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TWO-DIMENSIONAL DYNAMICS OF CUBIC MAPS

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Abstract. We investigate the global properties of two cubic maps on the plane, we try to explain the basic mechanisms of global bifurcations leading to the creation of nonconnected basins of attraction. It is shown that in some certain conditions the global structure of such systems can be simple. The main results here can be seen as an improvement of the results of stability and bifurcation analysis.

1. Introduction

Polynomial diffeomorphisms have been widely studied and they are fundamental to our understanding of dynamical systems. They are of great interest as approximations of more complicated maps with constant Jacobian, and some of them exhibit some of the familiar properties of the quadratic Hénon map.The single Hénon map: $(x', y') = (y + x^2 + a, cx)$ is the simplest polynomial map, and the simplest nontrivial diffeomorphism of the plane containing a single quadratic term as nonlinearity. This map is also known to display chaos for certain parameter values and initial conditions. Due to its simplicity, it has become a benchmark system and has received considerable attention because of its genericity, the complexity and richness of its dynamics, frequently used as an example for demonstrating schemes for analyzing and controlling chaotic behavior.

The set of polynomial maps with polynomial inverse is called the "affine Cremona group", very dynamically interesting maps. The structure of this group is well-known and understood for two-dimensional case; as remarked in Friedland-Milnor's classical work [2], they proved that any map in this group is conjugate to a composite of basic polynomial maps called generalized Hénon maps: $(x', y') = (y + f(x), cx)$, maps with constant and nonzero Jacobian and where $f(x)$ is a polynomial of degree $d \geq 2$. It

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follows that any composition of Hénon maps has an inverse which is a polynomial. Recently, different types of generalization of the standard Hénon map have been studied. Dullin and Meiss in [1] considered polynomial cubic maps. In a recent paper, Sarmah and Paul [7] examined a period doubling route to chaos for a similar model with constant Jacobian. For more details, see the survey of Sibony [8] and the references therein [10,11], where more light was shed. Silverman [9] studied arithmetic properties of quadratic Hénon maps.

Many of complex behaviors that are observed in dynamical systems are intimately associated with the presence of homoclinic or heteroclinic points of maps [2,3]. The global bifurcations involving invariant curves have been less investigated, and several open problems are still present. Homoclinic tangencies between stable and unstable invariant manifolds of the same saddle point play a very important role. The existence of transversal homoclinic intersections is considered as the universal criterium of the complexity for maps. At the same time, the presence of non-transversal homoclinic orbits (homoclinic tangencies) indicates an extraordinary richness of bifurcations of such systems and, what is very important, the principal impossibility of providing of a complete description of bifurcations. Therefore, when studying homoclinic bifurcations, the main problems are related to the analysis of their principal bifurcations and characteristic properties of dynamics as a whole.

This work presents a research in the study of cubic polynomial invertible and noninvertible maps of the plane carried out some techniques and numerical simulations. The motivation for studying such maps is, in part, due to the form of these maps which is a generalized version of Hénon map. This set is of fundamental importance in dynamical systems and yields a great deal of interesting characteristics. Our main concerns are the global dynamics characterizing the topological structure of initial conditions which generate interesting path in cubic maps. In addition to the analytical considerations, we also display certain numerical results by using computers to perform rigorous mathematical proofs.

This paper intends to give such a study, particularly to consider two cases of cubic diffeomorphisms. Therefore, it is structured in the following way. In Section 2, division of the parameter plane for the two-dimensional maps into domains of regular and chaotic attractors is studied numerically and analytically. Regularities in the occurrence of different behaviors and transitions are analyzed. The dynamics involves various transitions by bifurcations. In Section 3, we introduce the language mentioned in [5,6], to analyze these maps, and give some useful definitions. Section 4 focuses on the global dynamics. The impact of invariant manifolds on the structure of basins is investigated. Section 5 gives some results on basin structures of noninvertible maps and their bifurcations, and illustrates properties of homoclinic-heteroclinic bifurcations. We end the paper with a conclusion.

2. Division of Parameter Plane

Consider the one-dimensional endomorphism of the $(p+q-2)$ model

$$T_0(x) = ax^{p-1}(1-x)^{q-1}. \tag{2.1}$$

Here, the trivial fixed point $x=0$ is unstable for $1<p<2$ and it is stable for $p>2$, both cases for any $a>0$ and $q>1$. We have a special case for $p=2$, where $x=0$ is an unstable fixed point if $r>1$ and a stable fixed point if $0<a<1$, both cases for any $q>1$. Consequently the set defined by $S=\{(q,a)\in\mathbb{R}^2 : q>1, a>0 \text{ for } p=2\}$ is a bifurcation plane that characterizes the stability of the fixed point $x=0$ at the parameter space (p,q,a). We consider an imbedding of the model (2.1), which is a one-dimensional noninvertible map into a two-dimensional diffeomorphism rediscovered afresh each time and with a variety of results. We study this diffeomorphism in dependance of at least three parameters and uncover many fascinating dynamical characteristics, using both analytic perturbation theory and numerical methods.

The planar diffeomorphism associated with T_0 is the following:

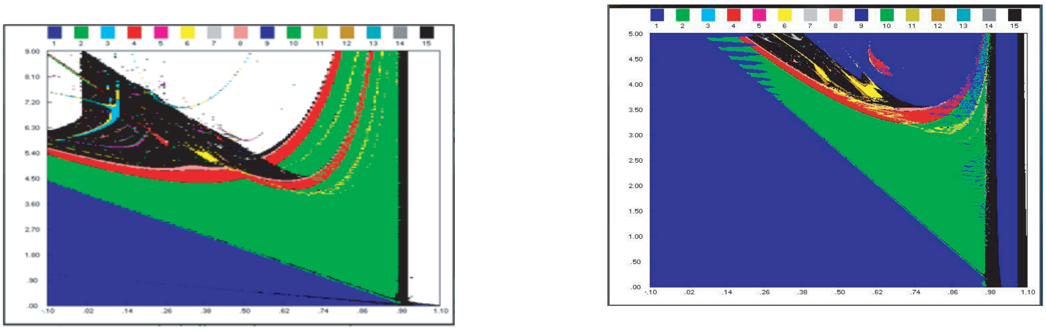
$$T_1 : \begin{cases} x' = T_0(x) + y, \\ y' = cx, \end{cases} \tag{2.2}$$

where x, y are real variables, a, p, q and c are real parameters. T_1 has a constant Jacobian determinant $detJ = -c$. We distinguish two types of cubic diffeomorphisms $(p+q-2=3)$, and each type gives different bifurcation diagrams. We only study the most interesting and principal peculiarities of the cubic maps ($p=3$, $q=2$ and $p=2$, $q=3$).

For $c=0$, the planar diffeomorphism (2.2) becomes the one-dimensional endomorphism (2.1). The model (2.2) possesses at most three fixed points depending upon the parameter values. To gain preliminary insight into the properties of the dynamical system (2.2) we conducted two-dimensional bifurcation analysis, which provides information on the dependance of the dynamics on parameters. This analysis is expected to reveal the type of attractor to which the dynamics will ultimately settle down after passing an initial transcient phase and within which the trajectory will remain forever. The parameters (c,a) are varied simultaneously to track bifurcations.

We indicate different attractors in different colors in the (c,a)-plane for which the mappings were expected to have simple dynamics in the case $p=3$ and $q=2$. The Figure 1 give the parameter value for which at least one fixed point is attractive (parameters located in the blue domain will be stabilized at a fixed point). More generally, the Figure $1(a,b)$ gives the regions of the (c,a)-plane for which at least a cycle of order k exists ($k=1,2,\ldots,14$). The black region ($k=15$) corresponds to the existence of bounded iterated sequences. Clearly, these figures exhibit the typical period doubling route to chaos obtained by increasing a for fixed c. We can recognize, in particular, two typical and well-known structures of the bifurcation diagrams in two-dimensional parameter plane, the so-called "saddle area" in the case $p=3$ and

$q = 2$, and saddle area with "cross-road area" in the case $p = 2$ and $q = 3$. The saddle area is special because associated with a "degenerate" bifurcation curve for $c = 1$.



(a) Bifurcation structure for $p = 2$ and $q = 3$

(b) Bifurcation structure for $p = 3$ and $q = 2$

FIGURE 1. two-dimensional bifurcation diagrams with colors obtained numerically according to the different orders observed in the plane (c, a).

3. Definitions and Fundamental Properties

In this section, we give precise notions in report with invertible polynomial maps, contact and homoclinic bifurcations, and some properties of increasing complexity that try to highlight the important concepts of nonlinear maps (refer to Mira et al. in [6]).

The polynomial map T of the plane has the form

$$(x',\ y') = T(x, y) = (f(x, y; \lambda), g(x, y; \lambda)),$$

where f et g are polynomials in x, y and λ is a real parameter-vector.The Jacobian determinant is defined as

$$\det J(f, g) = \det T(x, y) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}.$$

3.1. **General properties.** We assume that a closed and invariant set A is called an attracting set if some neighborhood U of A may exist such that $T(U) \subset U$ and $T^n(x, y) \to A$ as $n \to \infty$, for all $(x, y) \in U$. An attracting set A may contain one or several attractors (regular attractors are stable fixed points or cycles) coexisting with sets of repulsive points. The set $D = \cup_{n \geq 0} T^{-n}(U)$ is called the total basin of A,it is invariant under backward iteration T^{-1} of T, but not necessarily invariant by T

$$T^{-1}(\mathrm{D}) = \mathrm{D}, \quad T(\mathrm{D}) \subseteq \mathrm{D}.$$

An attracting set is called of order k if it is made up of k disjoint sets , $A = \cup_{i=1}^{k} A_i$, where each A_i is an attracting set of the map T^k.

When A is an attracting set of order $k = 1$, then its total basin is given by $D = D_0$ if it is connected, by $D = \cup_{n\geq 0} T^{-n}(\mathrm{D}_0)$ if it is nonconnected. When A is an attracting set of order $k > 1$, the immediate basin D_0 of A is the open set $D_0 = \cup_{i=1}^{n} D_{0,i}$, the $D_{0,i}$ being open disjoints basins of A_i. If A is connected attractor, the immediate basin D_0 of A is defined as the widest connected component of D containing A. When A is the widest attracting set of a map T, its basin D is the total basin of bounded iterates. That is, the open set D contains A such that D is the locus of points of the plane having bounded trajectories.

We assume that the existence of an attracting set A is observed through numerical methods.

Definition 3.1. Let S be a saddle fixed point of T, $W^s(S)$ and $W^u(S)$ denoting its stable and unstable sets. A point q is called homoclinic to S, if $q \in W^s(S) \cap W^u(S)$ and $q \neq S$. q is a transversal homoclinic point, so $W^s(S)$ intersects transversely $W^u(S)$.

Definition 3.2. One calls homoclinic orbit $O_o(q)$ associated with q, q belonging to a $U(S)$ of S, a set constituting of successive iterates of q, and its infinite sequence of preimages obtained by application of the local inverse map $T_l^{-1}$of T in $U(S)$, i.e., $O_o(q) = \left\{T_l^{-n}(q), q, T^n(q) : n > 0\right\} = \{\dots, q_{-n}, \dots, q_{-2}, q_{-1}, q, q_1, q_2, \dots, q_n, \dots\}$, where $q_n = T^n(q) \to S$ and $q_{-n} = T_l^{-n}(q) \to S$.

Definition 3.3. One calls heteroclinic orbit $\varepsilon(q)$ connecting S to S' associated with q, the one given by q together with its finite orbit and its infinite sequence of preimages obtained by application of the local inverse map $T_l^{-1}$of T in $U(S)$, i.e., $\varepsilon(q) = \left\{T_l^{-n}(q), q, T^n(q) : n > 0\right\} = \{\dots, q_{-n}, \dots, q_{-2}, q_{-1}, q, q_1, q_2, \dots, q_n, \dots\}$, where $q_n = T^n(q) \to S'$ and $q_{-n} = T_l^{-n}(q) \to S$.

3.2. **Generalized Hénon map properties.** First we recall the dynamics of the cubic diffeomorphism T_1

$$T_1(x, y) = (T_0(x) + y, cx).$$

$T_0(x)$ is a polynomial of degree-3 then T_1 is conjugate to Hénon map. We know some results which enable us to detect, predict, determine cycles and fixed points, and locate bifurcation curves in parameter plane. $T_0(x)$ can be equal to $x(1-x)^2$ or to $x^2(1-x)$.

Fixed point (x_*, y_*) of T_1 satisfies $y_* = cx_*$, and $(1-c)x_* = T_0(x_*)$, so that x_* is a root of the polynomial $q(x_*) = (c-1)x_* + T_0(x_*)$, thus all fixed points are located on the line $y - cx = 0$ in the plane.

The stability of these fixed points is determined by the Jacobian matrix

$$J_* = \begin{pmatrix} T_0'(x_*) & 1 \\ c & 0 \end{pmatrix},$$

which has trace $TrJ_* = T_0'(x_*)$ and determinant $\det J_* = -c$. The fixed point is stable when the eigenvalues of J_* are less than 1 in magnitude. This is true only when

J_* satisfies the three Jury conditions [4]: $1 - TrJ_* + \det J_* > 0,\ 1 + TrJ_* + \det J_* > 0,\ 1 - \det J_* > 0$.

It is easy to verify that T_1 can have bounded orbits only when there are fixed points of T_1^n.

It is sufficient to consider the case $|c| \leq 1$, since the inverse of a generalized Hénon map with $|c| > 1$ is conjugate to a generalized Hénon map with $|c| < 1$ under the reflection $r(x,y) = (y,x)$, and $r \circ T_1^{-1} \circ r = (y - T_0(\frac{x}{c}), \frac{x}{c}), T_1^{-1}(x,y) = (\frac{y}{c}, x - T_0(\frac{y}{c}))$.

Remark 3.1. For $c = 1$, the fixed points of T_1 are the roots of T_0. If $p = 3$ and $q = 2$, the determinant is equal to -1 and $T_0'(0) = 0$ with two eigenvalues -1,1. There is a fold-flip bifurcation for $O(0,0)$. For $p = 2$ and $q = 3$, $T_0'(1) = T_1(1) = 0$. These two cases are two nondegenerate codimension-2 bifurcations.

Theorem 3.1. *Suppose T_1 has no fixed points, then every orbits is unbounded.*

Proof. Suppose that T_1 has no fixed points, then the fixed point polynomial $q(x) = T_0(x) + cx - x$ is either positive or negative for all x. In the first case $q(x)$ is positive, consider $d(x,y) = x + y$, then $d(x',y') = d(x,y) + q(x)$ creases monotonically and must be unbounded. In the other case $q(x)$ is negative, $d(x',y')$ decreases monotonically and then either case there are no bounded orbits.

When there are fixed points, we can find a box that contains all these bounded orbits. □

Theorem 3.2. *Every bounded orbit of T_1 map is contained in the box*

$$\{(x,y) : |x| \leq M,\ |y| \leq |c|M\},$$

where M is the largest of the absolute values of the roots of $T_0(x) - (1 + |c|)|x|$.

Proof. See [1], more generally the polynomial determining M is the same as that for the fixed points, up to the absolute value signs. □

Proposition 3.1. *Concerning the existence of cycles of order 2, the following holds:*

- *cycles of order 2 occur for $T_1(x,y) = T_1^{-1}(x,y)$;*
- *they have to satisfy $T_0(x) + y = \frac{y}{c}$, $x - T_0\left(\frac{y}{c}\right) = cx$ and $(1-c)x - T_0\left(\frac{T_0(x)}{1-c)}\right) = 0$.*

Proof. Cycles of order-2 are given by the equation $T_1^2(x,y) = (x,y) = T_1^{-1} \circ T_1(x,y)$ and then it is easy to verify that $T_0(x) + y = \frac{y}{c}$, $x - T_0(\frac{y}{c}) = cx$, which is equivalent to $(1-c)x - T_0(\frac{T_0(x)}{1-c}) = 0$. This equation is divisible by $q(x)$ because fixed points are roots of both equations. Since T_1 is cubic then the equation giving 2-cycles is a polynomial of degree-6, there are at most three 2-cycles. □

Remark 3.2. By a way analogous to that in the proof of Proposition 3.1, we can determine without any difficulty the equations of cycles of higher order by using $T_1^n(x,y) = (x,y) = T_1^{-1} \circ T_1(x,y)$ which can be reduced to $T_1^{n-1}(x,y) = T_1^{-1}(x,y)$.

Similarly, 3-cycles are solutions of : $T_1^2(x,y) = (T_0(T_0(x) + y) + cx, cy + cT_0(x)) = T_1^{-1}(x,y) = (\frac{y}{c}, x - T_0(\frac{y}{c}))$. They are determined by the system $x_1 - c^2x_0 = T_0(x_0) - cT_0(x_1)$ and $x_0 - cx_1 = T_0(T_0(x_1) + cx_0)$, if we assume that $y = cx_0$.

4. Basins and Attractors for the Cubic Diffeomorphism

Now, we examine the behavior of T_1 on basin structure and its bifurcations. These bifurcations are characterized by the creation of heteroclinic and homoclinic connections or homoclinic tangles. Especially, we explain basin bifurcations which result from the contact between basin boundaries delimited by stable manifolds of the 2-cycle of saddle type and the nontrivial saddle fixed point (possibly a flip saddle).

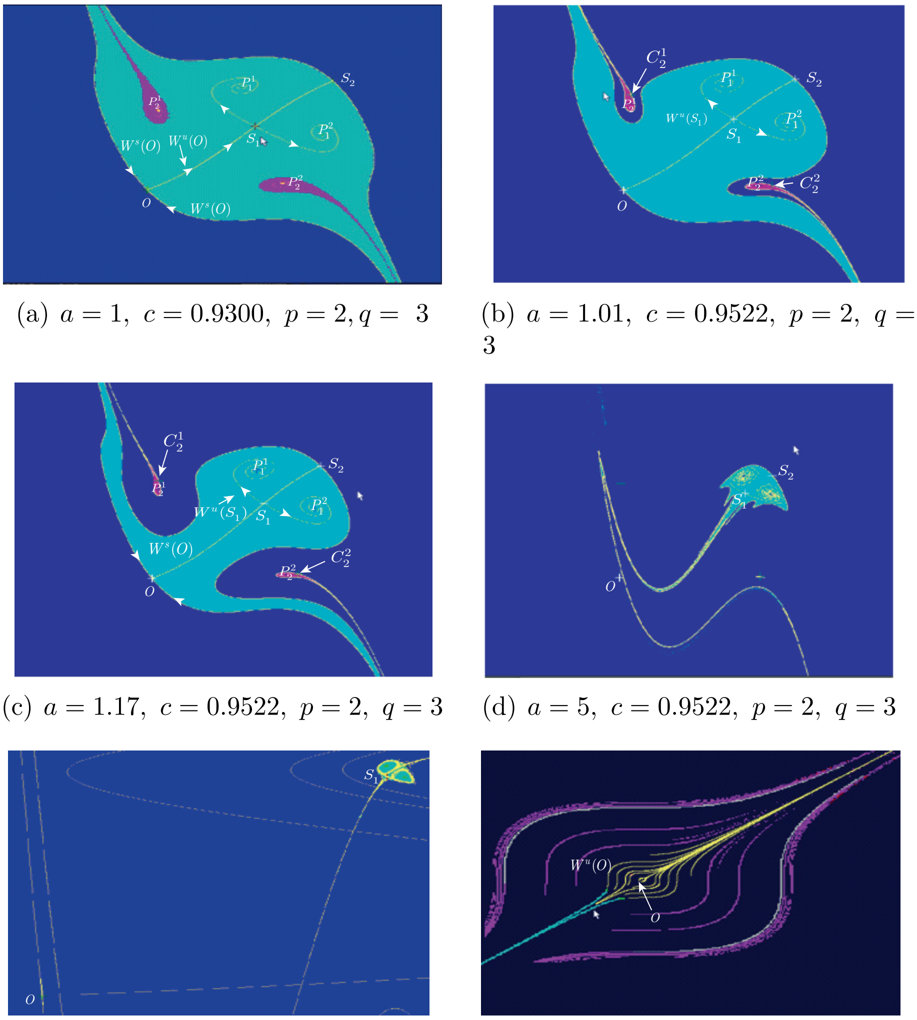
Figures 2 (a), (b), (c), (d), (e), (f) represent the existing attractors (fixed points and 2-cycles), invariant manifolds of saddle points and their basins. The evolution of attractors and their basins is given directly in figures, the parameters p, q have been chosen constant.

We start a qualitative description of bifurcations that are expected to occur as one parameter a or c is varied following a bifurcation path such c close of 1.0, we identify a very fascinating scenario in (a), (b), (c): two nontrivial fixed points are created by a saddle-node bifurcation and one of them (S_1) undergoes a period-doubling bifurcation and becomes a flip saddle. A further increase of the parameter a causes a contact between these two boundaries which marks changes in the basins of attraction from connected to nonconnected basins.

Here, if we consider $T_1 \circ T_1$, instead of T_1, points of 2-cycle correspond to fixed points of T_1^2 and then a flip bifurcation of T_1 corresponds to a pitchfork bifurcation of T_1^2. This implies that the same bifurcations are to be expected in the two cases.

The map generates many 2-cycles, we have three 2-cycles of which two are stable. We can see that the bifurcation which is put in evidence can be classified as a global bifurcation, only fixed points and 2-cycles exist and communicate through saddles. This kind of bifurcation involves attracting and repelling invariant curves issuing from saddles. Also, saddles on the boundary of basins play a major role because if they become outside the basins, thus transitions from “connected basin $\leftrightarrow$ nonconnected basin” occur. In particular, we remark that the sequence of bifurcations described in this work, cause the transition of a pair of 2-cycles from inside to outside a stable manifold associated with the saddle S_2. This invariant curve, involved in this global structure, exhibits different dynamic behaviors before and after the transition.

We can also see that in (e), (f) the basin associated with the 2 -cycle P_2 is destroyed and the trivial fixed point is outside, still exists and is unstable. In Figure 3 (a), (b), (c), (d), (e) all the fixed points are aligned but the single fixed point that always exists is stable, the other two fixed points are located on the boundary of the big basin and on the boundary of trivial fixed point basin. When the saddle point S_2 is outside then the basin becomes nonconnected, each point of 2-cycle has now its own basin. The stable manifold of the saddle point S_1 located on the boundary of the trivial fixed point $O(0,0)$ performs two loops and delimits after the basin of the unique attractor.



(a) $a = 1,\ c = 0.9300,\ p = 2, q = 3$ (b) $a = 1.01,\ c = 0.9522,\ p = 2,\ q = 3$

(c) $a = 1.17,\ c = 0.9522,\ p = 2,\ q = 3$ (d) $a = 5,\ c = 0.9522,\ p = 2,\ q = 3$

(e) $a = 16,\ c = 0.9,\ p = 2,\ q = 3$ (f) $a = 21,\ c = 1.0,\ p = 2,\ q = 3$

FIGURE 2. The three fixed points are unstable. The basin of the 2-cycle inside the big basin has a contact with the frontier of the big one, becomes outside and disappears.

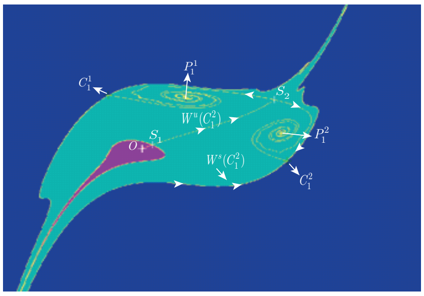
5. Bifurcations Basins for the Cubic Endomorphism

Let us consider now the noninvertible map T_2 defined by

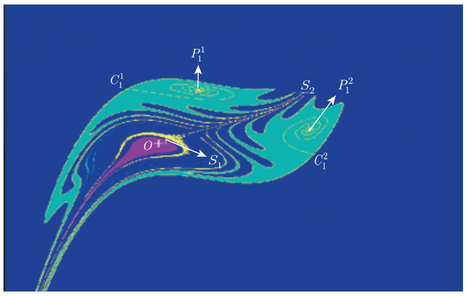
$$T_2 : \begin{cases} x' = T_0(x) + y, \\ y' = cx + dy, \end{cases}$$

where c, d are real parameters.

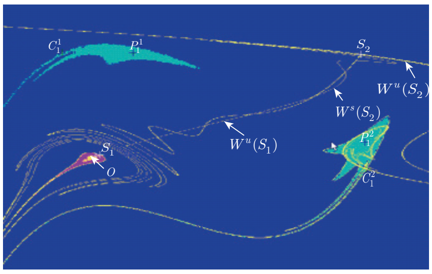
For $d \neq 0$, the system T_2 becomes again an endomorphism. We foresee that new phenomena are likely to occur for T_2. Figure 4 shows that the dynamics, influenced by the parameter d, revolves around fixed points and cycles of order-2 which exist



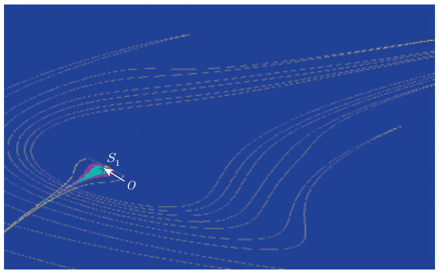
(a) $a = 1.50,\ c = 0.9,\ p = 3,\ q = 2$



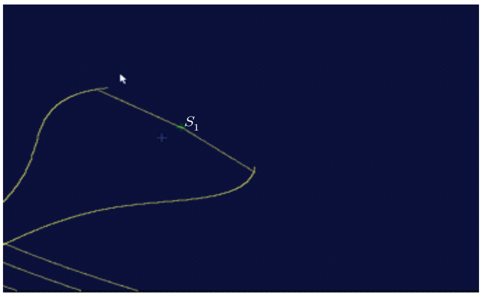
(b) $a = 1.96,\ c = 0.9,\ p = 3,\ q = 2$



(c) $a = 15,\ c = 0.95,\ p = 3,\ q = 2$



(d) $a = 15,\ c = 0.98,\ p = 3,\ q = 2$



(e) $a = 15,\ c = 0.98,\ p = 3,\ q = 2$

FIGURE 3. The red basin is associated with the trivial fixed point. The big attraction basin of a 2-cycle breaks after homoclinic-heteroclinic bifurcations.

respectively in blue and green domains for $p = 2$ and $q = 3$. Close enough to $c = 1$ (in this case $c = 0.952$) only 2-cycles are stable for $a = 1$, here fixed points exist but are unstable after a flip bifurcation.

5.1. **Study of the phase plane.** Our numerical evidence includes the following: for fixed parameter values, we plot attraction basins of attractors. Two types of basins are illustrated in this section. We first choose the parameters so that two attractors coexist. The two attractors do not undergo identical sets of bifurcations in the parameter plane. While one attractor can experience flip bifurcation, the second one undergoes fold bifurcation and we do this by having $c = 0.952$, and negative

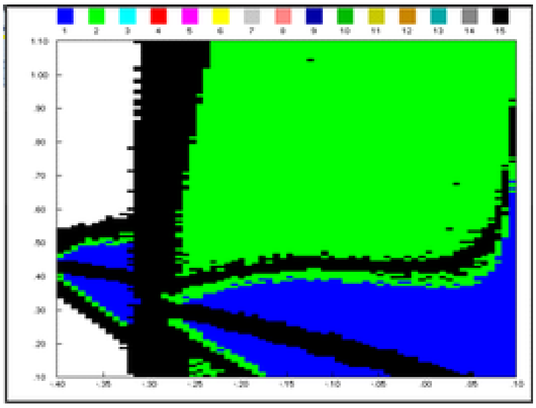


FIGURE 4. Bifurcation diagram in (d, a) parameter plane.

values of $d = -0.07$ which is instructive, with the occurrence of a change of type of bifurcations inside the same basin after heteroclinic bifurcations.

For the value $a = 1$, one has the following situation: two 2-cycles $(P_1^1,\ P_1^2)$ and $(P_2^1,\ P_2^2)$ which interact dynamically with a flip saddle point S_1 in the phase plane and their basins are delimited by stable manifolds of the two points of the 2-cycle of saddle type $(C_2^1,\ C_2^2)$ and the unstable manifold of the flip saddle point S_1.

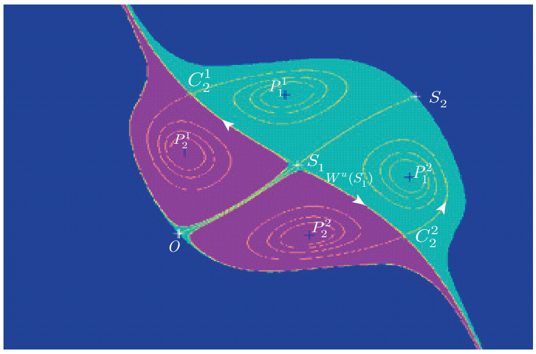


FIGURE 5. For the case $p = 2$, $q = 3$.

We decrease d, one has the following situation: the phase portrait of the recurrence T_2 at $d = -0.1$ is presented in Figure 6, the two stable 2-cycles exchange their associated saddles. It is in accordance with the bifurcation diagram in Figure 1 (a), the presence of cross-road area allows this change between attractors. For the case $p = 3$ and $q = 2$, we choose $c = 0.9$, $d = -0.32$, and $a = 1.5$, here also we have two 2-cycles which coexist with two flip saddle points and a regular saddle point located on their common frontier.

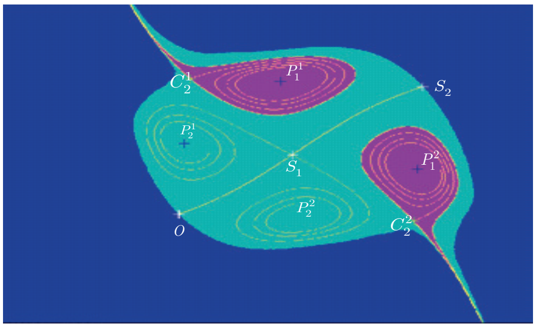


FIGURE 6. For the case $c = 0.952$, $d = -0.1$.

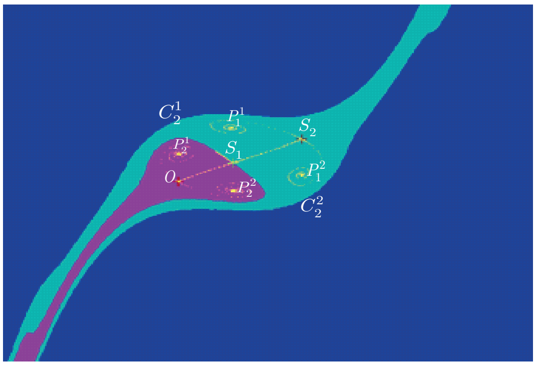


FIGURE 7. For the case $p = 3$, $q = 2$.

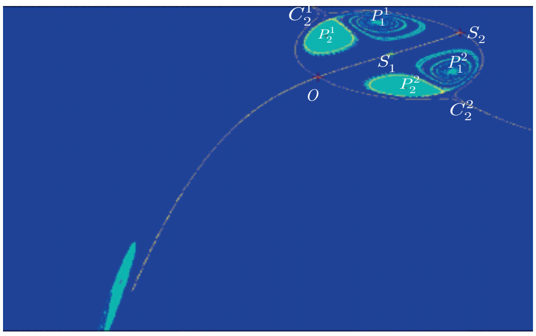


FIGURE 8. For the case $d = -0.355$, $p = 3$ and $q = 2$.

Here a, c are constant but $d = -0.355$, the two basins are now nonconnected and bounded, and a Hopf bifurcation takes place for the 2-cycle (P_2^1, P_2^2). We have a structural stable heteroclinic contour around basins.

6. Conclusion

Numerical explorations of cubic maps give interesting results, however, they reveal many intricate phenomena, that can only be understood by means of further specific investigation. A particularly rich bifurcation structure is detected near the limit value $c = 1$. Global bifurcations have important consequences as appearance of saddle connections and basins bifurcations. Heteroclinic bifurcations of saddle points, taking place on and inside the basins of attraction, this phenomenon provides a route for the appearance of nonconnected basins with saddles points located outside.

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