain the approximate solution of (1.1) according to (4.3).

5. Convergence and Error Analysis

In this section, we obtain an error bound for the approximate solution, then from which we conclude convergence of the method. We define the error function as

$$e_n(x,y) = u(x,y) - \hat{u}(x,y),$$

where $u(x, y)$ and $\hat{u}(x, y)$ denote the exact and approximate solutions, respectively.

Theorem 5.1. *Suppose $u(x, y) \in I$ and $e_n(x, y) = u(x, y) - u_n(x, y)$, $(x, y) \in I = [0, T) \times [0, T)$, where $u_n(x, y) = \sum_{i=0}^{n} u(ih, jh)\phi_{i,j}(x, y)$ is the generalized hat function expansion of $u(x, y)$. Then, we have*

$$\|e_n(x, y)\| \leq \frac{T^2}{2n^2}\|u''(x, y)\|, \tag{5.1}$$

and so the convergence is of order two, that is $\|e_n(x, y)\| = O\left(\frac{1}{n^2}\right)$.

Proof. See [17]. ☐

Theorem 5.2. *Suppose $u(x, y)$ as an exact solution of fractional integral* (1.1) *and $\hat{u}(x, y)$ show the approximate solution by Hat functions. If $|(x-s)^{\alpha-1}(y-t)^{\beta-1}k(x, y, s, t)| < N$, $u(x, y)$ and $k(x, y, s, t)$ are continuous functions and also, $G(u) = (u(x, t))^p$ satisfies Lipschitz condition $|G(u) - G(\hat{u})| \leq L|u - \hat{u}|$, then*

$$\|u - \hat{u}\| = \sup_{0\leq x,y\leq 1} |u(x, y) - \hat{u}(x, y)| = O\left(\frac{1}{n^2}\right).$$

Proof. We have

$$\begin{aligned}
&|u(x, y) - \hat{u}(x, y)|\\
=&\left|\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^y\int_0^x (y-s)^{\alpha-1}(x-t)^{\beta-1}k(x, y, s, t)(u(s, t) - \hat{u}(s, t))dtds\right|\\
\leq&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^y |(x-s)^{\alpha-1}(y-t)^{\beta-1}k(x, y, s, t)(u(s, t) - \hat{u}(s, t))|dsdt\\
\leq&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^y\int_0^x |(y-s)^{\alpha-1}(x-t)^{\beta-1}k(x, y, s, t)||(u(s, t) - \hat{u}(s, t))|dsdt\\
\leq&\frac{N}{\Gamma(\alpha)\Gamma(\beta)}\int_0^y\int_0^x |(u(s, t) - \hat{u}(s, t))|dsdt.
\end{aligned}$$

From (5.1), we conclude that

$$|u(x, y) - \hat{u}(x, y)| \leq \frac{NLT^2xy}{2n^2\Gamma(\alpha)\Gamma(\beta)} \leq \frac{NLT^2}{2n^2\Gamma(\alpha)\Gamma(\beta)}.$$

This completes the proof. ☐

Theorem 5.3. *The solving systems of partial 2DFVIE by using 2D-HFs converge if $0 < \theta < 1$, where $\theta = \frac{NLT^2}{2n^2\Gamma(\alpha)\Gamma(\beta)}$.*

Proof. If we assume $G(u) = D_x^\alpha u(x, y)$, we have $\|G(u) - G(u_m)\|_\infty \leq \|u - u_m\|_\infty$. From Theorem 5.2, we have

$$\|G(u) - G(u_m)\|_\infty \leq \frac{NLT^2}{2n^2\Gamma(\alpha)\Gamma(\beta)}\|u - u_m\|_\infty. \tag{5.2}$$

Inequality (5.2) implies that if $0 < \theta < 1$, then we have $\lim_{m\to\infty} \|G(u) - G(u_m)\|_\infty = 0$ and $\lim_{m\to\infty} \|u - u_m\|_\infty = 0$. □

6. Numerical Examples

To illustrate the effectiveness of the proposed method in the present paper, some test examples are carried out in this section.

Example 6.1. Consider the fractional partial volterra integro-differential equation [19]

$$D_x^{0.75}u(x,y) = \int_0^y \int_0^x (y+t)u(s,t)dsdt = \frac{6.4}{\Gamma(0.25)}yx^{5/4} - \frac{5}{18}x^3y^3,$$

where the exact solution is known and it is given by $u(x,y) = x^2y$, for $x, y \in [0,1]$ and with supplementary condition $u(0,y) = 0$. Numerical results are presented in Table 1.

Table 1. The absolute errors for Example 1.

	$m=n=4$	$m=n=4$	$m=n=5$	$m=n=5$
(x,y)	u_{2DLWs} [19]	u_{2DHFs}	u_{2DLWs} [19]	u_{2DHFs}
$(0.0, 0.7)$	0.1404×10^{-2}	0.1404×10^{-2}	0.3508×10^{-3}	0.2327×10^{-3}
$(0.1, 0.3)$	0.1636×10^{-3}	0.2584×10^{-2}	0.1342×10^{-3}	0.4158×10^{-3}
$(0.3, 0.8)$	0.1456×10^{-2}	0.3651×10^{-3}	0.8962×10^{-3}	0.1001×10^{-4}
$(0.4, 0.2)$	0.1087×10^{-3}	0.6521×10^{-3}	0.2700×10^{-4}	0.5057×10^{-4}
$(0.6, 0.6)$	0.3248×10^{-3}	0.1421×10^{-3}	0.6759×10^{-3}	0.5884×10^{-4}
$(0.7, 0.5)$	0.8878×10^{-3}	0.6250×10^{-3}	0.5285×10^{-4}	0.1019×10^{-4}
$(0.8, 0.4)$	0.7061×10^{-3}	0.7247×10^{-3}	0.4090×10^{-4}	0.1018×10^{-4}
$(0.9, 0.9)$	0.5898×10^{-3}	0.1997×10^{-3}	0.1974×10^{-3}	0.4108×10^{-4}

Example 6.2. Consider the linear two-dimensional fractional integro-differential equation [19]

$$D_x^{0.5}u(x,y) = \int_0^y \int_0^x (x^2y+s)u(s,t)dsdt = 4y\sqrt{\frac{x}{\pi}} - \frac{1}{2}x^4y^3 - \frac{1}{3}x^3y^2,$$

where the exact solution is known and given by $u(x,y) = 2xy$, for $x, y \in [0,1]$ and with supplementary condition $u(0,y) = 0$. Numerical results are presented in the Table 2.

Example 6.3. Consider the linear two-dimensional fractional integro-differential equation [19]

$$D_x^{0.5}u(x,y) = \int_0^y \int_0^x (x\cos(s)+yt)u(s,t)dsdt = f(x,y),$$

TABLE 2. The absolute errors for Example 2.

	$m=n=4$	$m=n=4$	$m=n=5$	$m=n=5$
(x,y)	u_{2DLWs} [19]	u_{2DHFs}	u_{2DLWs} [19]	u_{2DHFs}
$(0.1,0.8)$	0.1173×10^{-3}	0.1853×10^{-3}	0.1250×10^{-3}	0.4141×10^{-3}
$(0.2,0.6)$	0.1805×10^{-3}	0.9461×10^{-3}	0.2751×10^{-4}	0.4258×10^{-3}
$(0.3,0.8)$	0.9276×10^{-4}	0.9276×10^{-4}	0.1189×10^{-4}	0.1104×10^{-4}
$(0.4,0.6)$	0.2710×10^{-4}	0.3621×10^{-4}	0.1395×10^{-5}	0.1245×10^{-5}
$(0.5,0.5)$	0.7309×10^{-5}	0.1001×10^{-4}	0.4065×10^{-5}	0.7412×10^{-5}
$(0.6,0.5)$	0.3884×10^{-4}	0.3621×10^{-4}	0.1174×10^{-4}	0.3241×10^{-5}
$(0.7,0.3)$	0.3548×10^{-4}	0.5200×10^{-3}	0.9798×10^{-5}	0.4142×10^{-4}
$(0.8,0.4)$	0.9069×10^{-4}	0.3247×10^{-4}	0.2406×10^{-4}	0.3258×10^{-4}
$(0.9,0.9)$	0.6179×10^{-3}	0.1657×10^{-3}	0.1607×10^{-3}	0.4741×10^{-4}

where

$$\begin{aligned} f(x,y) =& \frac{2\sin(y)\sqrt{x}}{\sqrt{0.5}} + x\cos(x) - x^2\sin(x) - x\cos(y) + x\cos(x)\cos(y) \\ &+ x^2\sin(x)\cos(y) - \frac{1}{2}x^2y\sin(y) + \frac{1}{2}x^2y^2\cos(y), \end{aligned}$$

where the exact solution is known and given by $u(x,y) = x\sin(y)$, for $x,y\in[0,1]$ and with supplementary condition $u(0,y)=0$. Numerical results are presented in the Table 3.

TABLE 3. The absolute errors for Example 3.

	$m=n=3$	$m=n=3$	$m=n=4$	$m=n=4$
(x,y)	u_{2DLWs} [19]	u_{2DHFs}	u_{2DLWs} [19]	u_{2DHFs}
$(0.1,0.1)$	0.1599×10^{-3}	0.2514×10^{-2}	0.5398×10^{-4}	0.9841×10^{-3}
$(0.2,0.2)$	0.2155×10^{-3}	0.6251×10^{-3}	0.5185×10^{-4}	0.4625×10^{-4}
$(0.3,0.3)$	0.1566×10^{-3}	0.5210×10^{-3}	0.6503×10^{-4}	0.1984×10^{-4}
$(0.4,0.4)$	0.2122×10^{-3}	0.9654×10^{-3}	0.7688×10^{-4}	0.1962×10^{-4}
$(0.5,0.5)$	0.2477×10^{-3}	0.2014×10^{-3}	0.8809×10^{-4}	0.7620×10^{-4}
$(0.6,0.6)$	0.2971×10^{-3}	0.6521×10^{-3}	0.9899×10^{-4}	0.3021×10^{-4}
$(0.7,0.7)$	0.3662×10^{-3}	0.6214×10^{-3}	0.1226×10^{-3}	0.4142×10^{-4}
$(0.8,0.8)$	0.4738×10^{-3}	0.2147×10^{-3}	0.1599×10^{-3}	0.3108×10^{-4}
$(0.9,0.9)$	0.6344×10^{-3}	0.9651×10^{-3}	0.2246×10^{-3}	0.4748×10^{-3}

Example 6.4. Consider the two-dimensional fractional Volterra integral equation [1]

$$u(x,y) - \frac{1}{\Gamma(\frac{7}{2})\Gamma(\frac{5}{2})}\int_0^y\int_0^x (y-s)^{\frac{5}{2}}(x-t)^{\frac{3}{2}}(y^2+s)e^{-t}u(s,t)dsdt = f(x,y),$$

where

$$f(x,y) = x^2e^y - \frac{1024x^{\frac{11}{2}}y^{\frac{5}{2}}(6x+13y^2)}{2027025\pi},$$

where the exact solution is known and it is given by $u(x,y) = x^2e^y$. To solve this equation, we implement the HFs method for $\alpha = \frac{7}{2}$ and $\beta = \frac{5}{2}$. Numerical results are presented in Table 4 and Figure 1.

TABLE 4. The absolute errors for Example 4.

	$m=n=2$	$m=n=2$	$m=n=4$	$m=n=4$
$x=y$	u_{2DBPOM} [1]	u_{2DHFs}	u_{2DBPOM} [1]	u_{2DHFs}
0.0	2.090×10^{-4}	2.125×10^{-4}	4.086×10^{-4}	5.237×10^{-5}
0.1	2.532×10^{-4}	2.635×10^{-4}	4.181×10^{-4}	4.258×10^{-5}
0.2	6.967×10^{-5}	5.689×10^{-4}	4.471×10^{-4}	4.125×10^{-4}
0.3	2.602×10^{-4}	3.070×10^{-4}	4.970×10^{-4}	4.157×10^{-4}
0.4	3.346×10^{-4}	4.325×10^{-4}	5.656×10^{-4}	4.984×10^{-4}
0.5	2.778×10^{-4}	3.215×10^{-3}	6.474×10^{-4}	6.259×10^{-4}
0.6	1.701×10^{-3}	2.587×10^{-3}	7.316×10^{-4}	7.147×10^{-4}
0.7	2.090×10^{-3}	2.090×10^{-3}	7.817×10^{-4}	7.548×10^{-4}
0.8	3.542×10^{-3}	3.985×10^{-3}	6.788×10^{-4}	7.214×10^{-4}
0.9	1.137×10^{-3}	2.087×10^{-3}	1.004×10^{-4}	2.587×10^{-4}

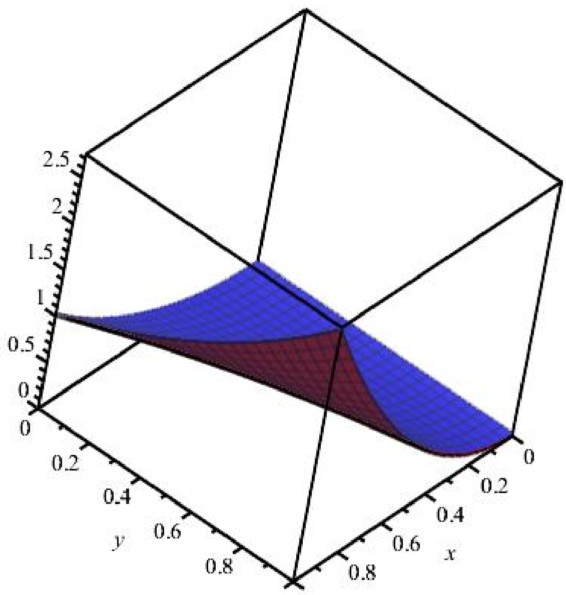

FIGURE 1. Exact and approximation solutions of Example 4.

Example 6.5. Consider the two-dimensional nonlinear fractional Volterra equation [20]

$$u(x,y) - \frac{1}{\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})}\int_0^y\int_0^x (y-s)^{\frac{1}{2}}(x-t)^{\frac{3}{2}}\sqrt{xyt}[u(s,t)]^2 dsdt = f(x,y),$$

where

$$f(x,y)=\sqrt{y}\left(\frac{-1}{180}x^3y^{\frac{7}{2}}+\sqrt{\frac{x}{3}}\right).$$

The exact solution is known and it is given by $u(x,y)=\frac{\sqrt{3xy}}{3}$. This example has been solved, for $\alpha=\frac{3}{2}$ and $\beta=\frac{5}{2}$. Numerical results for this a solution are presented in Table 5 and Figure 2.

TABLE 5. The numerical results for Example 5.

	Exact solution	$m=32$	$m=32$
$x=y$		u_{2DBPFs} [20]	u_{2DHFs}
0.0	0	0.009386	0.002541
0.1	0.05773	0.042121	0.042541
0.2	0.11547	0.124282	0.138744
0.3	0.17323	0.156905	0.144871
0.4	0.23094	0.239179	0.235487
0.5	0.28867	0.274574	0.275487
0.6	0.34641	0.354075	0.344872
0.7	0.40414	0.389848	0.404151
0.8	0.46188	0.468971	0.469874
0.9	0.50702	0.507021	0.507210

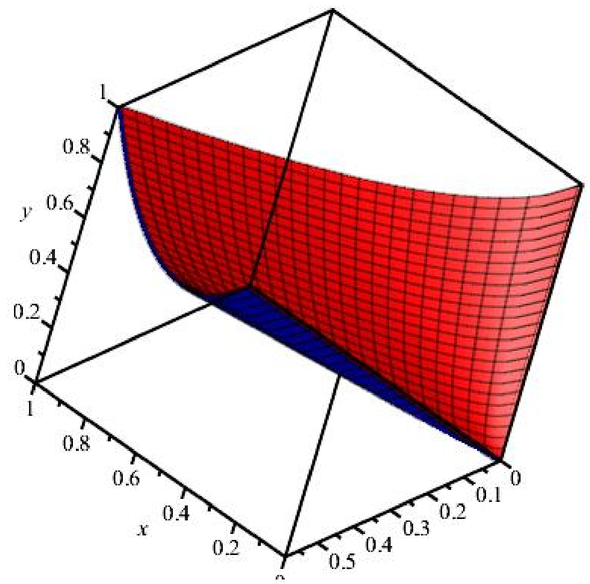

FIGURE 2. Comparison the exact solution and the presented method for Example 5.

7. Conclusion

In this paper, a Hat operational matrix of fractional order integration is obtained and it is used to solve the two-dimensional nonlinear fractional Volterra integro-differential equations. By properties of 2DHFs and using of operational matrices the possibility of reducing these equations to a system of algebraic equations are provided. Moreover, a general procedure of forming this matrix $P^{\alpha,\beta}_{m\times m}$ is summarized. For more investigation, some examples are presented. As the numerical results showed, the proposed method is an accurate and effective method for solving a fractional two-dimensional integral equation.

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